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CONVERSE RESULTS IN THE WALSH THEORY OF OVERCONVERGENCE (*)

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Résumé. — *Récemment, J. Szabados a obtenu un nouveau théorème réciproque dans la théorie de la sur-convergence de Walsh, fondé sur l'Interpolation de Lagrange. Ici, nous développons un théorème réciproque similaire, fondé sur l'Interpolation d'Hermite, qui généralise le résultat de Szabados.*

Abstract. — *Recently, J. Szabados has obtained a new converse theorem in the Walsh overconvergence theory, based on Lagrange interpolation. Here, we similarly develop a related converse theorem, based on Hermite interpolation, which generalizes Szabados' result.*

1. INTRODUCTION

Let A_ρ denote the collection of functions analytic in $|z| < \rho$, and, as usual, let π_m denote the collection of all complex polynomials of degree at most m. For any $f(z) \in A_\rho$ with $\rho > 1$, and for any positive integer n, let $L_{n-1}(z\,;f)$ denote the Lagrange polynomial interpolant in π_{n-1} of $f(z)$ in the n-th roots of unity, i.e.,

$$L_{n-1}(\omega\,;f) = f(\omega)\,, \tag{1.1}$$

where ω is any nth root of unity. With $f(z) := \sum_{k=0}^{\infty} a_k z^k$ in $|z| < \rho$, and for each positive integer l, set

$$Q_{n-1,l}(z\,;f) := \sum_{j=0}^{l-1} \sum_{k=0}^{n-1} a_{k+jn} z^k\,, \tag{1.2}$$

so that $Q_{n-1,l}(z\,;f)$ is also an element of π_{n-1}. Then, the original and oft-cited

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beautiful result of J. L. Walsh [6, p. 153] on overconvergence is the case $l = 1$ of

THEOREM A ([1]) : *For any $f(z) \in A_\rho$ with $\rho > 1$, and for any positive integer l,*

$$\lim_{n \to \infty} \{ L_{n-1}(z\,;f) - Q_{n-1,l}(z\,;f) \} = 0\,, \quad \text{for all} \quad |z| < \rho^{l+1}\,, \tag{1.3}$$

the convergence being uniform and geometric on any closed subset of $|z| < \rho^{l+1}$. Moreover, the result is best possible (in the sense that (1.3) is not valid at each point of $|z| = \rho^{l+1}$ for all $f(z)$ in A_ρ).

Now Theorem A, in the terminology of approximation theory, is a *direct* theorem in the Walsh overconvergence theory, in that the assumption $f(z) \in A_\rho$ leads to the overconvergence result of (1.3). Recently, Szabados [4] obtained the following interesting *converse* theorem to Theorem A. For notation, let $A_1\,C$ denote the collection of all $f(z)$ in A_1 which are continuous on $|z| = 1$.

THEOREM B ([4]) : *Assume that $f(z) \in A_1\,C$. If $\rho > 1$, if l is a positive integer, and if the sequence*

$$\{ L_{n-1}(z\,;f) - Q_{n-1,l}(z\,;f) \}_{n=1}^{\infty} \tag{1.4}$$

is uniformly bounded on every closed subset of $|z| < \rho^{l+1}$, then $f(z) \in A_\rho$.

It may be asked if the conclusion of Theorem B (namely, that $f(z) \in A_\rho$) is best possible, i.e., with the hypothesis of Theorem B, could $f(z) \in A_{\rho'}$ where $\rho' > \rho$, in general ? On considering the particular function $\hat{f}(z) := (\rho - z)^{-1}$ which, with (1.3) satisfies the hypothesis of Theorem B, one sees that $\hat{f}(z)$ is an element of A_ρ, but is clearly not an element of $A_{\rho'}$ for any $\rho' > \rho$. In this sense, Theorem B is best possible, as was remarked by Szabados [4].

There are now many known direct theorems in the Walsh overconvergence theory on the difference of interpolating polynomials (cf. [1], Rivlin [2], [5, chap. 4]). It is natural to ask if there are similar converse theorems which complement Szabados' Theorem B. Here, we show that such a converse theorem can be similarly derived for Hermite polynomial interpolation.

