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APPROXIMATION OF EIGENVALUES OF DIFFERENTIAL EQUATIONS WITH NON-SMOOTH COEFFICIENTS (*)

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Abstract. — *The eigenvalues of second order, one dimensional, generalized eigenvalue problem with non-smooth coefficients are approximated by using the $\mathscr{L}_2$ – Finite Element method This method was introduced in* [4] *in the context of approximation of the solution of differential equations with non-smooth coefficients In this paper, error estimates for eigenvalues as well as eigenvectors are derived*

Resume — *Les valeurs propres de second ordre, dans des problemes de valeurs propres généralisés à une dimension avec des coefficients non réguliers, sont approximées par la méthode de $\mathscr{L}_2$-elements finis Cette méthode a été introduite dans* [4], *dans le contexte d'approximation des solutions, de certaines équations différentielles dont les coefficients étaient non réguliers Des estimations de l'erreur sur les valeurs propres aussi bien que celles sur les vecteurs propres sont obtenues dans cet article*

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1. INTRODUCTION

The eigenvalue problem for differential equations with non-smooth coefficients arise in the analysis of vibrations in structures with abruptly changing or smoothly but rapidly changing material properties, e.g., in structures composed of composite material [10]. These problems also arise in many other physical situations ([1], [11]).

The corresponding eigenfunctions of such problems are non-smooth and it is well known that the usual finite element method (FEM) does not yield

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accurate approximation to the eigenvalues. The use of a mixed method, employing trigonometric polynomial approximating functions, was suggested by Nemat-Nasser [12], [13] and the effectiveness of the method was shown by a series of numerical experiments. A posteriori error bounds were proved in [14] under certain assumptions on the spectrum.

Babuska and Osborn studied this method along with a related method employing finite element approximating functions, and proved convergence, rate of convergence estimates for eigenvalue approximation [2].

Later Babuska and Osborn studied the approximation of the solution of the source problems with non-smooth coefficients [4] by a class of methods, which they referred to as Generalized Finite Element Methods. One of these methods, called the $\mathscr{L}_2$-FEM in their paper, is closely related to the mixed method discussed in the previous paragraphs. It differs, however, in that different finite element approximating functions are employed.

In this paper, we will study the approximation of eigenvalues of differential equations with non-smooth coefficients using the $\mathscr{L}_2$-FEM. This method was not covered in [2]. The type of finite element approximating functions employed in $\mathscr{L}_2$-FEM are, however, more natural from a computational point of view than those employed in [2]. We have derived error estimates for approximate eigenvalues obtained from the $\mathscr{L}_2$-FEM, which show that this method is very accurate and robust for problems with non-smooth coefficients.

In Section 2, we give preliminary notions and notations. In Section 3, some known results of spectral approximation have been stated. The $\mathscr{L}_2$-FEM is introduced in Section 4 along with two finite dimensional subspaces. In Section 5 and Section 6, we present the error estimate for approximate eigenvectors and approximate eigenvalues obtained from the $\mathscr{L}_2$-FEM respectively. Finally we give some numerical results and conclusions in Section 9.

2. PRELIMINARIES

We will study the approximation of eigenvalues of the problem,

$$(2.1)\qquad \begin{aligned} Lu \equiv -(au')' + cu = \lambda bu\ , \quad 0 < x < 1\ ,\\ u(0) = u(1) = 0\ , \end{aligned}$$

where the coefficients a, c and b are functions of bounded variation satisfying,

$$0 < \alpha \leqq a(.)\ ,\quad b(.) \leqq \beta\ ,\quad 0 \leqq c(.) \leqq \beta\ .$$

Functions of bounded variation will be assumed to be left continuous and the total variation of a function a over the interval [0, 1] will be denoted by $V_0^1(a)$.

Problem (2.1) is a self-adjoint, positive definite eigenvalue problem with simple eigenvalues λ_j and corresponding eigenfunctions u_j satisfying

$$0 < \lambda_1 < \lambda_2 < \dots \uparrow \infty ,$$

and

$$\int_0^1 b u_i \, u_j \, dx = \delta_{ij} .$$

If $c(.) \equiv 0$, we write L_0 instead of L in (2.1).

Let $I = (0, 1)$ and let $W_p^k(I)$ be the usual Sobolev spaces consisting of functions with derivatives upto order k in $L_p(I)$. $\mathring{W}_p^k(I)$ is the subspace of $W_p^k(I)$ consisting of functions which vanish together with their first $(k-1)$ derivatives at $x = 0$ and $x = 1$. On $W_p^k(I)$, we have the usual norm

$$\|u\|_{k,p,I} = \begin{cases} \left(\sum_{j=0}^{k} \|u^{(j)}\|_{L_p(I)}^p \right)^{1/p} , & 1 \leqq p < \infty , \\ \max_{0 \leqq j \leqq k} \|u^{(j)}\|_{L_\infty(I)} , & p = \infty . \end{cases}$$

$W_p^k(I)$ and $\|.\|_{k,p,I}$ will be written as W_p^k and $\|.\|_{k,p}$ where the context is clear.

It is well known that eigenvalue problems are closely related to source problems, and hence, though we are primarily concerned with the eigenvalue problem, we state the following source problem :

$$\text{(2.2)} \qquad \begin{aligned} -(au')' + cu &= bf , \quad 0 < x < 1 , \\ u(0) = u(1) &= 0 , \end{aligned}$$

where the coefficients a, c and b are as defined in problem (2.1).

With problem (2.2), we associate the following bilinear forms :

$$B_0(u, v) = \int_0^1 a u' \, v' \, dx , \quad B_1(u, v) = \int_0^1 c u v \, dx .$$

The exact solution of (2.2) is also characterized via variational formulation as the unique $u \in \mathring{W}_p^1$ satisfying

$$\text{(2.3)} \qquad B(u, v) \equiv B_0(u, v) + B_1(u, v) = \int_0^1 b f v dx , \quad \forall v \in \mathring{W}_q^1 ,$$

where $f \in L_p$, $1 \leqq p \leqq \infty$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

It is well known that $B(.,.)$ satisfies the stability condition,

$$\inf_{\substack{u \in \mathring{W}_p^1 \\ \|u\|_{1,p} = 1}} \; \sup_{\substack{v \in \mathring{W}_q^1 \\ \|v\|_{1,q} = 1}} |B(u,v)| \geqq C > 0 \tag{2.4}$$

for $1 \leqq p \leqq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, where C is a constant.

Let $T : L_2(I) \to L_2(I)$ be the solution operator of the problem (2.3) with $p = 2$, such that

$$\begin{aligned} &Tf \in \mathring{W}_2^1 \\ &B(Tf, v) = \int_0^1 bfv \, dx, \quad \forall v \in \mathring{W}_2^1 . \end{aligned} \tag{2.5}$$

The operator T is compact on $L_2(I)$ and is self-adjoint with respect to the inner product

$$[f, g]_b = \int_0^1 bfg \, dx .$$

The variational formulation of the eigenvalue problem (2.1) is

$$\begin{aligned} &u \in \mathring{W}_2^1, \quad \lambda \in \mathbb{R}, \quad u \neq 0 \\ &B(u, v) = \lambda \int_0^1 buv \, dx, \quad \forall v \in \mathring{W}_2^1 . \end{aligned} \tag{2.6}$$

We note that (2.1) and (2.6) are equivalent in the sense that (2.6) is satisfied if and only if u and $au' \in W_2^1$, and (2.1) is satisfied almost everywhere. We further note that (λ, u) is an eigenpair of (2.6) if and only if (μ, u) is an eigenpair of the operator T where $\mu = 1/\lambda$.

