MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

J. CAINZOS
M. LOBO-HIDALGO

Spectral perturbations in linear viscoelasticity of the Boltzmann type

Modélisation mathématique et analyse numérique, tome 19, nº 4 (1985), p. 559-572
<http://www.numdam.org/item?id=M2AN_1985__19_4_559_0>

MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

(vol. 19, n° 4, 1985, p. 559 à 572)

SPECTRAL PERTURBATIONS IN LINEAR VISCOELASTICITY OF THE BOLTZMANN TYPE (*)

by J. CAINZOS (1) and M. LOBO-HIDALGO (2)

Communiqué par E. SANCHEZ-PALENCIA

Abstract. — *We study the vibration frequencies of a viscoelastic body filling a bounded domain of* $\mathbb{R}^3$. *The viscosity term is multiplied by a small parameter* ε, *which is made to tend to* 0^+. *Let* ω_0 *be a vibration frequency of the elastic body* ($\varepsilon = 0$) *with algebraic multiplicity m. We prove that there exist, for small* ε, *vibration frequencies with total algebraic multiplicity m converging to* ω_0 *when* $\varepsilon \to 0^+$. *They constitute an algebroid singularity.*

Résumé. — *On considère les fréquences de vibration d'un corps viscoélastique remplissant un domaine borné de* $\mathbb{R}^3$. *Le terme de viscosité apparaît multiplié par un petit paramètre* ε, *que l'on fait tendre vers zéro. On établit que, si* ω_0 *est une fréquence propre de vibration de multiplicité m du corps élastique* ($\varepsilon = 0$), *pour* ε *suffisamment petit, il y a des fréquences de vibration du corps viscoélastique de multiplicité algébrique m qui convergent vers* ω_0 *et qui constituent une singularité algébroïde.*

1. INTRODUCTION

This paper deals with the study of the eigenvalues of a linear viscoelastic problem of Boltzmann type (see Dafermos [2], [3]) when the viscosity term is small.

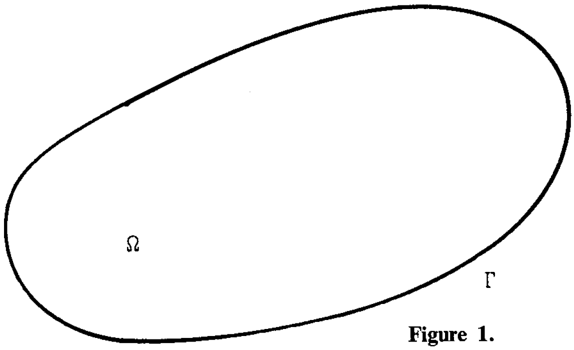

Figure 1.

(*) Received in October 1984.

(1) Departamento de Ecuaciones Funcionales. Facultad de Matematicas. Universidad de Santiago de Compostela (Spain).

(2) Departamento de Ecuaciones Funcionales. Facultad de Ciencias. Universidad de Santander (Spain).

Let Ω be a bounded domain of $\mathbb{R}^3$ with smooth boundary Γ and we consider in Ω a body that fills it.

The equilibrium equations are :

$$\frac{\partial}{\partial t}\left(\rho(\underline{x})\frac{\partial u_i^\varepsilon}{\partial t}\right) = \frac{\partial}{\partial x_j}(\sigma_{ij}^\varepsilon) \tag{1.1}$$

where :

$$\sigma_{ij}^\varepsilon(\underline{x}, t) = C_{ijkl}(\underline{x})\frac{\partial u_k^\varepsilon}{\partial x_l} - \varepsilon \int_{-\infty}^{t} G_{ijkl}(\underline{x}, t-\tau)\frac{\partial u_k^\varepsilon}{\partial x_l}(\tau)\, d\tau \tag{1.2}$$

and where $\varepsilon > 0$ is a small parameter.

The tensors $\{ C_{ijkl} \}$ and $\{ G_{ijkl} \}$ satisfy the usual elipticity and positivity conditions (see Dafermos [3]) for the differential and integro-differential terms in (1.2).

Some boundary conditions, either of Dirichlet type, or Neumann type, or mixed conditions, are associated with the problem (1.1). On the other hand, there are also some initial conditions as :

$$\underline{u}^\varepsilon(\underline{x}, t) = \underline{\phi}^\varepsilon(\underline{x}, t) \qquad t \in (-\infty, 0]\,. \tag{1.3}$$

The problem we study is that of vibration frequencies of the viscoelastic body, i.e. the search of functions $\zeta(\varepsilon)$ such that the problem (1.1) has solutions of the type :

$$\underline{u}^\varepsilon(\underline{x}, t) = \underline{u}(\underline{x})\, e^{-\zeta(\varepsilon)t} \tag{1.4}$$

under the boundary conditions imposed to the problem.

