Converse results in the Walsh theory of overconvergence


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CONVERSE RESULTS IN THE WALSH THEORY OF OVERCONVERGENCE (*)

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Abstract. — Recently, J. Szabados has obtained a new converse theorem in the Walsh overconvergence theory, based on Lagrange interpolation. Here, we similarly develop a related converse theorem, based on Hermite interpolation, which generalizes Szabados’ result.

1. INTRODUCTION

Let \( A_p \) denote the collection of functions analytic in \( |z| < \rho \), and, as usual, let \( \pi_m \) denote the collection of all complex polynomials of degree at most \( m \). For any \( f(z) \in A_p \) with \( \rho > 1 \), and for any positive integer \( n \), let \( L_{n-1}(z; f) \) denote the Lagrange polynomial interpolant in \( \pi_{n-1} \) of \( f(z) \) in the \( n \)-th roots of unity, i.e.,

\[
L_{n-1}(\omega; f) = f(\omega),
\]

where \( \omega \) is any \( n \)-th root of unity. With \( f(z) := \sum_{k=0}^{\infty} a_k z^k \) in \( |z| < \rho \), and for each positive integer \( l \), set

\[
Q_{n-1,l}(z; f) := \sum_{j=0}^{l-1} \sum_{k=0}^{n-1} a_{k+jn} z^k,
\]

so that \( Q_{n-1,l}(z; f) \) is also an element of \( \pi_{n-1} \). Then, the original and oft-cited

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beautiful result of J. L. Walsh [6, p. 153] on overconvergence is the case \( l = 1 \) of

**Theorem A** ([1]): For any \( f(z) \in A_\rho \) with \( \rho > 1 \), and for any positive integer \( l \),

\[
\lim_{n \to \infty} \{ L_{n-1}(z; f) - Q_{n-1,l}(z; f) \} = 0, \quad \text{for all } |z| < \rho^{l+1}, \tag{1.3}
\]

the convergence being uniform and geometric on any closed subset of \( |z| < \rho^{l+1} \). Moreover, the result is best possible (in the sense that (1.3) is not valid at each point of \( |z| = \rho^{l+1} \) for all \( f(z) \) in \( A_\rho \)).

Now Theorem A, in the terminology of approximation theory, is a direct theorem in the Walsh overconvergence theory, in that the assumption \( f(z) \in A_\rho \) leads to the overconvergence result of (1.3). Recently, Szabados [4] obtained the following interesting converse theorem to Theorem A. For notation, let \( A_1 C \) denote the collection of all \( f(z) \) in \( A_1 \) which are continuous on \( |z| = 1 \).

**Theorem B** ([4]): Assume that \( f(z) \in A_1 C \). If \( \rho > 1 \), if \( l \) is a positive integer, and if the sequence

\[
\{ L_{n-1}(z; f) - Q_{n-1,l}(z; f) \}_{n=1}^{\infty} \tag{1.4}
\]

is uniformly bounded on every closed subset of \( |z| < \rho^{l+1} \), then \( f(z) \in A_\rho \).

It may be asked if the conclusion of Theorem B (namely, that \( f(z) \in A_\rho \)) is best possible, i.e., with the hypothesis of Theorem B, could \( f(z) \in A_{\rho'} \) where \( \rho' > \rho \), in general? On considering the particular function \( \tilde{f}(z) := (\rho - z)^{-1} \) which, with (1.3) satisfies the hypothesis of Theorem B, one sees that \( \tilde{f}(z) \) is an element of \( A_\rho \), but is clearly not an element of \( A_{\rho'} \) for any \( \rho' > \rho \). In this sense, Theorem B is best possible, as was remarked by Szabados [4].

There are now many known direct theorems in the Walsh overconvergence theory on the difference of interpolating polynomials (cf. [1], Rivlin [2], [5, chap. 4]). It is natural to ask if there are similar converse theorems which complement Szabados' Theorem B. Here, we show that such a converse theorem can be similarly derived for Hermite polynomial interpolation.