3. A SPECTRAL APPROXIMATION RESULT

A general theory of approximation of eigenvalues and corresponding spaces of generalized eigenvectors of compact operators was developed in [7] and [15]. In [7], Bramble and Osborn took the Hilbert space setting to study the approximation of eigenvalues of non-self adjoint elliptic partial differential operators, whereas, in [15], Osborn established spectral approximation results for compact operators on Banach spaces. We will use the results from [15] and we state them without proof. An extensive work on these problem is also in [8].

Consider a compact operator $T : X \to X$ on a Banach space X. Let $\sigma(T)$ be the spectrum of T. The non-zero elements of $\sigma(T)$ are the eigenvalues of T. For non-zero $\mu \in \sigma(T)$, there is a smallest integer $\alpha \equiv \alpha(\mu)$, called the ascent of $\mu - T$, such that $E \equiv N((\mu - T)^{\alpha}) = N((\mu - T)^{\alpha+1})$, where N denotes the null space. E is finite dimensional and $m = \dim E$ is called the algebraic multiplicity of μ. The elements of E are called the generalized eigenvectors corresponding to μ, and those of $N(\mu - T)$ are the eigenvectors of T corresponding to μ. The $\dim N(\mu - T)$ is called the geometric multiplicity of T and is less than or equal to m.

Consider a sequence of compact operators $T_n : X \to X$ such that $\| T_n - T \|_{X \to X} \to 0$ as $n \to \infty$. Then it is well known that, if μ is a non-zero, isolated eigenvalue of T with algebraic multiplicity m, then exactly m eigenvalues of T_n, denoted by $\mu_n^1, \mu_n^2, \ldots, \mu_n^m$, will converge to μ for large n. Moreover, it is known that

$$\hat{\mu}_n = \left(\sum_{j=1}^{m} \mu_n^j \right) / m$$

is a better approximation to μ.

If μ is an eigenvalue of T with algebraic multiplicity m, then $\bar{\mu}$ is an eigenvalue of T^* with algebraic multiplicity m, where T^* is the adjoint of T. The ascent of $\bar{\mu} - T^*$ is α. Let E^* be the set of generalized eigenvector of T^* corresponding to $\bar{\mu}$.

We will now state the main results of [15] which provide estimates for $|\mu - \hat{\mu}|$ and for the error in eigenvectors. In order to state the eigenvector result, we need the concept of gap. Let L, M be two closed subspaces of X. Define

$$\delta(M, L) = \sup_{\substack{x \in M \\ \|x\| = 1}} \operatorname{dist}(x, L)$$

and

$$\hat{\delta}(M, L) = \max(\delta(M, L), \delta(L, M)) .$$

$\hat{\delta}(M, L)$ is the gap between M and L.

Let E_n be the direct sum of the generalized eigenvectors of T_n corresponding to $\{\mu_n^j\}_{j=1}^{m}$.

THEOREM 3.1 : *There exists a constant* $C > 0$ *such that*

$$\hat{\delta}(E, E_n) \leqq C \| (T - T_n)|_E \|_{X \to X} ,$$

for n large, where $(T - T_n)|_E$ *denotes the restriction of* $T - T_n$ to E. □

THEOREM 3.2 : *Let $\varphi_1, \varphi_2, \dots, \varphi_m$ be any basis for E. Then there exists $\varphi_1^*, \varphi_2^*, \dots, \varphi_m^*$ in E^* such that*

$$\mu - \hat{\mu}_n = \frac{1}{m} \sum_{j=1}^{m} [(T - T_n) \varphi_j, \varphi_j^*] + \varepsilon$$

where $|\varepsilon| \leqq C \| (T - T_n)|_E \|_{X \to X} \| (T^* - T_n^*)|_{E^*} \|_{X^* \to X^*}$. □

This result can be obtained by a slight modification of the proof of Theorem 3 in [15].

4. FINITE DIMENSIONAL APPROXIMATING SUBSPACES. THE $\mathscr{L}_2$-FEM

In this section, we will first describe two families of finite dimensional subspaces along with some of their properties. We will then describe the $\mathscr{L}_2$-FEM to approximate the solution of the source problem (2.2) and relevant error estimates. Finally we describe the $\mathscr{L}_2$-FEM to approximate the eigenvalues and eigenvectors of problem (2.1).

Let $\Delta = \{0 = x_0 < x_1 < \cdots < x_n = 1\}$, where $n = n(\Delta)$ is a positive integer, be an arbitrary mesh on $[0, 1]$ and set $I_j = (x_j, x_{j+1})$, $h_j = x_{j+1} - x_j$ for $j = 0, 1, \dots, n-1$, and $h = h(\Delta) = \max_{0 \leqq j \leqq n-1} h_j$.

For $r = 1, 2, \dots$, consider the following subspace or $W_1^1(I)$:

$$S_\Delta^r = \left\{ \varphi \in W_1^1(I) : \varphi|_{I_j} \in \mathscr{P}^r(I_j)\,, \quad j = 0, 1, \dots, n-1 \right\}$$

where $\mathscr{P}^r(I_j)$ is the space of polynomials on I_j of degree $\leqq r$. We write

$$S_{\Delta,0}^r = S_\Delta^r \cap \mathring{W}_1^1\,.$$

Suppose $u \in W_\infty^1(I)$ and let $I_\Delta u \in S_\Delta^r$ be the interpolant of u defined by

$$u(x_j) = (I_\Delta u)(x_j)\,, \quad j = 0, 1, \dots, n$$

$$\int_{I_j} (u - I_\Delta u)(x - x_j)^i \, dx = 0\,, \quad i = 0, 1, \dots, r-2\,,$$

$$j = 0, 1, \dots, n-2\,.$$

For $r = 1$, the second condition is empty. $I_\Delta u$ is called the S_Δ^r-interpolant of u.

We also define $(u)_\Delta$, which is the piecewise average of u, i.e.,

$$(u)_\Delta|_{I_j} = \left(\int_{I_j} u \, dx \right) / h_j\,.$$

We now state the following interpolation error estimates, which will be used in this paper.

THEOREM 4.1 (a) ([5], Theorem 4.1) : *Suppose* $u \in W_1^1(I)$ *such that* u' *is a function of bounded variation. Let* $I_\Delta u$ *be the* S_Δ^r*-interpolant of* u*. Then for* $l = 0, 1$ *and for* $1 \leqq p \leqq \infty$, *we have*

$$\|u - I_\Delta u\|_{l,p} \leqq C h^{1+(1/p)-l} V_0^1(u') .$$

(b) *If* u *is a function of bounded variation, then*

$$\|u - (u)_\Delta\|_{0,p} \leqq C h^{1/p} V_0^1(u) . \quad \square$$

The proof of (b) is simple. It can also be done by using Lemma 4.2 of [5].

