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ON THE REGULARITY OF THE VARIATIONAL SOLUTION OF THE TRICOMI PROBLEM IN THE ELLIPTIC REGION (*)

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Résumé. — *Nous considérons ici la solution faible du problème de Tricomi dans la région elliptique avec une condition aux limites non locale sur la ligne parabolique. Nous discutons la régularité de ses dérivées secondes dans des espaces de Sobolev avec poids. Nous prouvons l'existence d'une singularité monodimensionnelle et explicitons celle-ci. Cela est utile pour l'approximation par éléments finis de la solution faible.*

Abstract. — *We consider the weak solution of the Tricomi problem in the elliptic region which satisfies a non-local boundary condition on the parabolic line. We discuss the regularity of its second order derivatives in weighted Sobolev spaces. The existence of a one-dimensional space of singularity is proved and it is explicity found. This is useful in the finite element approximation of the weak solution.*

I. INTRODUCTION

It is well-known that transonic flows of gases are modelled by Tricomi equation, Bers [2] :

$$Tu \equiv yu_{xx} + u_{yy} = 0 . \tag{1.1}$$

We consider the above equation in a bounded domain G in the plane $\mathbb{R}^2$. We set

$$\Omega = G \cap \{ y > 0 \} .$$

We assume that the boundary of G consists of three parts Σ, Γ and Γ_1 where Σ lies in the upper half-plane, Γ and Γ_1 are characteristics through (0, 0) and (1, 0) respectively. We suppose further that

$$\Omega \text{ is convex} , \tag{1.2}$$

$$\Sigma \text{ coincides with the lines } x=0 \text{ and } x=1 \text{ for sufficiently small } y > 0 . \tag{1.3}$$

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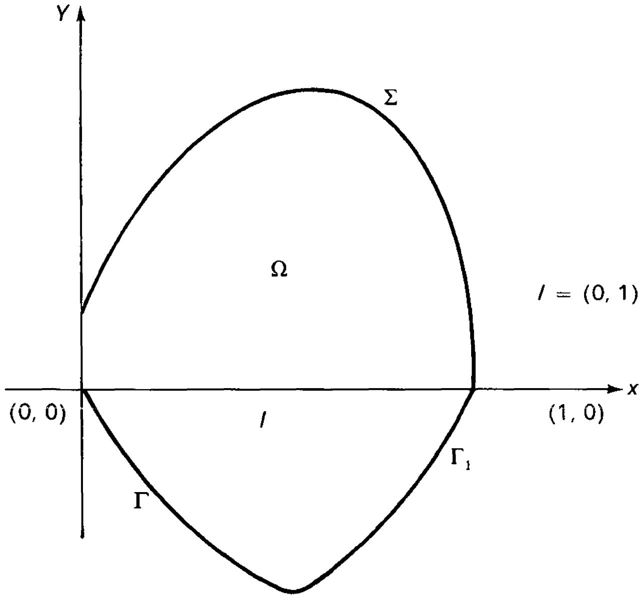
For details, see the following figure :



The Tricomi problem consists of supplementing (1.1) with some boundary conditions. We consider boundary conditions of Dirichlet type or of Neumann type :

$$u = \phi \quad \text{on} \quad \Sigma , \qquad u = \phi \quad \text{on} \quad \Gamma , \tag{1.4}$$

$$\frac{\partial u}{\partial \nu_T} = \phi \quad \text{on} \quad \Sigma , \qquad \frac{\partial u}{\partial \nu_T} = \phi \quad \text{on} \quad \Gamma , \tag{1.5}$$

where

$$\frac{\partial}{\partial \nu_T} \equiv y \nu_x \frac{\partial}{\partial x} + \nu_y \frac{\partial}{\partial y}$$

is the co-normal derivative associated with the operator T. $\nu = (\nu_x, \nu_y)$ denotes the outward unit normal vector on the boundary of G. Note that no condition is prescribed on Γ_1.

One of the approaches to the study of the Tricomi problem is to transform the boundary condition (1.4) or (1.5) on Γ to the parabolic line $y = 0$ and then concentrate on the problem in the elliptic region. The reader can, for instance, look into Bitsadze [3] where this is done for Dirichlet boundary condition. The resulting boundary condition on $y = 0$ is a non-local one.