2. STATEMENT OF A NEW RESULT.

We first state a direct theorem for Hermite interpolation in the Walsh overconvergence theory. To fix notations, for any $f(z) \in A_\rho$ with $\rho > 1$, for a fixed positive integer r, and for every positive integer n, let $h_{rn}(z\,;f)$ denote the Hermite polynomial interpolant in π_{rn-1} of $f, f', \dots, f^{(r-1)}$ in the nth roots of unity, i.e.,

$$h_{rn-1}^{(j)}(\omega\,;f) = f^{(j)}(\omega)\,, \qquad j = 0, 1, \dots, r-1\,, \tag{2.1}$$

where ω is any nth root of unity. Again, with $f(z) := \sum_{k=0}^{\infty} a_k z^k$ in $|z| < \rho$, and for any positive integer l, set

$$\tilde{Q}_{rn-1,l}(z;f) := \sum_{k=0}^{rn-1} a_k z^k + \sum_{j=1}^{l-1} \beta_{j,r}(z^n) \sum_{k=0}^{n-1} a_{k+(r+j-1)n} z^k, \tag{2.2}$$

where (cf. [1])

$$\beta_{j,r}(z) := \sum_{k=0}^{r-1} \binom{r+j-1}{k} (z-1)^k, \qquad j = 1, 2, \ldots, \tag{2.3}$$

and where the last sum in (2.2) is defined here, and subsequently, to be zero when $l = 1$. Note that $\tilde{Q}_{rn-1,l}(z)$ is also in π_{rn-1}. With these notations, a direct theorem for Hermite interpolation in the Walsh overconvergence theory is

THEOREM C ([1]) : *For any $f(z) \in A_\rho$ with $\rho > 1$, and for any positive integers r and l,*

$$\lim_{n\to\infty} \{ h_{rn-1}(z;f) - \tilde{Q}_{rn-1,l}(z;f) \} = 0, \quad \textit{for all} \quad |z| < \rho^{1+(l/r)}, \tag{2.4}$$

the convergence being uniform and geometric on any closed subset of $|z| < \rho^{1+(l/r)}$. Moreover, the result is best possible.

A new result, a converse result to Theorem C, is the following. For notation, for each positive integer r, let $A_1 C^{(r-1)}$ denote the collection of all $f(z)$ in A_1 for which $f(z)$, $f'(z)$, ..., and $f^{(r-1)}(z)$ are all continuous on $|z| = 1$. For any $f(z) \in A_1 C^{(r-1)}$ and for any $n \geqslant 1$, it is evident that the interpolatory polynomials $h_{rn-1}(z;f)$ and $\tilde{Q}_{rn-1,l}(z;f)$ of (2.1)-(2.2) are well-defined.

THEOREM 1 : *Assume that $f(z) \in A_1 C^{(r-1)}$. If $\rho > 1$, if l is a positive integer, and if the sequence*

$$\{ h_{rn-1}(z;f) - \tilde{Q}_{rn-1,l}(z;f) \}_{n=1}^{\infty} \tag{2.5}$$

is uniformly bounded on every closed subset of $|z| < \rho^{1+(l/r)}$, then $f(z) \in A_\rho$.

As the special case $r = 1$ of Theorem 1 reduces to Szabados' Theorem B, we remark that Theorem 1 then generalizes Theorem B.

The proof of Theorem 1 will be given in Section 3. Because it is needed in the proof of Theorem 1, we state, as in Theorem D below, a recent related result of Saff and Varga [3, theorem 2] on Hermite interpolation in the Walsh overconvergence theory.

THEOREM D ([3]) : *For each $f(z) \in A_\rho$, and for each pair of positive integers r and l, the sequence* (2.5) *can be bounded in at most $r + l - 1$ distinct points in $|z| > \rho^{1+(l/r)}$.*

3. PROOF OF THEOREM 1

With the notations from Section 2, we begin with the following result which, for $r = 1$, reduces to Lemma 1 of [4].