Another subspace of $W_1^1(I)$, that we are about to describe, was introduced in [4] to approximate the solution of (2.2). For $r = 2, 3, \ldots$, consider the subspace

$$\tilde{S}_\Delta^r = \left\{ \varphi \in W_1^1(I) : L_0 \varphi \mid_{I_j} \in \mathscr{P}^{r-2}(I_j) , \ j = 0, 1, \ldots, n-1 \right\} ,$$

where L_0 is as defined in Section 2. For $r = 1$, we define

$$\tilde{S}_\Delta^r = \left\{ \varphi \in W_1^1(I) : L_0 \varphi \mid_{I_j} = 0 , \ j = 0, 1, \ldots, n-1 \right\} .$$

We write, $\tilde{S}_{\Delta,0}^r = \tilde{S}_\Delta^r \cap \mathring{W}_1^1$.

It was shown in [4] that for $u \in W_1^1(I)$, there exists a unique $\tilde{u} \in \tilde{S}_\Delta^r$, such that

$$u(x_j) = \tilde{u}(x_j) , \quad j = 0, 1, \ldots, n$$

$$\int_{I_j} (u - \tilde{u}) (x - x_j)^i \, dx = 0 , \quad i = 0, 1, \ldots, r-2 ,$$

$$j = 0, 1, \ldots, n-1 .$$

For $r = 1$, the second condition is empty. $\tilde{u}$ is called the $\tilde{S}_\Delta^r$-interpolant of u.

We will now describe the $\mathscr{L}_2$-FEM to approximate the solution of (2.2), as introduced in [4]. The method is stated as

$$u_\Delta \in S_{\Delta,0}^r$$

$$(4.1) \qquad B_0(\tilde{u}_\Delta, v) + B_1(u_\Delta, v) = \int_0^1 b f v \, dx , \quad \forall v \in S_{\Delta,0}^r ,$$

where $\tilde{u}_\Delta$ is the $\tilde{S}_{\Delta,0}^r$-interpolant of u_Δ.

Let $R_j : \mathscr{P}^{r-1}(I_j) \to \mathscr{P}^{r-1}(I_j)$ be defined as

$$R_j(v) = P(v/a)$$

where P is the L_2-projection operator onto $\mathscr{P}^{r-1}(I_j)$. It was shown in [4] that (4.1) can be written as

$$\sum_{j=0}^{n-1} \int_{I_j} Q_j u'_\Delta v' \, dx + B_1(u_\Delta, v) = \int_0^1 bfv \, dx , \quad \forall v \in S^r_{\Delta,0} ,$$

where $Q_j = R_j^{-1}$. It is possible to find Q_j explicitly, and thus one avoids the use of the basis functions of $\tilde{S}^r_{\Delta,0}$ in actual computation. The basis functions of $S^r_{\Delta,0}$, which are also used in a standard finite element method are used in the implementation of $\mathscr{L}_2$-FEM.

The following error estimate for $\mathscr{L}_2$-FEM will be used in this paper.

THEOREM 4.2 ([5], Theorem 6.2) : *Suppose $V^1_0(a) < \infty$ and functions b, c are measurable such that $0 < \alpha \leqq b(.) \leqq \beta$ and $0 \leqq c(.) \leqq \beta$. If u_Δ is the solution of* (4.1) *with $r \geqq 1$, then for $1 \leqq p \leqq \infty$*

$$\|u - u_\Delta\|_{0,p} \leqq Ch^{1+(1/\underline{\mu})} \|bf\|_{0,t} ,$$

where $\dfrac{1}{\underline{\mu}} = \min\left(\dfrac{1}{t'}, \dfrac{1}{p}\right)$, $\dfrac{1}{t} + \dfrac{1}{t'} = 1$ and $1 \leqq t \leqq \infty$. □

The $\mathscr{L}_2$-FEM to approximate the eigenvalues and eigenvectors of (2.1) is given by

$$u_\Delta \in S^r_{\Delta,0} , \quad \lambda_\Delta \in \mathbb{R} , \quad u_\Delta \neq 0$$

$$B_0(\tilde{u}_\Delta, v) + B_1(u_\Delta, v) = \lambda_\Delta \int_0^1 bu_\Delta v \, dx , \quad \forall v \in S^r_{\Delta,0} , \tag{4.2}$$

where $\tilde{u}_\Delta$ is the $\tilde{S}^r_{\Delta,0}$-interpolant of u_Δ.

5. EIGENVECTOR APPROXIMATION BY THE $\mathscr{L}_2$-FEM

In this section we will establish the rate of convergence of approximate eigenvectors of (2.1) obtained from $\mathscr{L}_2$-FEM, i.e., from (4.2).

Let T_Δ be the solution operator of (4.1) defined by

$$T_\Delta f \in S^r_{\Delta,0}$$

$$B_0(\widetilde{T_\Delta f}, v) + B_1(T_\Delta f, v) = \int_0^1 bfv \, dx , \quad \forall v \in S^r_{\Delta,0}$$

where $\widetilde{T_\Delta f}$ is the $\tilde{S}^r_{\Delta,0}$-interpolant of $T_\Delta f$ and $u_\Delta = T_\Delta f$ is the $\mathcal{L}_2$-FEM approximation to $u = Tf$. From Theorem 4.2, with $p = 2$ and $t = 2$, we get

$$\|(T - T_\Delta) f\|_{0,2} = \|u - u_\Delta\|_{0,2} \leqq Ch^{3/2} \|bf\|_{0,2} \leqq C\beta h^{3/2} \|f\|_{0,2} .$$

Thus, $T_\Delta \to T$ in $L_2(I)$ and all the results from Section 3 apply. Moreover one can show that T_Δ is selfadjoint with respect to $[. , .]_b$. In fact, the operators T and T_Δ can also be considered as mappings from $L_p(I)$ to $L_p(I)$, and using Theorem 4.2, we see that $T_\Delta \to T$ in $L_p(I)$.

We observe that $(\lambda_\Delta, u_\Delta)$ is an eigenpair of (4.2) if and only if (μ_Δ, u_Δ) is an eigenpair of T_Δ, where $\mu_\Delta = 1/\lambda_\Delta$. Since $T_\Delta \to T$, we conclude that $\lambda_\Delta \to \lambda$ (see Section 3). $(\lambda_\Delta, u_\Delta)$ is referred to as the $\mathcal{L}_2$-FEM approximation to (λ, u).

The rate of convergence of approximate eigenvectors obtained from $\mathcal{L}_2$-FEM is estimated as follows :

THEOREM 5.1 : *Suppose $V_0^1(a) < \infty$ and $r \geqq 1$. Let (μ, u) be an eigenpair of T such that $\|u\|_{0,p} = 1$. Let μ_Δ be an eigenvalue of T_Δ such that $\mu_\Delta \to \mu$. Then there is an eigenvector u_Δ corresponding to T_Δ and μ_Δ such that $\|u_\Delta\|_{0,p} = 1$ and*

$$\|u - u_\Delta\|_{0,p} \leqq Ch^{1+(1/p)} ,$$

where the constant $C = C(\alpha, \beta, |\lambda|, V_0^1(a)) > 0$.