Trangenstein [19] provided a variational formulation of the associated elliptic sub-problem. The case of the Neumann boundary condition is treated in Vanninathan and Veerappa Gowda [23].

The problems in the elliptic region, which are solved in a weak sense are as follows :

$$\left.\begin{aligned} Tu \equiv y u_{xx} + u_{yy} &= 0 \quad \text{in} \quad \Omega\,, \\ u &= 0 \quad \text{on} \quad \Sigma\,, \\ u_y(x, 0) - k \frac{d}{dx} \int_0^x \frac{u(t, 0)\, dt}{(x - t)^{2/3}} &= \phi(x)\,, \quad 0 < x < 1\,. \end{aligned}\right\} \tag{1.6}$$

$$\left.\begin{aligned} Tu \equiv y u_{xx} + u_{yy} &= 0 \quad \text{in} \quad \Omega\,, \\ \frac{\partial u}{\partial \nu_T} &= 0 \quad \text{on} \quad \Sigma\,, \\ u_y(x, 0) - l \int_0^x \frac{u_x(t, 0)\, dt}{(x - t)^{2/3}} &= \phi(x)\,, \quad 0 < x < 1\,. \end{aligned}\right\} \tag{1.7}$$

Here k and l are the positive constants given by

$$k = \frac{2\pi}{3^{1/6}(\Gamma(1/3))^3}\,, \quad l = \frac{1}{2\pi}\left(\frac{4}{3}\right)^{2/3} \frac{\Gamma(5/6)^2}{\Gamma(5/3)}\,. \tag{1.8}$$

The functions ϕ in (1.6) and (1.7) (not necessarily the same) are obtained from the corresponding ϕ in (1.4) and (1.5) respectively. The weak solutions of the problems (1.6), (1.7) are found in the space

$$H_y^1(\Omega) = \{\, v \in L^2(\Omega)\,;\, y^{1/2}\, v_x \in L^2(\Omega),\, v_y \in L^2(\Omega) \,\}\,. \tag{1.9}$$

This paper deals with the question of the regularity of the second order derivatives of the weak solution when ϕ is sufficiently regular in (1.6) and (1.7). This is of interest when we study finite element approximations of the weak solution, *cf.* Ciarlet [4].

Similar study for the Lavrentiev-Bitsadze model was done by Osher [15], Deacon and Osher [5]. Existence and uniqueness of the weak solution for the problem (1.1), (1.4), (1.5) on the whole domain G were proved by Morawetz [13], [11], [12].

The question of regularity of the solution of Tricomi equation was object of study of Germain, Bader [7], Nocilla *et al.* [14] though not in the context of finite element approximation.

The plan of the paper is as follows : In § 2, we prove the existence of one-dimensional space of singularity and we obtain it explicitly. In § 3, we state some of the consequences of the analysis done in § 2 in the context of the finite element approximation. We conclude by passing some remarks on the Neumann boundary condition.

2. SINGULARITY AT (1, 0)

In this section, we consider the problem (1.6) with Dirichlet boundary condition. We rewrite this in a slightly generalized form :

$$\left.\begin{aligned} Tu \equiv yu_{xx} + u_{yy} &= f \quad \text{in} \quad \Omega , \\ u &= 0 \quad \text{on} \quad \Sigma , \\ Lu \equiv u_y(x, 0) - k \frac{d}{dx} \int_0^x \frac{u(t, 0)\, dt}{(x - t)^{2/3}} &= \phi(x) , \quad 0 < x < 1 . \end{aligned}\right\} \tag{2.1}$$