In Section 2.1 we study the abstract formulation of problem (1.1) within the framework of weak solutions following the techniques of contraction semigroups (see Dafermos [4]). We impose additional hypotheses of exponential decaying in ξ of the terms of the tensor $G_{ijkl}(\underline{x}, \xi)$, $\xi \in [0, \infty)$.

In section 2.2 we study the analytic dependence on ε of the eigenvalues $\zeta(\varepsilon)$. Also, we show the existence of a residual spectrum in a certain half-space $\operatorname{Re}\zeta \geqslant \frac{\mu}{2}$, where $\mu > 0$ is a characteristic exponent of the exponential decaying of $G_{ijkl}(\underline{x}, \xi)$.

Finally, in Section 3 we study the *reduction process* (see Kato [5]) within the framework of rigorous asymptotic analysis, to obtain the first terms of the asymptotic expansion of $\zeta(\varepsilon)$.

This work contains an improvement of the results obtained in Lobo-Hidalgo [7] : in that paper only the viscoelastic problem where the tensor $G_{ijkl}(\underline{x}, \xi)$ is factorizable as a product of functions $g(\xi)\, A_{ijkl}(\underline{x})$ was treated.

From the point of view of the computation of eigenvalues, a scalar equation was obtained in the factorizable case. We study now the general case when G_{ijkl} is not necessarily factorizable, and the eigenvalues equation becomes an equation in terms of operators in a certain Hilbert space V, of the form :

$$\left(\zeta^2 A - \varepsilon \int_0^\infty G(\xi)\, e^{\zeta\xi}\, d\xi\right) u + u = 0 \qquad u \in V\,, \quad u \neq 0 \qquad (1.5)$$

where A, $G(\xi) \in L(V, V)$.

The techniques we use are those of Ohayon and Sanchez-Palencia [9] and Sanchez-Palencia [10], where the elastic case is dealt with. For the reduction process, we have used similar techniques to those of Kato [5], chapter II. In Section 4, we give some comments and examples on the application of the abstract results previously obtained to different boundary condition problems.

2. SPECTRAL STUDY

2.1. Abstract functional setting

The problem (1.1), (1.3) with the appropriate boundary conditions can be considered in the following abstract setting.

Let V, H be two complex Hilbert spaces, $V \subset H$ with a dense and compact embedding, and let V' and H' be their antidual spaces.

Then we can write, with the appropriate identification

$$V \subset H \equiv H' \subset V' \qquad (2.1)$$

where each space is contained in the following one trough a dense and compact embedding.

We define a sesquilinear, hermitean, continuous and coercive form

$$a : V \times V \to \mathbb{C} \qquad (2.2)$$
$$a(u, u) \geqslant C_0 \, \| u \|_V^2 \, .$$

On the other hand, we consider a family of sesquilinear, hermitean and continuous forms depending on ξ :

$$\{ b(\xi, ., .) : V \times V \to \mathbb{C} / \xi \in [0, \infty) \} \qquad (2.3)$$

satisfying

$$b(\xi, u, u) \geqslant C(\xi) \, \| u \|_V^2 \qquad C(\xi) > 0\,, \quad \forall \xi \in [0, \infty) \qquad (2.4)$$

$$\forall u, v \text{ fixed}\,, \quad \xi \to b(\xi, u, v) \text{ is of class } C^1([0, \infty)) \qquad (2.5)$$

and modulus decreasing.

Now, let $\hat{A}$, $\hat{G}(\xi) \in L(V, V')$ be the operators associated with the forms $a(u, v)$, respectively $b(\xi, u, v)$. We make the following assumptions on $\hat{G}(\xi)$.

Let be $g(\xi) = \| \hat{G}(\xi) \|_{L(V;V')}$ such that there exist positive constant $\lambda_1, \lambda_2, \mu$ verifying :

$$\lambda_1 e^{-\mu\xi} \leqslant g(\xi) \leqslant \lambda_2 e^{-\mu\xi} \qquad \forall \xi \in [0, \infty) . \tag{2.6}$$

Under these hypotheses we pose the following problem :