Proof: From Theorem 3.1, we know that there exists $\underline{u}_\Delta \in E_\Delta$ such that

$$\|u - \underline{u}_\Delta\|_{0,p} \leqq C \|(T - T_\Delta)|_E\|_{L_p \to L_p} , \tag{5.1}$$

where E and E_Δ are the spaces of eigenvectors of T and T_Δ corresponding to μ and μ_Δ respectively. Set $u_\Delta = \underline{u}_\Delta / \|\underline{u}_\Delta\|_{0,p}$. Now

$$\|\underline{u}_\Delta - u_\Delta\|_{0,p} \leqq \|\underline{u}_\Delta\|_{0,p} \left| 1 - \frac{\|u\|_{0,p}}{\|\underline{u}_\Delta\|_{0,p}} \right| \leqq \|u - \underline{u}_\Delta\|_{0,p} .$$

Thus, from (5.1) we have

$$\begin{aligned} \|u - u_\Delta\|_{0,p} &\leqq \|u - \underline{u}_\Delta\|_{0,p} + \|\underline{u}_\Delta - u_\Delta\|_{0,p} \\ &\leqq 2C \|(T - T_\Delta)|_E\|_{L_p \to L_p} . \end{aligned} \tag{5.2}$$

By using Theorem 4.2 with $t = \infty$, we get

$$\|(T - T_\Delta)|_E\|_{L_p \to L_p} = \sup_{\substack{\varphi \in E \\ \|\varphi\|_{0,p} = 1}} \|(T - T_\Delta)\varphi\|_{0,p}$$

$$\leqq \sup_{\substack{\varphi \in E \\ \|\varphi\|_{0,p} = 1}} C h^{1 + (1/p)} \|b\varphi\|_{0,\infty}$$

$$\leqq \sup_{\substack{\varphi \in E \\ \|\varphi\|_{0,p} = 1}} C h^{1 + (1/p)} \|\varphi\|_{0,\infty} .$$

We can easily show, using (2.4), that

$$\|\varphi\|_{0,\infty} \leqq C|\lambda| \; \|\varphi\|_{0,p} ,$$

and hence, we get

$$\|(T - T_\Delta)|_E\|_{L_p \to L_p} \leqq C h^{1 + (1/p)} .$$

Using this in (5.2) we get the desired result. □

6. EIGENVALUE APPROXIMATION BY THE $\mathscr{L}_2$-FEM

We now start the discussion on the approximation of eigenvalues by $\mathscr{L}_2$-FEM. We will establish the rate of convergence of approximate eigenvalues for $r = 1$ and $r = 2$.

We first consider the case $r = 1$. In this case the methods of [15] can be applied directly to our problem to obtain convergence results. Consider the problem (4.2). We have seen in Section 5 that $T_\Delta \to T$ in $L_2(I)$ and hence $\mu_\Delta \to \mu$. Moreover μ is a simple eigenvalue of T.

THEOREM 6.1 : *Suppose* $r \geqq 1$, $V_0^1(a) < \infty$ *and* b, c *are measurable functions. Then there exist a constant* $C = C(\alpha, \beta, V_0^1(a)) > 0$, *such that*

$$|\lambda - \lambda_\Delta| \leqq C h^2 .$$

Proof: We know that T, T_Δ are self-adjoint with respect to the inner product $[\cdot\,,\cdot]_b$. Hence from Theorem 3.2, with $X = L_2(I)$, we get

$$(6.1) \qquad |\mu - \mu_\Delta| \leqq |[(T - T_\Delta)\varphi, \varphi]_b| + C\|(T - T_\Delta)|_E\|^2_{L_2 \to L_2}$$

where φ is any fixed vector in E with $[\varphi, \varphi]_b = 1$. Now,

$$\left|[(T - T_\Delta)\varphi, \varphi]_b\right| = \left|\int_0^1 b\varphi(T - T_\Delta)\varphi \, dx\right| \leqq \beta \|\varphi\|_{0,\infty} \|(T - T_\Delta)\varphi\|_{0,1}.$$

Since $V_0^1(a) < \infty$, we apply Theorem 4.2 with $p = 1$, $t = \infty$ to get

$$\|(T - T_\Delta)\varphi\|_{0,1} \leqq Ch^2 \|b\varphi\|_{0,\infty} \leqq C\beta h^2 \|\varphi\|_{0,\infty}.$$

From this and the fact that $\|\varphi\|_{0,\infty} \leqq C|\lambda| \|\varphi\|_{0,2}$ we get

$$|[(T - T_\Delta)\varphi, \varphi]_b| \leqq Ch^2 \|\varphi\|_{0,\infty}^2 \leqq Ch^2. \tag{6.2}$$

Also from Theorem 4.2 with $p = 2$, $t = 2$ we have

$$\|(T - T_\Delta)|_E\|_{L_2 \to L_2} = \sup_{\substack{\varphi \in E \\ \|\varphi\|_{0,2} = 1}} \|(T - T_\Delta)\varphi\|_{0,2} \leqq \sup_{\substack{\varphi \in E \\ \|\varphi\|_{0,2} = 1}} Ch^{3/2} \|\varphi\|_{0,2} \leqq Ch^{3/2}.$$

Hence from (6.1), (6.2) and the above inequality we get

$$|\mu - \mu_\Delta| \leqq Ch^2.$$

We know that $\lambda = 1/\mu$, $\lambda_\Delta = 1/\mu_\Delta$ and $|\lambda_\Delta| \leqq 2|\lambda|$ for $h = h(\Delta)$ small. Thus we get

$$|\lambda - \lambda_\Delta| \leqq C|\mu - \mu_\Delta| \leqq Ch^2. \quad \square$$

We will now consider the case $r = 2$ and the approximation of eigenvalues obtained from $\mathscr{L}_2$-FEM. In general, using $r = 2$ in (4.2), i.e., $\mathscr{L}_2$-FEM, one does not get a higher order of convergence with respect to h when functions a, b and c are of bounded variation. Nevertheless, a higher order of convergence can be shown in the special case of " non-coinciding singularities ", and we will present a result regarding one of these special cases.

Now onwards, for the sake of brevity, we will assume that $c(x) \equiv 0$. The nature of the results that we will prove, do not change with such an assumption, and at the end, we will state a result incorporating a non-negative, bounded function c of bounded variation.