2. Statement of a New Result.

We first state a direct theorem for Hermite interpolation in the Walsh overconvergence theory. To fix notations, for any \( f(z) \in A_\rho \) with \( \rho > 1 \), for a fixed positive integer \( r \), and for every positive integer \( n \), let \( h_{rn}(z; f) \) denote the Hermite polynomial interpolant in \( \pi_{r-1} \) of \( f, f', \ldots, f^{(r-1)} \) in the \( n \)th roots of unity, i.e.,

\[
h_{rn-1}^{(j)}(\omega; f) = f^{(j)}(\omega), \quad j = 0, 1, \ldots, r-1, \tag{2.1}
\]
where $\omega$ is any $n$th root of unity. Again, with $f(z) := \sum_{k=0}^{\infty} a_k z^k$ in $|z| < \rho$, and for any positive integer $l$, set

$$Q_{rn-1,l}(z; f) := \sum_{k=0}^{rn-1} a_k z^k + \sum_{j=1}^{l-1} \beta_{j,r}(z^n) \sum_{k=0}^{n-1} a_{k+(r+j-1)n} z^k, \quad (2.2)$$

where (cf. [1])

$$\beta_{j,r}(z) := \sum_{k=0}^{r-1} \left( \frac{r+j-1}{k} \right) (z-1)^k, \quad j = 1, 2, \ldots, \quad (2.3)$$

and where the last sum in (2.2) is defined here, and subsequently, to be zero when $l = 1$. Note that $Q_{rn-1,l}(z)$ is also in $\pi_{rn-1}$. With these notations, a direct theorem for Hermite interpolation in the Walsh overconvergence theory is

**Theorem C ([1]):** For any $f(z) \in A_\rho$ with $\rho > 1$, and for any positive integers $r$ and $l$, \[ \lim_{n \to \infty} \{ h_{rn-1}(z; f) - Q_{rn-1,l}(z; f) \} = 0, \quad \text{for all } |z| < \rho^{1+\frac{1}{(l/r)}}, \quad (2.4) \]

the convergence being uniform and geometric on any closed subset of $|z| < \rho^{1+\frac{1}{(l/r)}}$. Moreover, the result is best possible.

A new result, a converse result to Theorem C, is the following. For notation, for each positive integer $r$, let $A_1 C^{(r-1)}$ denote the collection of all $f(z)$ in $A_1$ for which $f(z)$, $f'(z)$, ..., and $f^{(r-1)}(z)$ are all continuous on $|z| = 1$. For any $f(z) \in A_1 C^{(r-1)}$ and for any $n \geq 1$, it is evident that the interpolatory polynomials $h_{rn-1}(z; f)$ and $Q_{rn-1,l}(z; f)$ of (2.1)-(2.2) are well-defined.

**Theorem 1:** Assume that $f(z) \in A_1 C^{(r-1)}$. If $\rho > 1$, if $l$ is a positive integer, and if the sequence

$$\{ h_{rn-1}(z; f) - Q_{rn-1,l}(z; f) \}_{n=1}^{\infty} \quad (2.5)$$

is uniformly bounded on every closed subset of $|z| < \rho^{1+\frac{1}{(l/r)}}$, then $f(z) \in A_\rho$.

As the special case $r = 1$ of Theorem 1 reduces to Szabados’ Theorem B, we remark that Theorem 1 then generalizes Theorem B.

The proof of Theorem 1 will be given in Section 3. Because it is needed in the proof of Theorem 1, we state, as in Theorem D below, a recent related result of Saff and Varga [3, theorem 2] on Hermite interpolation in the Walsh overconvergence theory.
THEOREM D ([3]) : For each \( f(z) \in A_p \), and for each pair of positive integers \( r \) and \( l \), the sequence (2.5) can be bounded in at most \( r + l - 1 \) distinct points in \( |z| > \rho^{1+(l/r)} \).

3. PROOF OF THEOREM 1

With the notations from Section 2, we begin with the following result which, for \( r = 1 \), reduces to Lemma 1 of [4].

**LEMMA 1** : If \( f(z) := \sum_{k=0}^{\infty} a_k z^k \) is an element of \( A_1 C^{(r-1)} \), then for each positive integer \( l \),

\[
\left( 2.5 \right) \quad h_{rn-1}(z; f) - \hat{Q}_{rn-1,l}(z; f) = h_{rn-1}(z; \sum_{k=0}^{\infty} a_k z^k).
\]

**Proof** : As \( h_{rn-1}(z; f) \) of (2.1) is necessarily a linear operator which reproduces all polynomials of degree at most \( rn - 1 \), then

\[
\begin{align*}
\left( 3.1 \right) \quad h_{rn-1}(z; f) - h_{rn-1}(z; \sum_{k=0}^{rn-1} a_k z^k) &= h_{rn-1}(z; \sum_{k=0}^{rn-1} a_k z^k) + h_{rn-1}(z; \sum_{k=rn}^{rn-l+n-1} a_k z^k) \\
&= \sum_{k=0}^{rn-1} a_k z^k + \sum_{k=rn}^{rn-l+n-1} a_k h_{rn-1}(z; z^k) \\
&= \sum_{k=0}^{rn-1} a_k z^k + \sum_{j=1}^{l-1} \sum_{k=0}^{n-1} a_k \beta_{j,r}(z^{k+(r+j-1)n}) \\
&= \sum_{k=0}^{rn-1} a_k z^k + \sum_{j=1}^{l-1} \sum_{k=0}^{n-1} \beta_{j,r}(z^{k+(r+j-1)n}).
\end{align*}
\]