LEMMA 1 : *If $f(z) := \sum_{k=0}^{\infty} a_k z^k$ is an element of $A_1 C^{(r-1)}$, then for each positive integer l,*

$$h_{rn-1}(z;f) - \tilde{Q}_{rn-1,l}(z;f) = h_{rn-1}\left(z;\sum_{k=(r+l-1)n}^{\infty} a_k z^k\right). \qquad (3.1)$$

Proof : As $h_{rn-1}(z;f)$ of (2.1) is necessarily a linear operator which reproduces all polynomials of degree at most $rn-1$, then

$$\begin{aligned} h_{rn-1}(z;f) - h_{rn-1}\left(z;\sum_{k=(r+l-1)n}^{\infty} a_k z^k\right) &= h_{rn-1}\left(z;\sum_{k=0}^{(r+l-1)n-1} a_k z^k\right) \\ &= h_{rn-1}\left(z;\sum_{k=0}^{rn-1} a_k z^k\right) + h_{rn-1}\left(z;\sum_{k=rn}^{(r+l-1)n-1} a_k z^k\right) \\ &= \sum_{k=0}^{rn-1} a_k z^k + \sum_{k=rn}^{(r+l-1)n-1} a_k h_{rn-1}(z;z^k) \\ &= \sum_{k=0}^{rn-1} a_k z^k + \sum_{j=1}^{l-1}\sum_{k=0}^{n-1} a_{k+(r+j-1)n} h_{rn-1}(z;z^{k+(r+j-1)n}). \end{aligned}$$

It is known (cf. [1, eq. (4.4)]) that

$$h_{rn-1}(z;z^{k+(r+j-1)n}) = z^k \beta_{j,r}(z^n), \quad \text{for} \quad j = 1, 2, \ldots, \qquad (3.2)$$

where $\beta_{j,r}(z)$ is defined in (2.3). Inserting the above identity into the previous display gives, with the definition of $\tilde{Q}_{rn-1,l}(z;f)$ in (2.2), the desired result of (3.1). $\square$

Szabados [4] has pointed out that his special case $r = 1$ of Lemma 1 gives an elementary proof of Theorem A. We remark that Lemma 1 similarly gives an elementary proof of Theorem C. As its proof follows along the lines of the proof of Theorem 1, we omit the details.

Next, as $\beta_{j,r}(z)$ from (2.3), is in π_{r-1}, we can write

$$\beta_{j,r}(z) := \sum_{\nu=0}^{r-1} C_{\nu,r}(j)\, z^\nu, \qquad (3.3)$$

where evidently

$$C_{\nu,r}(j) := \sum_{k=\nu}^{r-1} (-1)^{k-\nu} \binom{r+j-1}{k} \binom{k}{\nu}, \quad \text{for} \quad \nu = 0, 1, \ldots, r-1. \quad (3.4)$$

LEMMA 2 : *The polynomials*

$$C_{\nu,r}(x) := \sum_{k=\nu}^{r-1} (-1)^{k-\nu} \binom{x+r-1}{k} \binom{k}{\nu} = \sum_{k=\nu}^{r-1} (-1)^{k-\nu} \frac{(x+r-1)\cdots(x+r-k)}{(k-\nu)!\,\nu!}, \quad (3.5)$$

for $\nu = 0, 1, \ldots, r-1$, *form a Lagrangian basis for* π_{r-1}, *i.e., for any* $p_{r-1}(x) \in \pi_{r-1}$,

$$p_{r-1}(x) \equiv \sum_{j=0}^{r-1} p_{r-1}(j+1-r)\, C_{j,r}(x), \quad \textit{for all} \quad x. \quad (3.6)$$

In particular, choosing $p_{r-1}(x) \equiv 1$ *in* (3.5) *gives*

$$1 = \sum_{\nu=0}^{r-1} C_{\nu,r}(\lambda + l) \quad \textit{for any integers } \lambda \textit{ and } l. \quad (3.7)$$