It was shown in [4] that the $\mathscr{L}_2$-FEM is closely related to a mixed method obtained by discretizing a mixed variational formulation of (2.2). The equation (2.2) can be written as a system of two first order equations by introducing an auxiliary variable $s = au'$ and the associated variational formulation, known as a mixed variational formulation, is given by (for $c(x) \equiv 0$)

$$\begin{aligned} &u \in \mathring{W}_2^1(I), \quad s \in L_2(I) \\ (6.3) \qquad &a(s, \sigma) + b(\sigma, u) = 0, \quad \forall \sigma \in L_2(I), \\ &b(s, v) = -\int_0^1 bfv\, dx, \quad \forall v \in \mathring{W}_2^1(I), \end{aligned}$$

where

$$a(s, \sigma) = \int_0^1 (s\sigma/a)\, dx, \quad b(\sigma, u) = -\int_0^1 \sigma u'\, dx.$$

With Δ, an arbitrary mesh as defined before, a mixed method to approximate the solutions of (6.3) is given by

$$\begin{aligned} &u_\Delta \in S_{\Delta,0}^r, s_\Delta \in V_\Delta \\ (6.4) \qquad &a(s_\Delta, \sigma) + b(\sigma, u_\Delta) = 0, \quad \forall \sigma \in V_\Delta, \\ &b(s_\Delta, v) = -\int_0^1 bfv\, dx, \quad \forall v \in S_{\Delta,0}^r, \end{aligned}$$

where

$$V_\Delta = \left\{\sigma \in L_2(I) : \sigma\,|_{I_j} \in \mathscr{P}^{r-1}(I_j)\right\}.$$

Proceeding as with the source problem, the mixed variational formulation of (2.1), for $c(x) \equiv 0$, is given by

$$\begin{aligned} &u \in \mathring{W}_2^1(I), \quad s \in L_2(I), \quad \lambda \in \mathbb{R}, \\ (6.5) \qquad &a(s, \sigma) + b(\sigma, u) = 0, \quad \forall \sigma \in L_2(I), \\ &b(s, v) = -\lambda \int_0^1 buv\, dx, \quad \forall v \in \mathring{W}_2^1(I), \end{aligned}$$

and a mixed method for approximating the eigenvalues and eigenvectors is given by,

$$\begin{aligned} &u_\Delta \in S_{\Delta,0}^r, \quad s_\Delta \in V_\Delta, \quad \lambda_\Delta \in \mathbb{R}, \\ (6.6) \qquad &a(s_\Delta, \sigma) + b(\sigma, u_\Delta) = 0, \quad \forall \sigma \in V_\Delta, \\ &b(s_\Delta, v) = -\lambda_\Delta \int_0^1 bu_\Delta v\, dx, \quad \forall v \in S_{\Delta,0}^r. \end{aligned}$$

It was shown in [4] that u_Δ in (4.1) (for $c(x) \equiv 0$) is the same as u_Δ in (6.4). If the first equation of (6.4) was solved for s_Δ in terms of

u_Δ, and the result was substituted in the second equation of (6.4), it was shown in [4] that one obtained the equation (4.1), characterizing the $\mathscr{L}_2$-FEM. Thus the study of the approximation of solution of the source problem and likewise, the study of approximation of eigenvalues of the eigenvalue problem, using $\mathscr{L}_2$-FEM, is equivalent to the study of mixed methods given by (6.4) and (6.6) respectively.

Let T be the solution operator as in (2.5) with $c(x) \equiv 0$. Also let $\bar{T},\ S : L_2(I) \to L_2(I)$ be the solution operator of (6.3), i.e., $\bar{T}f = u$ and $Sf = s$ where u, s are as in (6.3). It is easily seen that $Tf = \bar{T}f$ and $a(Tf)' = Sf$. From now on we will use T instead of $\bar{T}$. Also if (λ, s, u) is an eigentriple of (6.5), then (μ, u) is an eigenpair of T where $\mu = 1/\lambda$.

Let $T_\Delta,\ S_\Delta :\ L_2(I) \to L_2(I)$ be the solution operators of (6.4), i.e., suppose $T_\Delta f = u_\Delta$ and $S_\Delta f = s_\Delta$, where u_Δ and s_Δ are as in (6.4). It can be shown that if $(\lambda_\Delta, s_\Delta, u_\Delta)$ is an eigentriple of (6.6), then (μ_Δ, u_Δ) is an eigenpair of T_Δ where $\mu_\Delta = 1/\lambda_\Delta$. We further remark that T_Δ, as defined here, is the same as the approximate solution operator of $\mathscr{L}_2$-FEM (with $c(x) \equiv 0$), which was defined in Section 5, and λ_Δ, as obtained from the mixed method defined here, is the same as the λ_Δ obtained from $\mathscr{L}_2$-FEM, i.e., from (4.2) with $c(x) \equiv 0$. We also know from Section 5 that $T_\Delta \to T$ in L_2-norm, and hence all the results of [14] are valid.

We will now prove two lemmas, which will be used to establish an upper bound for the first term on the right hand side of the equation in Theorem 3.2.

LEMMA 6.1 : *Let T, S, T_Δ, S_Δ be the operators defined in this section. Then, for $f,\ g \in L_2(I)$,*

$$[(T - T_\Delta) f, g]_b = \int_0^1 bf\{Tg - I_\Delta(Tg)\}\,dx + \int_0^1 bg\{Tf - I_\Delta(Tf)\}\,dx - a((S - S_\Delta) f, (S - S_\Delta) g)\,,$$

where $I_\Delta(Tf)$ and $I_\Delta(Tg)$ are the $S^r_{\Delta,0}$-interpolants of Tf and Tg respectively.

Proof: From (6.3), (6.4) and the definition of solution operators, we get,

$$a((S - S_\Delta) f, \sigma) + b(\sigma, (T - T_\Delta) f) = 0\,, \quad \forall \sigma \in V_\Delta\,,$$
$$b((S - S_\Delta) f, v) = 0\,, \quad \forall v \in S^r_{\Delta,0}\,,$$

and adding the above equations, we get

$$a((S - S_\Delta) f, \sigma) + b(\sigma, (T - T_\Delta) f) + b((S - S_\Delta) f, v) = 0\,, \tag{6.7}$$
$$\forall \sigma \in V_\Delta\,, \quad \forall v \in S^r_{\Delta,0}\,.$$

Now, using the first equation of (6.3) with $\sigma = (S - S_\Delta) f$ and the second equation of (6.3) with f replaced by g and $v = (T - T_\Delta) f$, we obtain

$$\begin{aligned}[(T - T_\Delta) f, g]_b - \\ &= \int_0^1 bg(T - T_\Delta) f\,dx \\ &= -b(Sg, (T - T_\Delta) f) \\ &= a(Sf, (S - S_\Delta) f) + b((S - S_\Delta) f, Tf) - b(Sg, (T - T_\Delta) f).\end{aligned}$$

Putting $\sigma = S_\Delta g$, $v = -T_\Delta f$ in (6.7) and adding it to the above equation, we get

$$\begin{aligned}[(T - T_\Delta) f, g]_b = a((S - S_\Delta) f, Sf + S_\Delta g) \\ + b((S - S_\Delta) f, Tf - T_\Delta f) \\ + b((S_\Delta - S) g, (T - T_\Delta) f).\end{aligned} \tag{6.8}$$

Now

$$\begin{aligned}a((S - S_\Delta) f, Sf + S_\Delta g) = a((S - S_\Delta) f, Sf) \\ + a((S - S_\Delta) f, (S_\Delta - S) g) \\ + a((S - S_\Delta) f, Sg).\end{aligned} \tag{6.9}$$