We introduce the space

$$H_y^2(\Omega) = \{ v \in H_y^1(\Omega) ; y^{1/2} v_{xx} \in L^2(\Omega), v_{xy} \in L^2(\Omega), y^{-1/2} v_{yy} \in L^2(\Omega) \} ,$$

in which we seek what we call strong solution. First of all, we have the following trace results : Uspenkii [22] :

$$\begin{gathered} H_y^2(\Omega) \to H^{4/3}(\mathrm{I}) \times H^{2/3}(\mathrm{I}) , \\ v \to (v(x, 0), v_y(x, 0)) , \end{gathered}$$

is continuous linear. Consider also the following spaces :

$$\begin{aligned} W &= \{ v \in H_y^2(\Omega) ; v = 0 \text{ on } \Sigma \} , \\ L^2(y^{-1/2}) &= \{ f \in \mathscr{D}'(\Omega) ; y^{-1/2} f \in L^2(\Omega) \} , \\ {}_0H^{2/3}(\mathrm{I}) &= \{ \psi \in H^{2/3}(\mathrm{I}) ; \psi(0) = 0 \} . \end{aligned}$$

Define now the operator $\mathscr{P}$ by

$$\begin{gathered} W \to L^2(y^{-1/2}) \times {}_0H^{2/3}(\mathrm{I}) , \\ v \to (Tv, Lv) , \end{gathered}$$

which is continuous by trace results stated above. The question discussed below is the following : Given $f \in L^2(y^{-1/2})$ and $\phi \in {}_0H^{2/3}(1)$, does the weak solution of the problem (2.1) belong to W? We show, in the sequel, that this is not true in general. In fact there exists a H_y^1 singularity near (1, 0) due to the lack of compatibility between the boundary conditions at that point which we now proceed to find explicitly.

First of all, it is clear that the possible singularity is on the x-axis. Because of our assumptions (1.2), (1.3), there is no loss of generality in supposing Ω is the unit square. We try to adapt the standard method of finding the singularity of the solution of a boundary value problem near a corner point, Grisvard [8]. Of course there are extra difficulties present in our problem like variable coefficients, degeneracy, non-local boundary conditions etc. We show below how we overcome these difficulties.

Let us now localize the problem around (1, 0). In fact, it is sufficient to construct a $v \in H_y^1$ but $v \notin H_y^2$ which has the following properties :

$$\left.\begin{aligned} &\text{(i)} \quad Tv \in L^2(y^{-1/2}), \\ &\text{(ii)} \quad v(1, y) = 0, \quad 0 < y < 1, \\ &\text{(iii)} \quad v_y(x, 0) - k\frac{d}{dx}\int_0^x \frac{v(t, 0)\, dt}{(x - t)^{2/3}} \in H^{2/3}(\mathrm{I}). \end{aligned}\right\} \tag{2.2}$$

We can then multiply by cut-off functions $\theta(x)$ and $\eta(y)$ which are one in a small neighbourhood of $x = 1$ and $y = 0$ respectively to produce a function u satisfying

$$\left.\begin{aligned} &u \in H_y^1(\Omega), \\ &Tu \in L^2(y^{-1/2}), \\ &u = 0 \quad \text{on} \quad \Sigma. \end{aligned}\right\} \tag{2.3}$$

To see that

$$Lu \in {}_0H^{2/3}(\mathrm{I}), \tag{2.4}$$

we use the fact that

$$\frac{d}{dx}\int_0^x \frac{\theta(t)\, v(t, 0)\, dt}{(x - t)^{2/3}} - \theta(x)\frac{d}{dx}\int_0^x \frac{v(t, 0)\, dt}{(x - t)^{2/3}} \in {}_0H^{2/3}(\mathrm{I}), \tag{2.5}$$

which can be proved as follows.

Proof of (2.5) : Firstly, we have the following trace result, Uspenskii [22] :

$$\begin{aligned} H_y^1(\Omega) &\to H^{1/3}(\mathrm{I}), \\ v &\to v(x, 0), \end{aligned} \tag{2.6}$$

is linear and continuous. Next, since extension by zero is a continuous linear operation from $H^{1/3}(\mathrm{I}) \to H^{1/3}(\mathbb{R})$, we can rewrite (2.5) as follows :

$$\frac{d}{dx}\int_{-\infty}^{x}\frac{\theta(t)\,v(t,0)\,dt}{(x-t)^{2/3}} - \theta(x)\frac{d}{dx}\int_{-\infty}^{x}\frac{v(t,0)\,dt}{(x-t)^{2/3}} \in {}_0H^{2/3}(\mathrm{I}). \quad (2.7)$$