Find $u_\varepsilon : (-\infty, \infty) \to V$ such that :

$$(u_\varepsilon''(t), v)_H + a(u_\varepsilon(t), v) - \varepsilon \int_{-\infty}^{t} b(t - \tau, u_\varepsilon(\tau), v)\, d\tau = 0 \quad \forall v \in V . \tag{2.7}$$

$$u_\varepsilon(t) = \phi(t) \qquad \forall t \in (-\infty, 0] .$$

The problem (2.7) can be written as an evolution equation in t (see Dafermos [4]). Calling $U = (u, v, w)$ with $v(t) = u'(t)$, $w(\xi, t) = u(t - \xi) - u(t)$, we can arrive at :

$$\frac{dU}{dt} + \mathscr{A}_\varepsilon U = 0 \tag{2.8}$$

$$U(0) = U_0$$

where $\mathscr{A}_\varepsilon$ is the operator defined as follows :

$$\mathscr{A}_\varepsilon U = \begin{cases} -v \\ \left(\hat{A} - \varepsilon \int_0^\infty \hat{G}(\xi)\, d\xi\right) u - \varepsilon \int_0^\infty \hat{G}(\xi)\, w(\xi)\, d\xi \\ \dfrac{\partial w}{\partial \xi} + v . \end{cases} \tag{2.9}$$

We consider the Hilbert space

$$H_\varepsilon = V \times H \times L_g^2(0, \infty ; V) \tag{2.10}$$

endowed with the scalar product :

$$(U, \hat{U})_\varepsilon = a(u, \hat{u}) - \varepsilon \int_0^\infty b(\xi, u, \hat{u})\, d\xi + (v, \hat{v})_H + \varepsilon \int_0^\infty b(\xi, w(\xi), \hat{w}(\xi))\, d\xi$$

and we consider the operator $\mathcal{A}_\varepsilon : D(\mathcal{A}_\varepsilon) \to H_\varepsilon$, where :

$$D(\mathcal{A}_\varepsilon) = \left\{ U \in H/v \in V, \left(\hat{A} - \varepsilon \int_0^\infty \hat{G}(\xi)\, d\xi\right) u - \varepsilon \int_0^\infty \hat{G}(\xi)\, w(\xi)\, d\xi \in H \right.,$$

$$\left. \frac{\partial w}{\partial \xi} \in L_g^2(0, \infty ; V), w(0) = 0 \right\} \quad (2.11)$$

then we arrive at the following existence and uniqueness results.

THEOREM 1 : *The operator* $- \mathcal{A}_\varepsilon$ *is the infinitesimal generator of a contraction semigroup* $\{ T_\varepsilon(t) \}_{t \geqslant 0}$ *that solves in a unique way the problem* (2.8) *with* $U_0 \in D(\mathcal{A}_\varepsilon)$.

Proof : The details can be seen in Dafermos [4] and Lobo-Hidalgo [7] for simpler cases, but the proof is essentially the same.

2.2. Eigenvalues equation

The search of the stationary solutions of exponential type given by the relation (1.4) leads us to the study of the point spectrum of the operator $\mathcal{A}_\varepsilon$ associated with the problem and given by the relation (2.9).

The spectral relation

$$\mathcal{A}_\varepsilon U = \zeta U \qquad \zeta \in \mathbb{C} \qquad U \equiv (u, v, w) \in H_\varepsilon \quad (2.12)$$

take us to the equivalent relation :

$$\zeta^2 (u, v)_H + a(u, v) - \varepsilon \int_0^\infty b(\xi, u, v)\, e^{\zeta\xi}\, d\xi = 0 \qquad \forall v \in V \quad (2.13)$$

where u is the first component of U.

The study of equation (2.13) shows the existence of a residual spectrum for $\mathcal{A}_\varepsilon$.

PROPOSITION 2.1 : *Under the assumptions* (2.1)-(2.6), *and for every sufficiently small* $\varepsilon \in (0, \varepsilon_0]$ *the half-plane* $\operatorname{Re} \zeta < 0$ *is a part of the resolvent set,* $\rho(\mathcal{A}_\varepsilon)$, *whereas the half-plane* $\operatorname{Re} \zeta \geqslant \frac{\mu}{2}$ *equals the residual spectrum of* $\mathcal{A}_\varepsilon$.

Proof : The details can be seen in Lobo-Hidalgo [7].

As a consequence of Proposition 2.1, the eigenvalues given by equation (2.12) lay in the strip

$$0 \leqslant \operatorname{Re} \zeta < \frac{\mu}{2}, \qquad \zeta \in \mathbb{C}. \quad (2.14)$$

Let us give an expression of the eigenvalues equation (2.13) in terms of bounded operators. In order to achieve that, let us consider in V the following scalar product (equivalent to the usual one)

$$(u, v)_V = a(u, v) \qquad \forall u, v \in V . \tag{2.15}$$

Then we have :

$$(u, v)_H = (Au, v)_V \qquad \forall u, v \in V \tag{2.16}$$

where $A : V \to V$ is a compact selfadjoint operator.

On the other hand

$$b(\xi, u, v) = (G(\xi)\, u, v)_V \qquad \forall u, v \in V \tag{2.17}$$

where $G(\xi) \in L(V, V)$.