It is known (cf. [1, eq. (4.4)]) that

\[
\left( 3.2 \right) \quad h_{rn-1}(z; z^{k+(r+j-1)n}) = z^k \beta_{j,r}(z^n), \quad \text{for} \quad j = 1, 2, ..., 
\]

where \( \beta_{j,r}(z) \) is defined in (2.3). Inserting the above identity into the previous display gives, with the definition of \( \hat{Q}_{rn-1,l}(z; f) \) in (2.2), the desired result of (3.1).

Szabados [4] has pointed out that his special case \( r = 1 \) of Lemma 1 gives an elementary proof of Theorem A. We remark that Lemma 1 similarly gives an elementary proof of Theorem C. As its proof follows along the lines of the proof of Theorem 1, we omit the details.

Next, as \( \beta_{j,r}(z) \) from (2.3), is in \( \pi_{r-1} \), we can write

\[
\left( 3.3 \right) \quad \beta_{j,r}(z) := \sum_{\nu=0}^{r-1} C_{\nu,r}(j) z^\nu,
\]

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LEMMA 2: The polynomials
\[ C_{\nu,r}(x) := \sum_{k=\nu}^{r-1} (-1)^{k-\nu} \binom{x+r-1}{k} \binom{k}{\nu} = \sum_{k=\nu}^{r-1} (-1)^{k-\nu} \frac{(x+r-1)\cdots(x+r-k)}{(k-\nu)!\nu!}, \]
for \( \nu = 0, 1, \ldots, r-1 \), form a Lagrangian basis for \( \pi_{r-1} \), i.e., for any \( p_{r-1}(x) \in \pi_{r-1} \),
\[ p_{r-1}(x) \equiv \sum_{j=0}^{r-1} p_{r-1}(j+1-r) C_{j,r}(x), \quad \text{for all } x. \] 
In particular, choosing \( p_{r-1}(x) \equiv 1 \) in (3.5) gives
\[ 1 = \sum_{\nu=0}^{r-1} C_{\nu,r}(\lambda + l) \quad \text{for any integers } \lambda \text{ and } l. \] 

Proof: It is evident from (3.5) that
\[ C_{\nu,r}(x + 1 - r) = \frac{x(x-1)\cdots(x-v+1)}{v!} \times \left\{ 1 + \sum_{k=1}^{r-v-1} (-1)^k \frac{(x-v)(x-v-1)\cdots(x-k-v+1)}{k!} \right\}. \] 
As the multiplier \( x(x-1)\cdots(x-(v-1)) \) in (3.8) vanishes for \( x = 0, 1, \ldots, v-1 \), then
\[ C_{\nu,r}(j+1-r) = 0 \quad \text{for } j = 0, 1, \ldots, v-1, \] 
while for \( x = v \), (3.8) gives \( C_{\nu,r}(v+1-r) = 1 \). Similarly, for \( x = v + l \) (where \( 1 \leq l \leq r-1 \)), the quantity in braces in (3.8) reduces to
\[ \left\{ 1 + \sum_{k=1}^{l} (-1)^k \frac{(l-1)\cdots(l-(k-1))}{k!} \right\}, \]
which is the binomial expansion of \((1-1)^l = 0\). Thus, we have shown that
\[ C_{\nu,r}(j+1-r) = \delta_{j,v}, \quad \text{for all } j = 0, 1, \ldots, r-1. \] 
Consequently, \( \{ C_{\nu,r}(x) \}_{\nu=0}^{r-1} \) forms a Lagrangian basis for \( \pi_{r-1} \), from which (3.6) and (3.7) directly follow. \[ \square \]