We now note that the operator

$$v \to \frac{d}{dx}\int_{-\infty}^{x}\frac{v(t)\,dt}{(x-t)^{2/3}}$$

is a pseudo differential operator of order 2/3. In fact

$$\left(\frac{d}{dx}\int_{-\infty}^{x}\frac{v(t)\,dt}{(x-t)^{2/3}}\right)^{\wedge}(\xi) = \Gamma(1/3)\,(i\xi)^{2/3}\,\hat{v}(\xi) \quad \text{for all} \quad v \in \mathcal{D}(\mathbb{R}). \quad (2.8)$$

See, for instance, Trangenstein [20]. The symbol is not c^{∞} at the origin but this will not trouble us. Applying the product rule for such operators, Treves [21], and their continuity between Sobolev spaces, we can conclude that the right side of (2.7) is in $H^{2/3}(\mathrm{I})$. That this is in ${}_0H^{2/3}(\mathrm{I})$ follows if we choose a vanishing $\theta(x)$ in a neighbourhood of $x = 0$. Thus (2.5) is proved and hence (2.4).

Construction of v satisfying (2.2)

First we consider the equation

$$y v_{xx} + v_{yy} = 0 .$$

Make the following change of variables as given in Ferrari and Tricomi [6] :

$$x = x, \quad z = \tfrac{2}{3} y^{3/2} .$$

This transforms the above equation into its canonical form :

$$v_{xx} + v_{zz} + \frac{1}{3\,z} v_z = 0 .$$

Now introduce the polar coordinates at (1, 0).

$$x = \gamma \cos \theta , \quad z = \gamma \sin \theta , \quad \gamma = [(x-1)^2 + z^2]^{1/2} , \quad \frac{\pi}{2} \leqslant \theta \leqslant \pi .$$

It can be shown that the variables, γ, θ can be separated in the resulting equation. Writing $v(\gamma, \theta) = X(\gamma)\, Y(\theta)$, we get the following equations for $X(\gamma)$ and $Y(\theta)$.

$$X''(\gamma) + \frac{4}{3\gamma} X'(\gamma) - \frac{\lambda}{\gamma^2} X(\gamma) = 0\,, \tag{2.9}$$

$$Y''(\theta) + \frac{1}{3} \cot \theta\, Y'(\theta) + \lambda Y(\theta) = 0\,, \tag{2.10}$$

where λ is a constant.

Note that (2.9) is an Euler equation and so we can search a solution of the form

$$X(\gamma) = \gamma^{\mu}\,. \tag{2.11}$$

We get a relation connecting μ and λ :

$$\lambda = \mu(\mu + 1/3)\,. \tag{2.12}$$

In order to solve for Y, we need to supplement (2.10) with boundary conditions. The condition at $\theta = \pi/2$ reads as follows :

$$y(\pi/2) = 0\,. \tag{2.13}$$

The condition at $\theta = \pi$ can be obtained by manipulations on the operator occuring in (2.2) (iii). This leads us to the hypergeometric functions defined by

$$F(a, b, c, z) = \frac{\Gamma(c)}{\Gamma(c - b)\,\Gamma(b)} \int_0^1 t^{b-1}(1 - t)^{c-b-1}(1 - zt)^{-a}\, dt \tag{2.14}$$

for $\operatorname{Re} c > \operatorname{Re} b > 0$ and for z in the complex plane with a cut from 1 to ∞ along the positive real axis. See for instance Spain and Smith [17]. We remark that the condition $v \in H_y^1 - H_y^2$ will imply that

$$-\frac{1}{6} < \mu \leqslant \frac{5}{6}, \tag{2.15}$$

$$\int_{\pi/2}^{\pi} (\sin \theta)^{1/3}\, Y(\theta)^2\, d\theta < \infty\,, \tag{2.16}$$

$$\int_{\pi/2}^{\pi} (\sin \theta)^{1/3}\, Y'(\theta)^2\, d\theta < \infty\,, \tag{2.17}$$

and hence $Y(\pi)$ is well-defined. The main properties of the hypergeometric functions used in the calculations are given below :

$$\left.\begin{array}{ll} \text{(i)} & \text{If } \operatorname{Re}(c-a-b)>0, \quad \text{then } \lim_{z\to 1} F(a,b,c,z) \text{ exists}, \\ \text{(ii)} & \dfrac{d}{dz}F(a,b,c,z) = \dfrac{ab}{c}F(a+1,b+1,c+1,z), \\ \text{(iii)} & F(a,b,c,z) = (1-z)^{c-b-a}F(c-a,c-b,c,z). \end{array}\right\} \quad (2.18)$$