PROPOSITION 2.2 : $G : [0, \infty) \to L(V, V)$ *is a continuous map, and the integral* :

$$\int_0^\infty G(\xi)\, e^{\zeta\xi}\, d\xi \tag{2.18}$$

exist for $\operatorname{Re} \zeta < \frac{\mu}{2}$ *and defines an operator of* $L(V, V)$.

Proof : The continuity and integrability of $G(\xi)\, e^{\zeta\xi}$ for $\operatorname{Re} \zeta < \frac{\mu}{2}$, follow immediately from the properties of $b(\xi, u, v)$ given in (2.3)-(2.6).

As a consequence, equation (2.13) admits the following expression :

$$\left(\zeta^2 A - \varepsilon \int_0^\infty G(\xi)\, e^{\zeta\xi}\, d\xi\right) u + u = 0 , \qquad u \in V , \quad u \neq 0 . \tag{2.19}$$

2.3. Existence of eigenvalues

Let $\omega_0 \in \mathbb{C}$ be such that $-\omega_0^2$ is an eigenvalue of the operator A.

Remark 2.1 : Since A is compact and selfadjoint, its eigenvalues form a bounded sequence $\{ \alpha_n^2 \}$ of positive real numbers. Therefore $\omega_0 = \pm \alpha_n i$ for some $n \in \mathbb{N}$.

Now, let γ be a simple closed curve that surrounds ω_0, and such that it does not surround any other value ω for which $-\omega^2$ is an eigenvalue of A, thus $-\gamma^2 \subset \rho(A)$.

On the other hand, let us consider the operator $T(\varepsilon, \omega)$ given by :

$$T(\varepsilon, \omega) = A - \varepsilon\omega^2 \int_0^\infty G(\xi)\, e^{\xi/\omega}\, d\xi . \tag{2.20}$$

It is obvious that $T(\varepsilon, \omega)$ is obtained formally from the involved in (2.19) through the change $\zeta = \frac{1}{\omega}$.

PROPOSITION 2.3 : *The curve* $-\gamma^2 \subset \mathbb{C}$ *is contained in the resolvent set of* $T(\varepsilon, \omega)$ *for sufficiently small* $\varepsilon > 0$.

Proof : The proof is divided in several steps.

a) $T(\varepsilon, \omega)$ is jointly holomorphic, in (ε, ω). This assertion is a consequence of the holomorphy on each of the variables (*). The domain of holomorphy D of the operator $T(\varepsilon, \omega)$ is the following :

$$D = \{ (\varepsilon, \omega) \in \mathbb{C} \times \mathbb{C} / \omega \notin B_\mu \} \tag{2.21}$$

$$B_\mu = \left\{ \omega \in \mathbb{C} \middle/ \left| \omega - \frac{1}{2\mu} \right| \leqslant \frac{1}{2\mu} \right\}. \tag{2.22}$$

Then, taking γ of sufficiently small radius (see *fig.* 2), the holomorphy of $T(\varepsilon, \omega)$ on a neighbourhood U_γ of γ follows.

b) We claim $-\gamma^2 \subset \rho(T(\varepsilon, \omega))$ for $(\varepsilon, \omega) \in D$ such that ε is sufficiently small. Let $\eta \in -\gamma^2$, and write

$$R(\eta) = (A - \eta)^{-1}, \qquad B(\omega) = \int_0^\infty G(\xi)\, e^{\xi/\omega}\, d\xi . \tag{2.23}$$

We arrive at the following expression

$$T(\varepsilon, \omega) - \eta = [I - \varepsilon\omega^2 B(\omega) R(\eta)] (A - \eta) . \tag{2.24}$$

Finally

$$(T(\varepsilon, \omega) - \eta)^{-1} = R(\eta) [I - \varepsilon\omega^2 B(\omega) R(\eta)]^{-1} . \tag{2.25}$$

For the existence of the right hand member in (2.25) it suffices with the estimate

$$\| \varepsilon\omega^2 B(\omega) R(\eta) \| < 1 ; \tag{2.26}$$

(*) We refer here to the Hartogs theorem for maps with values in an infinite dimensional space. (See for instance Noverraz [8].)