Proof of Theorem 1: Let \( f(z) = \sum_{k=0}^{\infty} a_k z^k \) be any element in \( A_1 C^{(r-1)} \) satisfying the hypothesis of Theorem 1, and let \( R \) be any number satisfying
\[ 1 < R < p^{1+(l/r)}. \]
Now, the boundedness hypothesis of (2.5) implies, from (3.1) of Lemma 1, that there is a constant $M(R)$ such that
\[
\max_{|z|=R} \left| h_{rs-1} \left( \sum_{k=(r+l-1)s}^{\infty} a_k z^k \right) \right| \leq M(R) < \infty , \quad (3.10)
\]
for any $s \geq 1$. In particular, choosing $s = 2$ in (3.10) gives
\[
\max_{|z|=R} \left| h_{2r-1} \left( z; \sum_{k=2(r+l-1)n}^{\infty} a_k z^k \right) \right| \leq M(R) . \quad (3.11)
\]
Next, setting
\[
h_{2r-1} \left( z; \sum_{k=2(r+l-1)n}^{2rn-1} a_k z^k \right) := \sum_{k=0}^{2rn-1} b_k z^k , \quad (3.12)
\]
the bound from (3.11), along with Cauchy's formula, implies
\[
| b_k | \leq M(R) . R^{-k} , \quad k = 0, 1, ..., 2rn - 1 . \quad (3.13)
\]
Since the set of $2n$th roots of unity includes all $n$th roots of unity, we obtain (cf. (2.1)) the identity :
\[
h_{2n}(z ; g) = h_{2n-1}(z ; h_{2n-1}(z ; g)) , \quad (3.14)
\]
for any $g(z) \in A_{1} C^{(r-1)}$. Choosing $g(z) := \sum_{k=2(r+l-1)n}^{\infty} a_k z^k$, then $g(z)$ is just $f(z)$, minus a polynomial, and is hence in $A_{1} C^{(r-1)}$, for any $n \geq 1$. Using in succession the identity of (3.14), the definition of (3.12), the fact that $h_{2n-1}$ is a linear operator which reproduces polynomials in $\rho_{2n-1}$, and the identity (3.2), we obtain the chain of equalities :
\[
h_{2n-1} \left( z; \sum_{k=2(r+l-1)n}^{\infty} a_k z^k \right) = h_{2n-1} \left( z; \sum_{k=0}^{2rn-1} b_k z^k \right) = \sum_{k=0}^{rn-1} b_k z^k + \sum_{k=0}^{rn-1} b_{k+r} h_{2n-1}(z ; z^{k+r})
\]
\[
= \sum_{k=0}^{rn-1} b_k z^k + \sum_{k=0}^{rn-1} \sum_{\lambda=0}^{r-1} b_{k+(r+\lambda)n} h_{2n-1}(z ; z^{k+(r+\lambda)n})
\]
\[
= \sum_{k=0}^{rn-1} b_k z^k + \sum_{\lambda=0}^{r-1} \sum_{k=0}^{n-1} b_{k+(r+\lambda)n} \beta_{\lambda+1,r}(z^n) z^k , \quad \text{i.e.,}
\]
\[
h_{2n-1} \left( z; \sum_{k=2(r+l-1)n}^{\infty} a_k z^k \right) = \sum_{k=0}^{rn-1} b_k z^k + \sum_{\lambda=0}^{r-1} \sum_{k=0}^{n-1} b_{k+(r+\lambda)n} \beta_{\lambda+1,r}(z^n) z^k . \quad (3.15)
\]
Now, it follows from the definition in (2.3) that
\[ |\beta_{\lambda+1,r}(z^n)| \leq 2^{r+\lambda}(|z|^n + 1)^{r-1} \text{ for all } z, \quad \text{and all } \lambda \geq 0, \]
from which it easily follows that
\[ \max_{|z|=R} |\beta_{\lambda+1,r}(z^n)| \leq 2^{2r+\lambda} R^{nr}, \text{ for all } \lambda \geq 0. \quad (3.16) \]
Applying the bounds of (3.16) and (3.13) to the terms of (3.15) gives, after an easy calculation, that
\[ \max_{|z|=R} |h_{r-1}(z; \sum_{k=2(r+1-1)n}^{\infty} a_k z^k)| \leq n 2^{3r} M(R). \quad (3.17) \]
This can be used as follows. By linearity again,
\[ h_{r-1}(z; \sum_{k=2(r+1-1)n}^{(2(r+1-1)n-1)} a_k z^k) = h_{r-1}(z; \sum_{k=2(r+1-1)n}^{\infty} a_k z^k) - h_{r-1}(z; \sum_{k=2(r+1-1)n}^{\infty} a_k z^k), \]
so that with (3.17) and (3.10) (for the case s = n),
\[ \max_{|z|=R} |h_{r-1}(z; \sum_{k=2(r+1-1)n}^{2(r+1-1)n-1} a_k z^k)| \leq (n 2^{3r} + 1) M(R). \quad (3.18) \]
Using in succession again the linearity of the operator \( h_{r-1} \), the identity of (3.2), and (3.4), we obtain
\[ h_{r-1}(z; \sum_{k=2(r+1-1)n}^{2(r+1-1)n-1} a_k z^k) = \sum_{k=0}^{n-1} \sum_{\lambda=0}^{r-1} a_k+(r+\lambda+l-1)n h_{r-1}(z; z^k+(r+\lambda+l-1)n) \]
\[ = \sum_{k=0}^{n-1} \sum_{\lambda=0}^{r-1} a_k+(r+\lambda+l-1)n z^k \beta_{l+\lambda}(z^n) \]
\[ = \sum_{k=0}^{n-1} \sum_{v=0}^{r-1} z^{k+v} z^{r-2} \sum_{\lambda=0}^{r-1} C_{v,r}(\lambda+1) a_k+(r+\lambda+l-1)n. \]
Applying Cauchy's formula and the bound of (3.18) to the above expression gives
\[ \left| \sum_{\lambda=0}^{r-1} C_{v,r}(\lambda+1) a_k+(r+\lambda+l-1)n \right| \leq \frac{(n 2^{3r} + 1) M(R)}{R^{k+v}}, \quad (3.19) \]
for all \( k = 0, 1, \ldots, n-1; v = 0, 1, \ldots, r-1. \)
Suppose we set
$$
\sum_{k=0}^{r+l-2} C_v, (k + l) a_{k+(r+l-1)n} := \mu_{k,v,n}, \tag{3.20}
$$
for \( k = 0, 1, \ldots, n - 1 \); \( v = 0, 1, \ldots, r - 1 \), where from (3.19),