Using all these, we find

$$\begin{aligned} Lv = \lim_{\theta\uparrow\pi} & \left(-\frac{3}{2}\right)^{1/3} (\sin\theta)^{1/3}\,\gamma^{\mu-2/3}\,Y'(\theta) - \\ & - 3\,kY(\pi)\left\{\frac{1}{3}(1-\gamma)^{-2/3}F\left(-\mu,1,\frac{4}{3},1-\gamma\right)\right. \\ & \left. - \frac{3\mu}{4}(1-\gamma)^{1/3}\gamma^{\mu-2/3}F\left(\mu+\frac{4}{3},\frac{1}{3},\frac{7}{3},1-\gamma\right)\right\}. \end{aligned}$$

Thus it is natural to impose the following condition on Y :

$$\lim_{\theta\uparrow\pi}(-3/2)^{1/3}(\sin\theta)^{1/3}\,Y'(\theta) + \frac{9}{4}\mu kF\left(\mu+\frac{4}{3},\frac{1}{3},\frac{7}{3},1\right)Y(\pi) = 0. \quad (2.19)$$

Thus the variables γ, θ can be separated in the boundary condition also.

Consider now the system of equations (2.10), (2.13) and (2.19) with μ as a parameter. We show below that there exists a value of μ such that

$$0 < \mu < 2/3, \quad (2.20)$$

for which the system admits a non-zero solution $Y(\theta)$. In order to fulfil the condition (2.2) (iii), we must show that the error term $E(\gamma)$ viz,

$$\begin{aligned} E(\gamma) = & \frac{1}{3}(1-\gamma)^{-2/3}F\left(-\mu,1,\frac{4}{3},1-\gamma\right) - \\ & - \frac{3\mu}{4}\gamma^{\mu-2/3}\left\{(1-\gamma)^{1/3}F\left(\mu+\frac{4}{3},\frac{1}{3},\frac{7}{3},1-\gamma\right) - F\left(\mu+\frac{4}{3},\frac{1}{3},\frac{7}{3},1\right)\right\} \end{aligned} \quad (2.21)$$

has the following regularity

$$E(\gamma) \in H^{2/3}(\mathrm{I}) \quad \text{near} \quad \gamma = 0. \quad (2.22)$$

We give some indications as to how this can be achieved. Take, for instance, the second term on the right side of (2.21) which is more troublesome. Put

$$G(\gamma) = (1-\gamma)^{1/3} F\left(\mu + \frac{4}{3}, \frac{1}{3}, \frac{7}{3}, 1-\gamma\right).$$

Then

$$G(\gamma) - G(0) = \int_0^1 \frac{d}{d\theta} G(\theta\gamma)\, d\theta = \gamma \int_0^1 G'(\theta\gamma)\, d\theta .$$

Thus it is enough to show that

$$\int_0^1 (1-\theta\gamma)^{1/3} F\left(1-\mu, 2, \frac{10}{3}, 1-\theta\gamma\right) \theta^{-\mu-1/3}\, d\theta \in H^{2/3} \quad \text{near} \quad \gamma = 0 . \tag{2.23}$$

$$\int_0^1 \gamma^{\mu+2/3}(1-\theta\gamma)^{-2/3} F\left(\mu + \frac{4}{3}, \frac{1}{3}, \frac{7}{3}, 1-\theta\gamma\right) d\theta \in H^{2/3} \quad \text{near} \quad \gamma = 0 . \tag{2.24}$$

This can be shown by observing that the first derivative of these functions are in $H^{-1/3}$ near $\gamma = 0$. In fact, the first order derivatives involve a singularity of the form $\gamma^{\mu-2/3}$ near $\gamma = 0$. The inequality (2.20) can be used in proving these assertions.