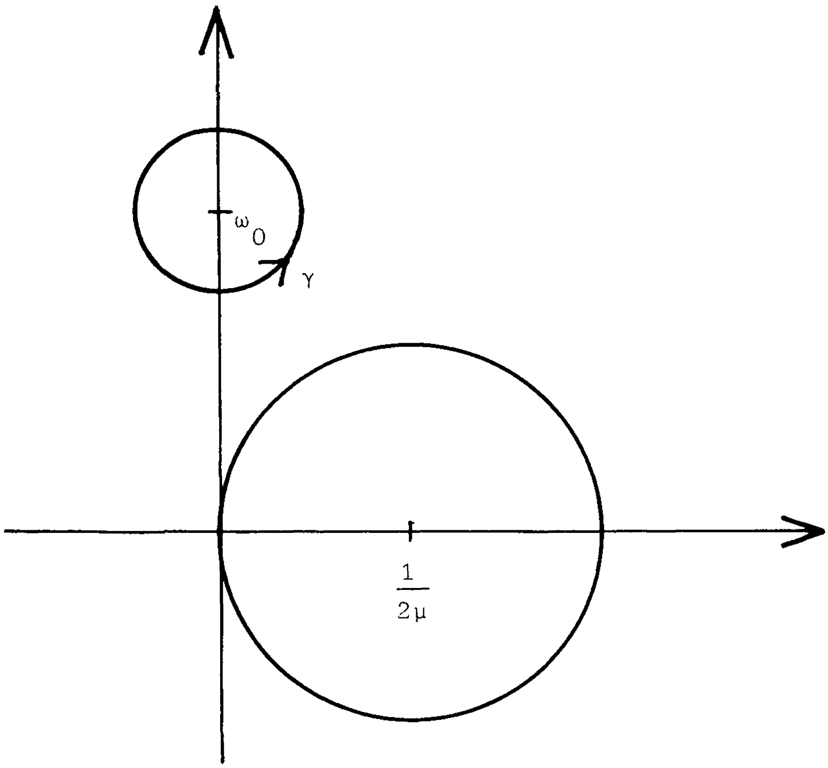

Figure 2.

The proof of Proposition 2.3 now follows from (2.26), using the compacteness of γ in $\mathbb{C}$ and the estimate of $\| \omega^2 B(\omega) \|$ in U_γ.

Thus we arrive at the following theorem of existence and analytic dependence on ε, for the eigenvalues of problem (2.12) (2.13).

THEOREM 2 : *If α_n^2 is an eigenvalue of multiplicity m of the operator A, then for sufficiently small ε, there exist m holomorphic branches $\omega_j^+(\varepsilon)$ (resp. $\omega_j^-(\varepsilon)$) $j = 1, \ldots, m$, each branch counted as many times as its algebraic multiplicity; such that $1/\omega_j^+$ (resp. $1/\omega_j^-$) are eigenvalues of the integro-differential problem that converge to $i\alpha_n$ (resp. $- i\alpha_n$) when $\varepsilon \to 0$.*

Proof : We are going to follow the method of Sanchez-Palencia (see [10]) for a differential problem of the scattering frequencies of an elastic system.

Let $\omega_0 \in \mathbb{C}$ be such that $- \omega_0^2 = \alpha_n^2$ is an eigenvalue of A and let γ be a curve that surrounds ω_0 like in Proposition 2.3. Making in (2.19) the change of variable $\zeta = 1/\omega$ on a neighborhood of the region bounded by γ, we obtain

$$(T(\varepsilon, \omega) + \omega^2) u = 0 \qquad u \in V \quad u \neq 0 . \tag{2.27}$$

From Proposition 2.3, the projection

$$P(\varepsilon, \omega) = \frac{1}{2 \pi i} \int_{-\gamma^2} (T(\varepsilon, \omega) - \lambda)^{-1} \, d\lambda \tag{2.28}$$

exist and is holomorphic in (ε, ω).

This allows us to write problem (2.27) as

$$P(\varepsilon, \omega) (T(\varepsilon, \omega) + \omega^2) P(\varepsilon, \omega) u = 0 \qquad u \in V , \quad u \neq 0 \tag{2.29}$$

in a neighbourhood of ω_0 for sufficiently small ε.

Now using the transformation function $U(\varepsilon, \omega)$, (see Kato [5], p. 102, Remark 4.4) that is holomorphic in (ε, ω) again for sufficiently small ε, we arrive at the reduction of the problem to finite dimension.

$$P_0 \, U^{-1}(\varepsilon, \omega) (T(\varepsilon, \omega) + \omega^2) \, U(\varepsilon, \omega) u = 0 \qquad u \, \varepsilon \, P_0 \, V , \quad u \neq 0 \tag{2.30}$$

where

$$P_0 = P(0, \omega) = \frac{1}{2 \pi i} \int_{-\gamma^2} (A - \lambda)^{-1} \, \mathrm{d}\lambda . \tag{2.31}$$

The problem of eigenvalues in a neighborhood of $\zeta_0 = 1/\omega_0$ is transformed trough (2.30) into a system of linear equation with holomorphic coefficients in (ε, ω) of dimension m; that is : The dimension of the eigenspace of ω_0, $P_0 \, V$, (Algebraic multiplicity).