Proof : It is evident from (3.5) that

$$C_{\nu,r}(x+1-r) = \frac{x(x-1)\cdots(x-\nu+1)}{\nu!} \times$$

$$\times \left\{ 1 + \sum_{k=1}^{r-\nu-1} (-1)^k \frac{(x-\nu)(x-\nu-1)\ldots(x-k-\nu+1)}{k!} \right\}. \quad (3.8)$$

As the multiplier $x(x-1)\ldots(x-(\nu-1))$ in (3.8) vanishes for $x=0, 1, \ldots, \nu-1$, then $C_{\nu,r}(j+1-r) = 0$ for $j = 0, 1, \ldots, \nu-1$, while for $x = \nu$, (3.8) gives $C_{\nu,r}(\nu+1-r) = 1$. Similarly, for $x = \nu + l$ (where $1 \leqslant l \leqslant r-1$), the quantity in braces in (3.8) reduces to $\left\{ 1 + \sum_{k=1}^{l} (-1)^k \frac{l(l-1)\ldots(l-(k-1))}{k!} \right\}$, which is the binomial expansion of $(1-1)^l = 0$. Thus, we have shown that

$$C_{\nu,r}(j+1-r) = \delta_{j,\nu}, \quad \text{for all} \quad j = 0, 1, \ldots, r-1.$$

Consequently, $\{ C_{\nu,r}(x) \}_{\nu=0}^{r-1}$ forms a Lagrangian basis for π_{r-1}, from which (3.6) and (3.7) directly follow. □

Proof of Theorem 1 : Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be any element in $A_1\, C^{(r-1)}$ satisfying the hypothesis of Theorem 1, and let R be any number satisfying

$$1 < R < \rho^{1+(l/r)}. \quad (3.9)$$

Now, the boundedness hypothesis of (2.5) implies, from (3.1) of Lemma 1, that there is a constant $M(R)$ such that

$$\max_{|z|=R} \left| h_{rs-1}\left(\sum_{k=(r+l-1)s}^{\infty} a_k z^k \right) \right| \leqslant M(R) < \infty, \tag{3.10}$$

for any $s \geqslant 1$. In particular, choosing $s = 2n$ in (3.10) gives

$$\max_{|z|=R} \left| h_{2rn-1}\left(z ; \sum_{k=2(r+l-1)n} a_k z^k \right) \right| \leqslant M(R). \tag{3.11}$$

Next, setting

$$h_{2rn-1}\left(z ; \sum_{k=2(r+l-1)n}^{\infty} a_k z^k \right) := \sum_{k=0}^{2rn-1} b_k z^k, \tag{3.12}$$

the bound from (3.11), along with Cauchy's formula, implies

$$|b_k| \leqslant M(R).R^{-k}, \qquad k = 0, 1, \ldots, 2rn - 1. \tag{3.13}$$

Since the set of 2 nth roots of unity includes all nth roots of unity, we obtain (*cf.* (2.1)) the identity :

$$h_{rn}(z ; g) = h_{rn-1}(z ; h_{2rn-1}(z ; g)), \tag{3.14}$$

for any $g(z) \in A_1 C^{(r-1)}$. Choosing $g(z) := \sum_{k=2(r+l-1)n}^{\infty} a_k z^k$, then $g(z)$ is just $f(z)$, minus a polynomial, and is hence in $A_1 C^{(r-1)}$, for any $n \geqslant 1$. Using in succession the identity of (3.14), the definition of (3.12), the fact that h_{rn-1} is a linear operator which reproduces polynomials in π_{rn-1}, and the identity (3.2), we obtain the chain of equalities :

