MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

J. CAINZOS

Spectral properties of a type of integro-differential stiff problems

Modélisation mathématique et analyse numérique, tome 19, nº 2 (1985), p. 179-193

<http://www.numdam.org/item?id=M2AN_1985__19_2_179_0>

MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE
(vol 19, n° 2, 1985, p 179 à 193)

SPECTRAL PROPERTIES OF A TYPE OF INTEGRO-DIFFERENTIAL STIFF PROBLEMS (*)

by J. CAINZOS ([1])

Communiqué par E SANCHEZ-PALENCIA

Abstract — *We consider an integro-differential stiff problem, which can be used as a model for the vibrations of a body with linear viscoelasticity at large times First, the problem is introduced in the framework of the theory of semigroups in Hilbert spaces and then, an asymptotic expansion of the solution is obtained Finally, it is shown the double convergence, when* $\varepsilon \to 0^+$, *of the eigenvalues of the problem to the eigenvalues of two associated problems These problems are related to the behaviour of the stiff problem on two different domains*

Résumé — *On considère un problème raide intégro-différentiel qui modélise les vibrations d'un corps viscoélastique à mémoire longue On introduit le problème dans le cadre de la théorie des semi-groupes de contraction et on obtient un developpement asymptotique de la solution On étudie le spectre du generateur et on demontre une double convergence, lorsque* $\varepsilon \to 0^+$, *des valeurs propres du probleme vers les valeurs propres de deux problèmes associes qui rendent compte du comportement du problème raide dans deux domaines différents*

0. INTRODUCTION

Let Ω_1, Ω_2 be two connected bounded domains of $\mathbb{R}^n$ with smooth boundaries, located as shown in figure 1. We also consider the " total domain " $\Omega = \Omega_1 \cup \Gamma_1 \cup \Omega_2$

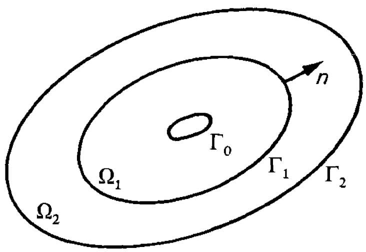

Figure 1.

(*) Received in January 1984, revised in mars 1984

([1]) Departamento de Ecuaciones Funcionales, Facultad de Matematicas, Universidad de Santiago de Compostela (Spain)

We define n as the outer unit normal to Γ_1.
Let us introduce the spaces :

$$H_0^1(\Omega_1, \Gamma_0) = \{ v \in H^1(\Omega_1)/v\,|_{\Gamma_0} = 0 \}$$

and

$$H_0^1(\Omega_2, \Gamma_2) = \{ v \in H^1(\Omega_2)/v\,|_{\Gamma_2} = 0 \}.$$

We consider the following stiff problem :

Problem P_ε : Find two functions of (t, x), defined on $(-\infty, \infty) \times \Omega$:

$$u_1 : (-\infty, \infty) \to H_0^1(\Omega_1, \Gamma_0)$$
$$u_2 : [0, \infty) \to H_0^1(\Omega_2, \Gamma_2)$$

satisfying :

$$\frac{\partial^2 u_1}{\partial t^2}(t) - \Delta u_1(t) + \int_{-\infty}^{t} g(t-\tau)\, \Delta u_1(\tau)\, d\tau = f_1(t) \quad \text{in} \quad \Omega_1$$

$$\varepsilon \frac{\partial^2 u_2}{\partial t^2} - \varepsilon\, \Delta u_2 = f_2 \quad \text{in} \quad \Omega_2$$

with the transmission conditions :

$$u_1(t) = u_2(t) \quad \text{on} \quad \Gamma_1$$

$$\frac{\partial u_1}{\partial n}(t) - \int_{-\infty}^{t} g(t-\tau) \frac{\partial u_1}{\partial n}(\tau)\, d\tau = \varepsilon \frac{\partial u_2}{\partial n}(t) \quad \text{on} \quad \Gamma_1$$

and the initial conditions :

$$u_1(t) = \phi(t) \in H_0^1(\Omega_1, \Gamma_0) \qquad \forall t \in (-\infty, 0]$$

$$u_2(0) = \phi_0 \in H_0^1(\Omega_2, \Gamma_2), \qquad \frac{\partial u_2}{\partial t}(0) = \phi_1 \in L^2(\Omega_2).$$

The integro-differential operator which acts on the domain Ω_1 has the same characteristics as the one used by Dafermos in his papers on viscoelasticity (See Dafermos [2, 3 and 4]).

In § 1, the problem is reduced to the theory of contraction semigroups in Hilbert spaces, using the method of Dafermos [4]. Furthermore, following Lions [5], some results on the existence of a regular asymptotic expansion are given.

Section 2 is devoted to study the spectrum of the problem P_ε. It is well known that the eigenvalues ζ are related to solutions of the following form :

$$u(x, t) = u(x)\, e^{-\zeta t}$$

i.e. steady solutions of exponential type.

This study is carried out using a modified version of Lobo-Sanchez's method [7]. It is shown that there is a double convergence of the eigenvalues of the problem P_ε, when $\varepsilon \to 0^+$, to the eigenvalues of two related problems. The first of these two problems, satisfying a Neumann type condition, shows a viscoelastic behaviour on the domain Ω_1. The other problem, satisfying a Dirichlet condition, exhibits a merely elastic behaviour on the domain Ω_2.

Similar problems without the integro-differential term have been studied by Lions [5] and their spectral study has been carried out by Lobo-Sanchez [7].