Thus, the existence of eigenvalues for the integro-differential problem follows from the roots of the scalar equation :

$$F(\varepsilon, \omega) = 0 \tag{2.32}$$

where $F(\varepsilon, \omega)$ is the determinant of the system (2.30) ; and hence a holomorphic function with a root $(0, \omega_0)$ $\omega_0 = \alpha_n \, i$ (resp. $\omega_0 = - \alpha_n \, i$). The Weierstrass preparation theorem and the properties of algebroid singularities (see Bochner and Martin [1], chap. IX and Knopp [6], chap. V), allow us to end the proof of the theorem.

3. REDUCTION PROCESS

It is possible to obtain more information on the asymptotic behaviour of the $\omega_j(\varepsilon)$, and hence of the $\zeta_j(\varepsilon)$ when $\varepsilon \to 0$.

Taking account of the selfadjointess of the operator A, a process analogous to that of Reduction for families of operators depending of a single parameter ε (see Kato [5] chap. II) holds, and we arrive at the following results.

THEOREM 3 : *Under the same hypotheses of Theorem* 2, *the* $\omega_j^+(\varepsilon)$, (*resp.* $\omega_j^-(\varepsilon)$) *can be expressed in the following way*

$$\omega_j(\varepsilon) = \omega_0 + \varepsilon\omega_{1,j} + o_j(\varepsilon) \tag{3.1}$$

where the $o_j(\varepsilon) \to 0$ *when* $\varepsilon \to 0$, *depending on the determination* ω_j, *and* $\omega_{1,j}$ *is an eigenvalue of the problem*

$$\left(\int_0^\infty G(\xi)\, e^{\xi/\omega_0}\, d\xi - 2\frac{\omega_1}{\omega_0}\right) u = 0 \qquad u \in P_0 V\,, \quad u \neq 0\,. \tag{3.2}$$

Proof :

a) We begin by posing two equivalent problems

Problem A : Find $\lambda = \lambda(\varepsilon)$ such that the system of equations (3.3) admits non trivial solutions

$$P_0\, U^{-1}[T(\varepsilon, \omega_0 + \varepsilon\lambda) + (\omega_0 + \varepsilon\lambda)^2]\, Uv = 0 \qquad v \in P_0 V\,, \quad v \neq 0\,. \tag{3.3}$$

Problem B : Find $\lambda = \lambda(\varepsilon)$ such that the system of equations (3.4) admits non trivial solutions

$$P_0\, U^{-1}\,\frac{1}{\varepsilon}\,[T(\varepsilon, \omega_0 + \varepsilon\lambda) + (\omega_0 + \varepsilon\lambda)^2]\, P(\varepsilon, \omega_0 + \varepsilon\lambda)\, Uv = 0$$

$$v \in P_0 V\,, \quad v \neq 0\,. \tag{3.4}$$

It is straightforward to see that the Problem A and Problem B are equivalent, by using the fact that $P = UP_0\, U^{-1}$ and substituting in (3.4) (resp. (3.3)) to prove the double implication.

b) Let us consider next

$$\tilde{F}(\varepsilon, \lambda) = \det P_0\, U^{-1}\,\frac{1}{\varepsilon}\,[T(\varepsilon, \omega_0 + \varepsilon\lambda) + (\omega_0 + \varepsilon\lambda)^2]\, P(\varepsilon, \omega_0 + \varepsilon\lambda)\, UP_0 \tag{3.5}$$

and we have only to compute the $\lambda(\varepsilon)$ such that

$$\tilde{F}(\varepsilon, \lambda(\varepsilon)) = 0\,. \tag{3.6}$$

c) We claim $\tilde{F}$ is holomorphic in (ε, λ). Since $U = U(\varepsilon, \omega)$ is holomorphic in (ε, ω), it suffices to prove that the following operator is also holomorphic

$$\frac{1}{\varepsilon}\left[T(\varepsilon, \omega_0 + \varepsilon\lambda) + (\omega_0 + \varepsilon\lambda)^2\right] P(\varepsilon, \omega_0 + \varepsilon\lambda). \tag{3.7}$$

Substitute in (3.7) the expression of $T(\varepsilon, \omega)$ given by (2.20) and expand $B(\omega_0 + \varepsilon\lambda)$ as a Taylor series to obtain

$$T(\varepsilon, \omega_0 + \varepsilon\lambda) = A - \varepsilon\omega_0^2 B(\omega_0) + \varepsilon^2 r(\varepsilon, \lambda) \tag{3.8}$$

where $r(\varepsilon, \lambda)$ is a holomorphic function in (ε, λ)

$$r(\varepsilon, \lambda) = \sum_{n=0}^{\infty} T^{(n)} \varepsilon^n \lambda^{n+1} \tag{3.9}$$

where the $T^{(n)}$ are the following

$$T^{(0)} = -\omega_0^2 B^{(1)}(\omega_0) - 2\omega_0 B(\omega_0) \tag{3.10}$$

$$T^{(n)} = -\omega_0^2 \frac{B^{(n+1)}(\omega_0)}{(n+1)!} - 2\omega_0 \frac{B^{(n)}(\omega_0)}{n!} - \frac{B^{(n-1)}(\omega_0)}{(n-1)!} \qquad n \geqslant 1. \tag{3.11}$$