From the equations (6.3), (6.4) and noting that $b(v, \varphi - I_\Delta \varphi) = 0$, $\forall v \in V_\Delta$ (from the definition of I_Δ), we get

$$\begin{aligned}a((S - S_\Delta) f, Sf) &= -b((S - S_\Delta) f, Tf) \\ &= -b((S - S_\Delta) f, Tf - I_\Delta(Tf)) \\ &= -b(Sf, Tf - I_\Delta(Tf)) \\ &= \int_0^1 bf\{Tf - I_\Delta(Tf)\}\,dx.\end{aligned} \tag{6.10}$$

Similarly, by replacing f with g in equations (6.3), (6.4) one can show,

$$a((S - S_\Delta) f, Sg) = \int_0^1 bf\{Tg - I_\Delta(Tg)\}\,dx.$$

Thus from (6.9) and (6.10) we have

$$\begin{aligned}a((S - S_\Delta) f, Sf + S_\Delta g) &= \int_0^1 bf\{Tf - I_\Delta(Tf)\}\,dx \\ &+ \int_0^1 bf\{Tg - I_\Delta(Tg)\}\,dx \\ &+ a((S - S_\Delta) f, (S_\Delta - S) g).\end{aligned} \tag{6.11}$$

Again using the same arguments as before

$$(6.12)\quad \begin{aligned} b((S-S_\Delta)f, Tf-T_\Delta f) &= b((S-S_\Delta)f, Tf-I_\Delta(Tf)) \\ &= b(Sf, Tf-I_\Delta(Tf)) \\ &= -\int_0^1 bf\{Tf-I_\Delta(Tf)\}\,dx\,, \end{aligned}$$

and similarly,

$$b((S-S_\Delta)g, Tf-T_\Delta f) = \int_0^1 bg\{Tf-I_\Delta(Tf)\}\,dx\,.$$

Hence combining (6.8), (6.11), (6.12) and the above equality we get the desired result. □

The last term of the right hand side of the result in Lemma 6.1 can be bounded by an application of results in [9]. Problem (6.3) is the same as the problem discussed in [9] with $V = H = L_2(I)$, $W = \mathring{W}_2^1(I)$, and u, ψ in [9] is the same as s and u respectively in (6.3). Take $W_h = S^r_{\Delta,0}$ and $V_h = V_\Delta$.

LEMMA 6.2 : *Suppose* $r = 2$. *Also suppose* s, s_Δ *are as in* (6.3) *and* (6.4), *and* s' *is a function of bounded variation. Then*

$$\|s - s_\Delta\|_{0,2} \leqq C h^{3/2} V_0^1(s')\,.$$

Proof: We will use the Theorem 2 of [9] to prove this result. We first verify the assumptions necessary to apply Theorem 2 of [9].

The boundedness of the bilinear forms $a(.,.)$ and $b(.,.)$ is obvious. We now verify the hypotheses H1-H5 of [9]. H1 and H2 are trivial in our case since (6.3) has a unique solution for all $f \in L_2(I)$ and $a(.,.)$ is symmetric. H3 is immediate since

$$a(v,v) = \int_0^1 (v^2/a)\,dx \geqq \|v\|_{0,2}^2/\beta\,, \quad \forall v \in V_\Delta\,,$$

and hence is true for all $v \in Z_h$, where

$$Z_h = \{v \in V_\Delta : b(v,\varphi) = 0,\ \forall \varphi \in S^r_{\Delta,0}\}\,.$$

H4 is trivial since $H = V = L_2(I)$. To see that H5 hold, let the operator $\pi_h : Y \to V_\Delta$ be the L_2-projection of Y onto V_Δ, where $Y = \text{span}\ \{s\}$. In our case π_h is the L_2-projection operator onto C^{-1}, piecewise linear functions. Then we see that

$$b(s - \pi_h s, v) = 0\,, \quad \forall v \in S^r_{\Delta,0}\,.$$

So H1-H5 are satisfied. Also, it can be shown that $Z_h \subset Z$, where,

$$Z = \left\{ v \in L_2(I) : b(v, \varphi) = 0 , \quad \forall \varphi \in \overset{\circ}{W}{}_2^1(I) \right\} .$$

Now from Theorem 2 of [9] we have

$$\|s - s_\Delta\|_{0,2} \leqq [1 + (C/\beta)] \|s - \pi_h s\|_{0,2} ,$$

and since $\pi_h s$ is the L_2-projection of s onto V_Δ, hence using Theorem 4.1, we get

$$\begin{aligned} \|s - s_\Delta\|_{0,2} &\leqq C \|s - I_\Delta s\|_{0,2} \\ &\leqq C h^{3/2} V_0^1(s') , \end{aligned}$$

where $I_\Delta s$ is the $S^1_{\Delta,0}$-interpolant of s. □

We now prove the rate of convergence result for approximate eigenvalues obtained from $\mathscr{L}_2$-FEM, in one of the special cases of " non-coinciding singularity ".

THEOREM 6.2 : *Suppose* $0 < \varepsilon < 0.25$ *and* $r = 2$. *Suppose* $V_0^1(a)$, $V_0^1(b) < \infty$ *such that* $a'(.)$ *and* $b'(.)$ *exists in* $[0.5 - \varepsilon, 1]$ *and* $[0, 0.5 + \varepsilon]$ *respectively, and are bounded by* $M > 0$ *in these intervals. Then,*

$$|\lambda - \lambda_\Delta| \leqq C h^3 ,$$

where

$$C = C(\alpha, \beta, V_0^1(a), V_0^1(b)) > 0 .$$

Proof: Since T is self adjoint with respect to $[.,.]_b$, from Theorem 3.2 and Lemma 6.1, we have

$$\begin{aligned} |\lambda - \lambda_\Delta| &\leqq |[(T - T_\Delta)\varphi, \varphi]_b| + C \|(T - T_\Delta)|_E\|^2_{L_2 \to L_2} \\ &\leqq 2 \left| \int_0^1 b\varphi \{T\varphi - I_\Delta(T\varphi)\} \, dx \right| + \frac{1}{\alpha} \|s - s_\Delta\|^2_{0,2} \\ &\quad + C \|(T - T_\Delta)|_E\|^2_{L_2 \to L_2} , \end{aligned} \tag{6.13}$$

where φ is an eigenvector of T corresponding to $1/\lambda$, $[\varphi, \varphi]_b = 1$ and $s = a\varphi'$.

For h small enough, there exists a positive integer k, such that $I_k \subset [0.5 - \varepsilon, 0.5 + \varepsilon]$. Then for $j \leqq k$, b is smooth on I_j and for $j \geqq k$, a is smooth on I_j.