Let us now turn to the resolution of $Y(\theta)$. We make one more change of variables. Introduce t by substituting

$$t = \frac{1}{2}(1 + \cos\theta) .$$

This takes the system (2.10), (2.13) and (2.19) to the following one, $z(t) = Y(\theta)$:

$$\left.\begin{aligned} t(1-t)\, z''(t) + \left[\frac{2}{3} - \frac{4}{3} t\right] z'(t) + \mu\left(\mu + \frac{1}{3}\right) z(t) = 0 , \\ 0 < t < \frac{1}{2}, \quad z(1/2) = 0 , \\ \lim_{t \to 0} 3^{1/3}\, t^{2/3}\, z'(t) + \frac{9}{4} \mu k F\left(\mu + \frac{4}{3}, \frac{1}{3}, \frac{7}{3}, 1\right) z(0) = 0 . \end{aligned}\right\} \tag{2.25}$$

The question of existence of non-zero solution $z(t)$ to the above system is answered in the following result and there by construction of v satisfying (2.2) is finished.

Lemma : *There exists a* μ, $0 < \mu < 2/3$ *for which* (2.25) *admits a non-zero solution.*

Proof : The equation in (2.25) is the hypergeometric equation and the two independent solutions are provided by

$$A_\mu(t) = F\left(-\mu, \mu + \frac{1}{3}, \frac{2}{3}, t\right),$$

$$B_\mu(t) = t^{1/3} F\left(-\mu + \frac{1}{3}, \mu + \frac{2}{3}, \frac{4}{3}, t\right).$$

We can then easily write down the necessary and sufficient condition in order that (2.25) has a non-zero solution. This is given as follows :

$$3^{-2/3} A_\mu(1/2) - \frac{9}{4}\mu k F\left(\mu + \frac{4}{3}, \frac{1}{3}, \frac{7}{3}, 1\right) B_\mu(1/2) = 0 .$$

This can be simplified further by using some properties of A_μ and B_μ. See Abramowitz and Stegun [1] p. 556. We get the following non-linear equation for μ :

$$\frac{\Gamma(2/3)}{3^{2/3}\,\Gamma\left(\frac{1-\mu}{2}\right)\Gamma\left(\frac{4+3\mu}{6}\right)} - \frac{9}{4}\mu k \frac{\Gamma(4/3)\,\Gamma(7/3)\,\Gamma(2/3-\mu)}{2^{2/3}\,\Gamma\left(\frac{5-3\mu}{6}\right)\Gamma\left(\frac{\mu+2}{2}\right)\Gamma(1-\mu)} = 0 . \quad (2.26)$$

We can easily see that there is a value of μ in the interval $0 < \mu < 2/3$ which is a solution of (2.26). In fact, the left side of (2.26) as a function of μ is continuous in the interval $0 \leqslant \mu < 2/3$ taking positive value at $\mu = 0$ and it goes to $-\infty$ as $\mu \to 2/3$.

For the value of k given by (1.8), the equation (2.26) can be solved exactly and the root is found to be

$$\mu = 1/3 . \quad (2.27)$$

Few questions now arise naturally. Is there any other value of μ satisfying (2.26) giving another singularity ? It is possible to prove the uniqueness of the solution of (2.26) in the interval $0 \leqslant \mu < 2/3$. It can also be seen that there is no root of (2.26) in $2/3 \leqslant \mu \leqslant 5/6$. Is there any other form of singularity at (1, 0) ? What about the point (0, 0) ? One can think of applying the method of Kondratiev [10]. But the following simple procedure bypasses this and provides an answer to all the questions above. We remark that the uniqueness question discussed above is not the same as the one proved by Pashkoviskii [16].