And $B^{(i)}(\omega_0)$ are the coefficients of the Taylor expansion of $B(\omega_0 + \varepsilon\lambda)$. Now we can write

$$(T - \eta)^{-1} = R(\eta)\left[I + (T - A) R(\eta)\right]^{-1}, \qquad T = T(\varepsilon, \omega_0 + \varepsilon\lambda). \tag{3.12}$$

Since

$$\begin{aligned}\left[I + (T - A) R(\eta)\right]^{-1} &= \sum_{p=0}^{\infty} \left[-(T - A) R(\eta)\right]^p \\ &= \sum_{p=0}^{\infty} \varepsilon^p\left[(\omega_0^2 B(\omega_0) - \varepsilon r(\varepsilon, \lambda)) R(\eta)\right]^p.\end{aligned} \tag{3.13}$$

Therefore, it follows that :

$$(T - \eta)^{-1} = R(\eta) + \varepsilon\omega_0^2 R(\eta) B(\omega_0) R(\eta) + \varepsilon^2 Q(\varepsilon, \lambda) \tag{3.14}$$

where $Q(\varepsilon, \lambda)$ is obviously a holomorphic function of (ε, λ).

Then the projection $P(\varepsilon, \omega_0 + \varepsilon\lambda)$ takes the following form :

$$P(\varepsilon, \lambda) = P_0 + \varepsilon\omega_0^2 P_1 + \varepsilon^2 \hat{Q}(\varepsilon, \lambda). \tag{3.15}$$

Recall that $P_0 = P(0, \omega_0)$ and

$$P_1 = \frac{1}{2\pi i} \int_{-\gamma^2} R(\eta) B(\omega_0) R(\eta) \, d\eta . \tag{3.16}$$

The selfadjointess of A implies that $-\omega_0^2$ is a semisimple eigenvalue and hence

$$R(\eta) = -(\eta + \omega_0^2)^{-1} P_0 + \sum_{n=0}^{\infty} (\eta + \omega_0^2)^n S^{n+1} \tag{3.17}$$

where

$$S = -\frac{1}{2\pi i} \int_{-\gamma^2} \xi^{-1}(A - \xi)^{-1} \, d\xi \tag{3.18}$$

and $P_0 S = SP_0 = 0$ (see Kato [5] chap. I, 5.3)
Therefore :

$$P_1 = P_0 B(\omega_0) S + SB(\omega_0) P_0 . \tag{3.19}$$

From (3.8) and (3.15) it follows that

$$\frac{1}{\varepsilon}[T + (\omega_0 + \varepsilon\lambda)^2] P(\varepsilon, \omega_0 + \varepsilon\lambda) = (A + \omega_0^2)[\omega_0^2 P_1 + \varepsilon\hat{Q}(\varepsilon, \lambda)] - (\omega_0^2 B(\omega_0) - 2\omega_0\lambda) P(\varepsilon, \omega_0 + \varepsilon\lambda) + \varepsilon[r(\varepsilon, \lambda) + \lambda^2] P(\varepsilon, \omega_0 + \varepsilon\lambda) \tag{3.20}$$

and hence $\tilde{F}(\varepsilon, \lambda)$ is a holomorphic function of (ε, λ).

d) If $\omega_1 \in C$ is an eigenvalue of the problem (3.2), then we claim that

$$\tilde{F}(0, \omega_1) = \det[P_0(A + \omega_0^2)\omega_0^2 P_1 - (\omega_0^2 B(\omega_0) - 2\omega_0\omega_1) P_0] = 0 . \tag{3.21}$$

Let be $v \in P_0 V$, $v \neq 0$, an eigenvector associated to ω_1, then using (3.19)

$$[P_0(A + \omega_0^2)\omega_0^2 P_1 - (\omega_0^2 B(\omega_0) - 2\omega_0\omega_1) P_0] v = \left[\omega_0^2 P_0(A + \omega_0^2)(P_0 B(\omega_0) S + SB(\omega_0) P_0) - \omega_0^2\left(B(\omega_0) - 2\frac{\omega_1}{\omega_0}\right) P_0\right] v . \tag{3.22}$$