If $X_\Delta = (b\varphi)_\Delta$, i.e., the piecewise average of $b\varphi$, then, observing that φ is an eigenfunction of T, we get

$$\left|\int_0^1 b\varphi\{T\varphi - I_\Delta(T\varphi)\}\,dx\right| = \frac{1}{\lambda}\left|\int_0^1 b\varphi(\varphi - I_\Delta\varphi)\,dx\right|$$

(6.14)

$$\leqq \frac{1}{\lambda}\sum_j \int_{I_j} |(b\varphi - X_\Delta)(\varphi - I_\Delta\varphi)|\,dx\,.$$

For $j \geqq k$, $a|_{I_j}$ is smooth and hence φ'' exists and is bounded in I_j. Thus, using a standard interpolation result and Theorem 4.1(b) with I replaced by I_j and with $\Delta = \{x_j, x_{j+1}\}$, we get for $j \geqq k$,

$$\int_{I_j} |(b\varphi - X_\Delta)(\varphi - I_\Delta\varphi)|\,dx \leqq \|(b\varphi - X_\Delta)\|_{0,1,I_j}\|\varphi - I_\Delta\varphi\|_{0,\infty,I_j}$$

$$\leqq Ch^3 V_{x_j}^{x_{j+1}}(b\varphi)\|\varphi\|_{2,\infty,I_j}\,.$$

Now from (2.1) with $c(x) \equiv 0$,

$$\varphi''|_{I_j} = [-(\lambda b\varphi + a'\varphi')/a]|_{I_j}\,,$$

from which we get

$$\|\varphi\|_{2,\infty,I_j} \leqq C\|\varphi\|_{1,\infty}\,.$$

Thus for $j \geqq k$

$$\int_{I_j} |(b\varphi - X_\Delta)(\varphi - I_\Delta\varphi)|\,dx \leqq Ch^3 V_{x_j}^{x_{j+1}}(b\varphi)\|\varphi\|_{1,\infty}\,. \tag{6.15}$$

For $j < k$, b is smooth on I_j and hence using a standard interpolation result and Theorem 4.1(a) with I replaced by I_j and with $\Delta = \{x_j, x_{j+1}\}$, we get

$$\int_{I_j} |(b\varphi - X_\Delta)(\varphi - I_\Delta\varphi)|\,dx \leqq \|b\varphi - X_\Delta\|_{0,\infty,I_j}\|\varphi - I_\Delta\varphi\|_{0,1,I_j}$$

$$\leqq Ch^3 V_{x_j}^{x_{j+1}}(\varphi')\|b\varphi\|_{1,\infty,I_j}$$

$$\leqq Ch^3 V_{x_j}^{x_{j+1}}(\varphi')\|\varphi\|_{1,\infty}\,.$$

Thus from (6.14) and (6.15) we have,

$$\left|\int_0^1 b\varphi\{T\varphi - I_\Delta(T\varphi)\}\,dx\right| \leqq Ch^3\|\varphi\|_{1,\infty}\left[\sum_{j\geqq k} V_{x_j}^{x_{j+1}}(b\varphi) + \sum_{j<k} V_{x_j}^{x_{j+1}}(\varphi')\right]$$

(6.16)

$$\leqq Ch^3\|\varphi\|_{1,\infty}[V_0^1(b\varphi) + V_0^1(\varphi')]\,.$$

Now,

$$V_0^1(\varphi') = V_0^1(a\varphi'/a) \leqq \|a\varphi'\|_{0,\infty} V_0^1(a^{-1}) + \alpha^{-1} V_0^1(a\varphi')$$
$$\leqq \beta \|\varphi\|_{1,\infty} V_0^1(a^{-1}) + \alpha^{-1} \|(a\varphi')'\|_{0,1}$$
$$\leqq C \|\varphi\|_{1,\infty}$$

where C depends on α, β and $V_0^1(a^{-1})$. Using (2.4) and the above inequality, we get

$$V_0^1(\varphi') \leqq C \|\varphi\|_{0,2}. \tag{6.17}$$

Also,

$$V_0^1(b\varphi) \leqq \|\varphi\|_{0,\infty} V_0^1(b) + \beta V_0^1(\varphi)$$
$$\leqq C \|\varphi\|_{1,\infty} \leqq C \|\varphi\|_{0,2},$$

where C depends on β and $V_0^1(b)$. From (6.16) and the above inequalities, we have

$$\left| \int_0^1 b\varphi \{T\varphi - I_\Delta(T\varphi)\} \, dx \right| \leqq Ch^3 \|\varphi\|_{0,2}, \tag{6.18}$$

where C depends on α, β, $V_0^1(a)$ and $V_0^1(b)$.

Also from Lemma 6.2, we have

$$\|s - s_\Delta\|_{0,2} \leqq Ch^{3/2} V_0^1(s').$$

But,

$$V_0^1(s') = V_0^1[(a\varphi')'] = V_0^1(\lambda b\varphi) \leqq C \|\varphi\|_{0,2}$$

where C depends on α, β, $V_0^1(b)$. Thus

$$\|s - s_\Delta\|_{0,2} \leqq Ch^{3/2} \|\varphi\|_{0,2}. \tag{6.19}$$

Moreover, as in Theorem 5.1, it can be shown that

$$\|(T - T_\Delta)|_E\|_{L_2 \to L_2} \leqq Ch^{3/2} \|\varphi\|_{0,2}.$$

Thus from (6.13), (6.18) and (6.19) we get the desired result. □

We will now state a result similar to Theorem 6.2, without proof, when the function c is positive, bounded and of bounded variation.

THEOREM 6.3: *Suppose $0 < \varepsilon < 0.25$ and $r = 2$. Suppose a is a function of bounded variation such that $a'(.)$ exists in $[0.5 - \varepsilon, 1]$ and $|a'(x)| < M$ for $x \in [0.5 - \varepsilon, 1]$. Also suppose b, c are functions of bounded variations* *such that, $a'(.)$, $b'(.)$ are bounded in* $[0, 0.5 + \varepsilon]$ *and* $|b'(x)|$, $|c'(x)| < M$ *for* $x \in [0, 0.5 + \varepsilon]$. *If* λ_Δ *is the least eigenvalue obtained from* $\mathscr{L}_2$-FEM, *then,*

$$|\lambda - \lambda_\Delta| \leqq Ch^3 ,$$

where λ *is the least eigenvalue of* (2.1).

As another example of " non-coinciding singularities ", we state that if a, b, c are step functions with finite number of jumps such that the jumps of a are distinct from the jumps of b and c, then again one gets an $O(h^3)$ convergence of eigenvalues as in Theorem 6.3.

It is also possible to establish a lower bound for the error in approximate eigenvalues obtained from $\mathscr{L}_2$-FEM for the problem (2.1) with $c(x) \equiv 0$ and with a, b defined as (coinciding singularity) :

$$a(x) = \begin{cases} a_L , & x \leqq \xi , \\ a_R , & x > \xi , \end{cases} \tag{6.20}$$
$$b(x) = \begin{cases} b_L , & x \leqq \xi , \\ b_R , & x > \xi , \end{cases}$$

where $\xi \in (0, 1)$ and a_L, a_R, b_L, b_R are positive constants such that $0 < \alpha \leqq a_L, a_R, b_L, b_R \leqq \beta$. If the mesh Δ is taken such that $\xi \in [x_{J_0}, x_{J_0+1})$, $\xi - x_{J_0} = \theta h_{J_0}$, $0 \leqq \theta < 1$ and $\theta = \theta_\Delta$, then one can show that

(6.21)

$$|\lambda - \lambda_\Delta| \geqq |\lambda| \, |b_L - b_R| \, \frac{|a_L - a_R|}{8 \, a_L \, a_R} \, \theta^2 (1 - \theta)^2 \left| \frac{1}{2} - \theta \right| |u(\xi)| \, |au'(\xi)| \; h^2$$

where λ_Δ is obtained by $\mathscr{L}_2$-FEM with $r = 2$ and u is an eigenvector corresponding to λ with $[u, u]_b = 1$.