$$\begin{aligned} h_{rn-1}\left(z ; \sum_{k=2(r+l-1)n}^{\infty} a_k z^k \right) &= h_{rn-1}\left(z ; h_{2rn-1}\left(z ; \sum_{k=2(r+l-1)n}^{\infty} a_k z^k \right)\right) = \\ &= h_{rn-1}\left(z ; \sum_{k=0}^{2rn-1} b_k z^k \right) = \sum_{k=0}^{rn-1} b_k z^k + \sum_{k=0}^{rn-1} b_{k+rn} h_{rn-1}(z ; z^{k+rn}) \\ &= \sum_{k=0}^{rn-1} b_k z^k + \sum_{k=0}^{n-1} \sum_{\lambda=0}^{r-1} b_{k+(r+\lambda)n} h_{rn-1}(z ; z^{k+(r+\lambda)n}) \\ &= \sum_{k=0}^{rn-1} b_k z^k + \sum_{\lambda=0}^{r-1} \beta_{\lambda+1,r}(z^n) \sum_{k=0}^{n-1} b_{k+(r+\lambda)n} z^k, \quad \text{i.e.,} \end{aligned}$$

$$h_{rn-1}\left(z ; \sum_{k=2(r+l-1)n}^{\infty} a_k z^k \right) = \sum_{k=0}^{rn-1} b_k z^k + \sum_{\lambda=0}^{r-1} \beta_{\lambda+1,r}(z^n) \sum_{k=0}^{n-1} b_{k+(r+\lambda)n} z^k. \tag{3.15}$$

Now, it follows from the definition in (2.3) that

$$| \beta_{\lambda+1,r}(z^n) | \leqslant 2^{r+\lambda}(| z |^n + 1)^{r-1} \quad \text{for all} \quad z, \qquad \text{and all} \quad \lambda \geqslant 0,$$

from which it easily follows that

$$\max_{|z|=R} | \beta_{\lambda+1,r}(z^n) | \leqslant 2^{2r+\lambda} R^{nr}, \quad \text{for all} \quad \lambda \geqslant 0. \tag{3.16}$$

Applying the bounds of (3.16) and (3.13) to the terms of (3.15) gives, after an easy calculation, that

$$\max_{|z|=R} \left| h_{rn-1}\left(z ; \sum_{k=2(r+l-1)n}^{\infty} a_k z^k \right) \right| \leqslant n\, 2^{3r} M(R). \tag{3.17}$$

This can be used as follows. By linearity again,

$$h_{rn-1}\left(z ; \sum_{k=(r+l-1)n}^{2(r+l-1)n-1} a_k z^k \right) = h_{rn-1}\left(z ; \sum_{k=(r+l-1)n}^{\infty} a_k z^k \right) - \\ - h_{rn-1}\left(z ; \sum_{k=2(r+l-1)n}^{\infty} a_k z^k \right),$$

so that with (3.17) and (3.10) (for the case $s = n$),

$$\max_{|z|=R} \left| h_{rn-1}\left(z ; \sum_{k=(r+l-1)n}^{2(r+l-1)n-1} a_k z^k \right) \right| \leqslant (n\, 2^{3r} + 1) M(R). \tag{3.18}$$

Using in succession again the linearity of the operator h_{rn-1}, the identity of (3.2), and (3.4), we obtain

$$\begin{aligned} h_{rn-1}\left(z ; \sum_{k=(r+l-1)n}^{2(r+l-1)n-1} a_k z^k \right) &= \sum_{k=0}^{n-1} \sum_{\lambda=0}^{r+l-2} a_{k+(r+\lambda+l-1)n}\, h_{rn-1}(z ; z^{k+(r+\lambda+l-1)n}) \\ &= \sum_{k=0}^{n-1} \sum_{\lambda=0}^{r+l-2} a_{k+(r+\lambda+l-1)n}\, z^k \beta_{l+\lambda}(z^n) \\ &= \sum_{k=0}^{n-1} \sum_{\nu=0}^{r-1} z^{k+\nu n} \sum_{\lambda=0}^{r+l-2} C_{\nu,r}(\lambda + l)\, a_{k+(r+\lambda+l-1)n} . \end{aligned}$$

Applying Cauchy's formula and the bound of (3.18) to the above expression gives

$$\left| \sum_{\lambda=0}^{r+l-2} C_{\nu,r}(\lambda + l)\, a_{k+(r+\lambda+l-1)n} \right| \leqslant \frac{(n\, 2^{3r} + 1) M(R)}{R^{k+\nu n}}, \tag{3.19}$$

for all $k = 0, 1, \ldots, n-1$; $\nu = 0, 1, \ldots, r-1$.