Since $(A + \omega_0^2) P_0 \equiv 0$ and $\left(B(\omega_0) - 2\frac{\omega_1}{\omega_0}\right) v = 0$, (3.22) is reduced to :

$$2\,\omega_0\,\omega_1\,P_0(A + \omega_0^2)\,SP_0\,v\,. \tag{3.23}$$

But, as we quoted previously, $SP_0 \equiv 0$ and thus $\tilde{F}(0, \omega_1) = 0$.

e) The complex variables theorems we quoted in Theorem 2, allow us to obtain (3.1) ; we can even write explicitely the function $o_j(\varepsilon)$, then obtain

$$\omega_j(\varepsilon) = \omega_0 + \varepsilon\omega_{1,j} + \varepsilon^{1+\frac{1}{p}}\omega_{2,j} + \cdots \tag{3.24}$$

so that the proof is finished.

4. FINAL REMARKS

The abstract framework we described in § 2, contain as we pointed out in the introduction the problem of vibrations of a finite viscoelastic body with a fixed boundary (Dirichlet problem)

$$u_i\,|_\Gamma = 0 \qquad i = 1, 2, 3\,. \tag{4.1}$$

It suffices to take $V = (H_0^1(\Omega))^3$, $H = (L^2(\Omega))^3$, and as hermitean forms :

$$a(u, v) = \int_\Omega C_{ijkl}\,\frac{\partial u_k}{\partial x_l}\,\overline{\frac{\partial v_i}{\partial x_j}}\,dx \tag{4.2}$$

$$b(\xi, u, v) = \int_\Omega G_{ijkl}(\xi, \underline{x})\,\frac{\partial u_k}{\partial x_l}\,\overline{\frac{\partial v_i}{\partial x_j}}\,dx\,. \tag{4.3}$$

Of course, considering these forms (4.2) and (4.3) over spaces $V \subset (H^1(\Omega))^3$, different from $(H_0^1(\Omega))^3$, gives place to another type of boundary problems, that can be included in the abstract framework of section 2.

As for the Neumann problem, where $V = (H^1(\Omega))^3$ and where the form (4.2) is not coercive, the same remarks as Lobo-Hidalgo [7] can be made in order to solve the problem.

On other hand, other types of dependence on ε can be studied, specifically :

$$\sigma_{ij}^{\varepsilon} = C_{ijkl} - \frac{1}{\varepsilon}\int_{-\infty}^{t} G_{ijkl}\left(\frac{t - \tau}{\varepsilon}, \underline{x}\right) d\tau\,. \tag{4.4}$$

This kind of dependence has been studied with different techniques by

Turbe (see [11]) and Lobo-Hidalgo (see [7]) dealing with the convergence of solutions when $\varepsilon \to 0$. The spectral study, analytic dependence on ε and reduction process that we described in § 3 can also be applied to problems of type (4.4).

REFERENCES

[1] S. BOCHNER and W. T. MARTIN, *Several Complex Variables,* Princeton, Univ. Press, Princeton (1948).

[2] C. M. DAFERMOS, *An abstract Volterra equation with application to linear viscoelasticity,* J. Diff. Equations, 7 (1970), pp. 554-569.

[3] C. M. DAFERMOS, *Asymptotic stability in viscoelasticity,* Arch. Rat. Mech. Anal., 37 (1970), pp. 297-308.

[4] C. M. DAFERMOS, *Contraction semigroups and trend to equilibrium in continuum mechanics,* Lect. Notes Math., 503, Springer, Berlin (1975), pp. 295-306.

[5] T. KATO, *Perturbation Theory for Linear Operators,* Springer, Berlin (1966).

[6] K. KNOPP, *Theory of Function,* II, Dover, New York (1947).

[7] M. LOBO-HIDALGO, *Propriétés spectrales de certaines équations différentielles intervenant en viscoélasticité,* Rend. Sem. Mat. Univ. Polit. Torino, 39 (1981), pp. 33-51.

[8] Ph. NOVERRAZ, *Fonctions plurisousharmoniques et analytiques dans les espaces vectoriels topologiques complexes,* Ann. Inst. Fourier, t. XIX, fasc. 2 (1970), pp. 419-493.

[9] R. OHAYON and E. SANCHEZ-PALENCIA, *On the vibration problem for an elastic body surrounded by a slightly compressible fluid,* R.A.I.R.O. Analyse Numérique, 17, n° 3 (1983), pp. 311-326.

[10] E. SANCHEZ-PALENCIA, *Fréquences de diffusion dans le problème de vibration d'un corps élastique plongé dans un fluide compressible de petite densité,* C. R. Acad. Sc. Paris, t. 295 (1982), pp. 197-200.

[11] N. TURBE, *On two scales method for a class of integrodifferential equations appearing in viscoelasticity,* Int. Jour. Engin. Sci. (1979), pp. 857-868.