$$
|\mu_{k,v,n}| \leq \frac{(n 2^{3r} + 1) M(R)}{R^{k+vn}}. \tag{3.21}
$$

On summing both sides of (3.20) with respect to \( v \) and using the identity of (3.7), we can write

$$
2(r+l-1) \sum_{j=0}^{r-1} a_{k+jn} = \sum_{v=0}^{r-1} \mu_{k,v,n},
$$
so that

$$
\left| \sum_{j=0}^{2(r+l-1)-1} a_{k+jn} \right| \leq \sum_{v=0}^{r-1} |\mu_{k,v,n}|.
$$

Applying the upper bound of (3.21) then gives

$$
\left| \sum_{j=0}^{2(r+l-1)-1} a_{k+jn} \right| \leq \frac{r(n 2^{3r} + 1) M(R)}{R^n}, \tag{3.22}
$$

for all \( k = 0, 1, \ldots, n - 1 \), all \( n \geq 1 \).

We now state a result which is implicit in the work of Szabados [4].

**Lemma 3 ([4]):** If \( g(z) = \sum_{k=0}^{\infty} \alpha_k z^k \) is an element of \( A_1 C \), and if, for each positive integer \( s \) and each \( R \) with \( 1 < R < p^{l+1} \), there is a constant \( M(R) \) such that

$$
\left| \sum_{j=s}^{2s-1} \alpha_{k+jn} \right| \leq \frac{(2n + 1) M(R)}{R^k}, \quad \text{for all } k = 0, 1, \ldots, n - 1, \quad \text{all } n \geq 1,
$$

then

$$
\lim_{n \to \infty} \left| \alpha_n \right|^{1/n} \leq \begin{cases} R^{-1/2}, & \text{if } s = 1 \\ R^{-(3s^2+1)}, & \text{if } s > 1 \end{cases} < 1. \tag{3.23}
$$

Lemma 3 can be applied as follows. As \( f(z) \), by hypothesis an element in \( A_1 C^{(r-1)} \), is necessarily in \( A_1 C \), and as (3.22) holds, then (3.24) of Lemma 3 with \( s = r + l - 1 \) gives that

$$
\lim_{n \to \infty} \left| a_n \right|^{1/n} < 1. \tag{3.25}
$$
This last inequality ensures, as in [4], that \( f(z) \) can be analytically continued from \( |z| \leq 1 \) into a larger circle. Let \( \rho > 1 \) be the maximal radius for which \( f(z) \) is analytic in \( |z| < \rho \), so that \( f(z) \) has a singularity on \( |z| = \rho \). But, by Theorem D, the sequence (2.6) can be bounded in at most \( r + l - 1 \) distinct points in \( |z| > \rho^{1+\ell/(\ell r)} \). As the hypothesis of Theorem 1 ensures that this sequence is uniformly bounded on every closed subset of \( |z| < \rho^{1+\ell/(\ell r)} \), it is evident that \( \rho \leq \rho' \), showing that \( f(z) \in A_{\rho} \). □

To conclude, we mention some open questions. It would be interesting to see if similar converse results hold for lacunary interpolation in the roots of unity, or for Rivlin’s case [2] of \( l_2 \)-convergence. Moreover, the above proof of Theorem 1 depends on the use of Saff and Varga’s Theorem D. Is it possible to prove Theorem 1 without the use of Theorem D?

REFERENCES