We consider the map $\mathscr{P}$ defined earlier :

$$W \to L^2_\pi(y^{-1/2}) \times {}_0H^{2/3}(\mathrm{I}),$$
$$\mathscr{P}(v) = (Tv, Lv) \quad \text{for} \quad v \in W.$$

THEOREM : *$\mathscr{P}$ is a continuous, one-to-one operator with closed range and the range has codimension exactly equal to unity.*

Proof : Pashkoviskii [16] has proved the following inequality

$$\| v \|_W \leqslant c(\| Tv \| + \| Lv \|) \quad \text{for all} \quad v \in W. \tag{2.28}$$

This shows that $\mathscr{P}$ is one-to-one and has closed range. Now define the operators $\mathscr{P}_\theta$ for $0 \leqslant \theta \leqslant 1$ as follows :

$$W \xrightarrow{\mathscr{P}_\theta} L^2(y^{-1/2}) \times {}_0H^{2/3}(\mathrm{I}),$$
$$\mathscr{P}_\theta(v) = (Tv, L_\theta\, v),$$

where

$$L_\theta\, v(x) = v_y(x, 0) - \theta k \frac{d}{dx} \int_0^x \frac{v(t, 0)\, dt}{(x - t)^{2/3}}.$$

An inequality similar to (2.28) can be established for $\mathscr{P}_\theta$. Thus $\{ \mathscr{P}_\theta \}, 0 \leqslant \theta \leqslant 1$ defines a homotopy of semi-Fredholm operators between $\mathscr{P}_0$ and $\mathscr{P}_1 \equiv \mathscr{P}$. Since the index is invariant under homotopy Kato [9], we get that the codimensions of the ranges of $\mathscr{P}$ and $\mathscr{P}_0$ are equal.

Let us see what is the problem corresponding to $\mathscr{P}_0$: given $g \in L^2(y^{-1/2})$. $\psi \in {}_0H^{2/3}(\mathrm{I})$, find $v \in W$ satisfying

$$\left.\begin{aligned} Tv &= g \quad \text{in} \quad \Omega, \\ v &= 0 \quad \text{on} \quad \Sigma, \\ v_y(x, 0) &= \psi(x) \quad \text{on} \quad \mathrm{I}. \end{aligned}\right\}$$

A method for solving similar problems is given in Vanninathan and Veerappa Gowda [23]. The result is that $v \in W$ if and only if $\psi \in {}_0H^{2/3}(\mathrm{I})$. with $\psi(1) = 0$. This subspace has co-dimension exactly equal to one in ${}_0H^{2/3}(\mathrm{I})$. This completes the proof of the theorem.

Thus we reach the following conclusion : there is exactly one-dimensional space of H^1_y singularity and this is situated at (1, 0). Also this has been found explicity.

3. FINITE ELEMENT APPROXIMATION

A weak formulation and a finite element method for (2.1) has been described by Trangenstein [19]. The presence of singularity at (1, 0) has been slightly overlooked in the above mentioned paper. We content ourselves by simply observing certain consequences of the conclusion reached in § 2.

We use Q_1 finite elements, *cf.* Ciarlet [4] at least for three reasons :

a) It is a c^0 finite element and so the finite element space V_h is a subspace of H_y^1.

b) While passing from the reference finite element to an arbitrary finite element, since there exists a diagonal affine map, the space H_y^1 is preserved. This is not true if we use triangular elements.

c) To prove the results of the type Bramble-Hilbert Lemma, we need the inclusion $H_y^1 \to L^2$ to be compact. This is proved only for rectangular elements.

If u_h denotes the approximate solution in V_h then we have the convergence result :

$$\| u - u_h \|_{H_y^1} \to 0 \quad \text{as} \quad h \to 0 .$$

In general, there is no order of convergence. However, if we augment our finite element space V_h by the inclusion of the one dimensional space of singularity obtained in § 2 and assume $f \in L^2(y^{-1/2})$ and $\phi \in {}_0H^{2/3}(\mathrm{I})$, then it is standard, Strang and Fix [18], that we obtain $0(h)$ estimate for the error in H_y^1 norm.

4. REMARKS ON NEUMANN CONDITION

We pass certain remarks about the Neumann problem (1.7). A result analogous to the Theorem in § 2 is true in this case and it is proved in Vanninathan and Veerappa Gowda [23]. This means that there is a compatibility condition between the data f and ϕ, though it is difficult to obtain it explicitly. For this, one may have to solve the dual problems in a weak sense.

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