The proof of this result is obtained using Lemmas 6.1, 6.2, the lower bound aspect of Theorem 3.2 and also noting that in this case it is possible to find out the eigenvector u analytically and hence also $I_\Delta u$. A detailed proof is omitted because of its lengthy and tedious, otherwise trivial algebra and can be found in [6].

So, if $a_L \neq a_R$, $b_L \neq b_R$ and if the family of meshes is such that $\theta = \theta_\Delta$ satisfies $\theta_\Delta \geqq \gamma > 0$, $\theta_\Delta \leqq 1 - \gamma$ and $\left| \frac{1}{2} - \theta_\Delta \right| \geqq \gamma$, then from (6.21) it is clear that

$$|\lambda - \lambda_\Delta| \geqq Ch^2 .$$

Using again the results of this section, one can show that

$$|\lambda - \lambda_\Delta| \leqq Ch^3 ,$$

for a mesh family for which $\theta_\Delta = 0$ or $\frac{1}{2}$. The result for $\theta_\Delta = 0$ is not surprising since the jump in a, b are at a node of the mesh (one gets $|\lambda - \lambda_\Delta| \leqq Ch^4$ by standard methods). For $\theta_\Delta = \frac{1}{2}$, the result is not obvious and indicates an underlying symmetry.

7. NUMERICAL RESULTS AND CONCLUSION

In this section we present some numerical results and concluding remarks.

We first consider the example (6.20) with $b(x) \equiv 1$, $a_L = 1$, $a_R = 100$, and $\xi = 1/2$. We calculated λ_Δ using the $\mathscr{L}_2$-FEM with $r = 2$ and $\theta = 1/3$. It can be shown that exact eigenvalues of (6.20) are solutions λ of the equation

$$a_L \sqrt{\lambda b_L/a_L} \tan\left(\sqrt{\lambda b_R/4\, a_R}\right) - a_R \sqrt{\lambda b_R/a_R} \tan\left(\sqrt{\lambda b_L/4\, a_L}\right) = 0\,.$$

We have solved this equation by bisection method using double precision with a tolerance of 10^{-11} to get the " exact " eigenvalue λ. In figure 1, (in

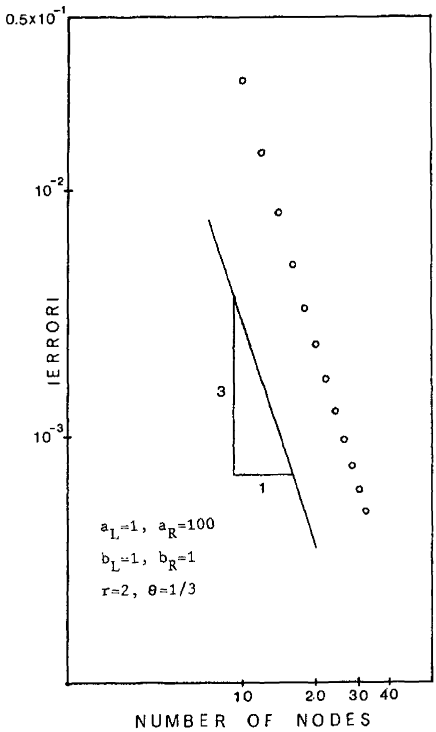

Figure 1.

other figures also) we have plotted the $|\text{error}| = |\lambda - \lambda_\Delta|$ against the number of mesh points. The absolute value of the slope of the graph should correspond to the order of convergence. In this case the error is of $O(h^3)$ as predicted in Theorem 6.2.

We next present the example (6.20) with $b_L = 100$, $b_R = 1$, $\xi = 1/2$ and same $a(x)$ as before. We calculated λ_Δ using the $\mathscr{L}_2$-FEM with $r = 1$ and $\theta = 1/3$. The results are given in figure 2 which shows that the error is of $O(h^2)$ as predicted in Theorem 6.1.

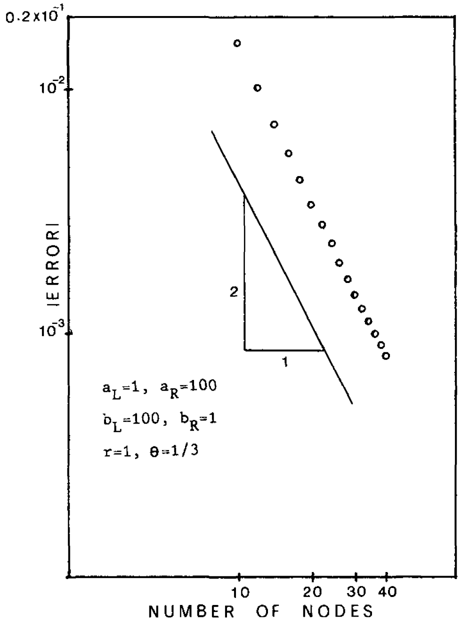

Figure 2.

This example does not show a higher order of convergence for $r = 2$ with arbitrary mesh. But for a special mesh with $\theta = 1/2$ (i.e. ξ is the midpoint of a subinterval), λ_Δ converges with $O(h^3)$. The results are given in figure 3.

Finally we conclude that the approximate eigenvalues obtained from the $\mathscr{L}_2$-FEM are more accurate than those obtained from the standard finite element method for these problems. Moreover the method is very robust in the sense that the constants that occur in error estimates depend on the bounds and the total variation of a, b and c. We further remark that the computational effort involved in the $\mathscr{L}_2$-FEM is same as the standard finite element method and thus should be preferred for problems with non-smooth coefficients.

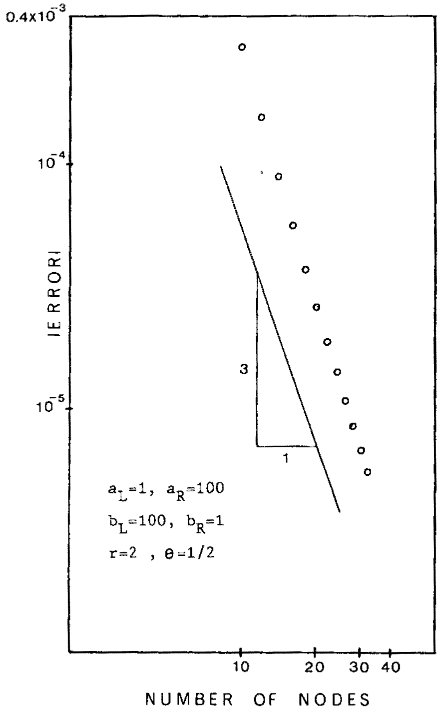

Figure 3.

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