Suppose we set

$$\sum_{\lambda=0}^{r+l-2} C_{\nu,r}(\lambda + l)\, a_{k+(r+\lambda+l-1)n} := \mu_{k,\nu,n}\,, \tag{3.20}$$

for $k = 0, 1, \ldots, n-1$; $\nu = 0, 1, \ldots, r-1$, where from (3.19),

$$|\,\mu_{k,\nu,n}\,| \leqslant \frac{(n\, 2^{3r} + 1)\, M(R)}{R^{k+\nu n}}\,. \tag{3.21}$$

On summing both sides of (3.20) with respect to ν and using the identify of (3.7), we can write

$$\sum_{j=r+l-1}^{2(r+l-1)-1} a_{k+jn} = \sum_{\nu=0}^{r-1} \mu_{k,\nu,n}\,,$$

so that

$$\left|\sum_{j=r+l-1}^{2(r+l-1)-1} a_{k+jn}\right| \leqslant \sum_{\nu=0}^{r-1} |\,\mu_{k,\nu,n}\,|\,.$$

Applying the upper bound of (3.21) then gives

$$\left|\sum_{j=r+l-1}^{2(r+l-1)-1} a_{k+jn}\right| \leqslant \frac{r(n\, 2^{3r} + 1)\, M(R)}{R^{k}}\,, \tag{3.22}$$

for all $k = 0, 1, \ldots, n-1$, all $n \geqslant 1$.

We now state a result which is implicit in the work of Szabados [4].

LEMMA 3 ([4]) : *If* $g(z) = \sum_{k=0}^{\infty} \alpha_k z^k$ *is an element of* $A_1\, C$, *and if, for each positive integer* s *and each* R *with* $1 < R < \rho^{l+1}$, *there is a constant* $M(R)$ *such that*

$$\left|\sum_{j=s}^{2s-1} \alpha_{k+jn}\right| \leqslant \frac{(2\,n + 1)\, M(R)}{R^k}\,, \quad \text{for all} \quad k = 0, 1, \ldots, n-1\,, \quad \text{all} \quad n \geqslant 1\,, \tag{3.23}$$

then

$$\overline{\lim_{n\to\infty}}\, |\,\alpha_n\,|^{1/n} \leqslant \left\{\begin{array}{ll} R^{-1/2}\,, & \text{if} \quad s = 1 \\ R^{-(3s^2+1)}\,, & \text{if} \quad s > 1 \end{array}\right\} < 1\,. \tag{3.24}$$

Lemma 3 can be applied as follows. As $f(z)$, by hypothesis an element in $A_1\, C^{(r-1)}$, is necessarily in $A_1\, C$, and as (3.22) holds, then (3.24) of Lemma 3 with $s = r + l - 1$ gives that

$$\overline{\lim_{n\to\infty}}\, |\,a_n\,|^{1/n} < 1\,. \tag{3.25}$$

This last inequality ensures, as in [4], that $f(z)$ can be analytically continued from $|z| \leqslant 1$ into a larger circle. Let $\bar{\rho} > 1$ be the *maximal* radius for which $f(z)$ is analytic in $|z| < \bar{\rho}$, so that $f(z)$ has a singularity on $|z| = \bar{\rho}$. But, by Theorem D, the sequence (2.6) can be bounded in at most $r + l - 1$ distinct points in $|z| > \bar{\rho}^{1+(l/r)}$. As the hypothesis of Theorem 1 ensures that this sequence is uniformly bounded on every closed subset of $|z| < \rho^{1+(l/r)}$, it is evident that $\rho \leqslant \bar{\rho}$, showing that $f(z) \in A_\rho$. $\square$

To conclude, we mention some open questions. It would be interesting to see if similar converse results hold for lacunary interpolation in the roots of unity, or for Rivlin's case [2] of l_2-convergence. Moreover, the above proof of Theorem 1 depends on the use of Saff and Varga's Theorem D. Is it possible to prove Theorem 1 *without* the use of Theorem D ?

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