1. EXISTENCE AND UNIQUENESS OF THE SOLUTION. ASYMPTOTIC EXPANSION

In the sequel, we shall assume that the influence function $g : \mathbb{R}^+ \to \mathbb{R}$ satisfies :

$$g \in C^1[0, \infty), \quad g' \leqslant 0, \quad g \in L^1(0, \infty), \quad c = 1 - \| g \|_{L^1(0,\infty)} > 0 \quad (1.1)$$

as well as the following exponential majorization :

$$\lambda_1 e^{-\mu\xi} \leqslant g(\xi) \leqslant \lambda_2 e^{-\mu\xi} \quad \lambda_1, \lambda_2, \mu > 0 . \quad (1.2)$$

1.1. Existence and uniqueness of the solution

Following the method used by Dafermos [4], it is possible to reduce the problem P_ε to the theory of contraction semigroups in Hilbert spaces. So, by making the following change of variable : $t - \tau = \xi$, and introducing the auxiliar variables $v = u'$ and $w(t, \xi) = u(t - \xi) - u(t)$, which sums up the history of u, we can formulate the following equivalent problem :

$$\frac{dU}{dt} + \mathscr{A}_\varepsilon U = F \quad (1.3)$$

$$U(0) = U_0$$

where $U = (u, v, w)$ and $U_0 = (u_0, v_0, w_0)$, with $u_0 |_{\Omega_1} = \phi(0)$

$$u_0 |_{\Omega_2} = \phi_0 , \quad v_0 |_{\Omega_1} = \phi'(0) , \quad v_0 |_{\Omega_2} = \phi_1 \quad \text{and} \quad w_0(\xi) = \phi(-\xi) - \phi(0) .$$

Both, U and U_0 belong to the following Hilbert space :

$$H_\varepsilon = H_0^1(\Omega) \times L^2(\Omega) \times L_g^2(0, \infty ; H_0^1(\Omega_1, \Gamma_0)) \tag{1.4}$$

endowed with the inner product :

$$\langle U, U^* \rangle_\varepsilon = c \int_{\Omega_1} \nabla u \nabla \overline{u}^* \, dx + \varepsilon \int_{\Omega_2} \nabla u \nabla \overline{u}^* \, dx + \int_{\Omega_1} v \overline{v}^* \, dx + \\ + \varepsilon \int_{\Omega_2} v \overline{v}^* \, dx + \int_0^\infty g(\xi) \int_{\Omega_1} \nabla w(\xi) \nabla \overline{w}^*(\xi) \, dx \, d\xi \tag{1.5}$$

The operator $\mathscr{A}_\varepsilon$ is defined by :

$$\mathscr{A}_\varepsilon U = \begin{bmatrix} -v \\ A_\varepsilon u - \int_0^\infty g(\xi) A w(\xi) \, d\xi \\ w' + v|_{\Omega_1} \end{bmatrix} \tag{1.6}$$

where $A_\varepsilon \in L(H_0^1(\Omega), H^{-1}(\Omega))$ is the operator associated with the form :

$$a_\varepsilon : (u, v) \in H_0^1(\Omega) \times H_0^1(\Omega) \to a_\varepsilon(u, v) =$$

$$= c \int_{\Omega_1} \nabla u \nabla \overline{v} \, dx + \varepsilon \int_{\Omega_2} \nabla u \nabla \overline{v} \, dx \in \mathbb{C} \tag{1.7}$$

and $A \in L(H_0^1(\Omega_1, \Gamma_0), H^{-1}(\Omega))$ is the one associated with the form :

$$a : (u, v) \in H_0^1(\Omega_1, \Gamma_0) \times H_0^1(\Omega) \to a(u, v) = \int_{\Omega_1} \nabla u \nabla \overline{v} \, dx \in \mathbb{C}. \tag{1.8}$$

Finally, the domain of $\mathscr{A}_\varepsilon$ is :

$$D(\mathscr{A}_\varepsilon) = \Big\{ U \in H_\varepsilon / v \in H_0^1(\Omega), A_\varepsilon u - \int_0^\infty g(\xi) A w(\xi) \, d\xi \in L^2(\Omega), \\ w' \in L_g^2(0, \infty ; H_0^1(\Omega_1, \Gamma_0)), w(0) = 0 \Big\}. \tag{1.9}$$

Remark 1.1 : When we define A_ε, we are considering that $H_0^1(\Omega)$ is a subspace of $L_\varepsilon^2(\Omega)$ i.e. $L_\varepsilon^2(\Omega)$ is the space $L^2(\Omega)$ endowed with the following inner product :

$$(u, v)_\varepsilon = \int_{\Omega_1} u\bar{v}\, dx + \varepsilon \int_{\Omega_2} u\bar{v}\, dx \quad \forall u, v \in L^2(\Omega) .$$

The previous choice of the space H_ε and the definition of the operator $\mathscr{A}_\varepsilon$ allow us to prove the following proposition, using already known results (see Brezis [1] and other authors).

PROPOSITION 1.1 : *The operator* $-\mathscr{A}_\varepsilon$ *is the infinitesimal generator of a contraction semigroup* $\{ T_\varepsilon(t) \}_{t \geq 0}$ *that allows us to obtain the unique solution of the problem* (1.3).

1.2. Asymptotic expansion

Let us denote by u_1 (resp. u_2) the restriction of u to Ω_1 (resp. Ω_2).

The search for an asymptotic expansion of the form :

$$\frac{1}{\varepsilon} u^{-1} + u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + \cdots + \varepsilon^k u^k + \cdots \tag{1.10}$$

leads us, by formal substitution in the variational formulation of the problem P_ε, to obtain coupled equations. These equations, independent of ε, allow us to calculate the terms of the above expansion (1.10).

In short, the functions u_1^k (resp. u_2^k), restriction of u^k to Ω_1 (resp. Ω_2), are the solution of :

Case $k = -1$: u_1^{-1} is the solution of the problem :

$$\frac{\partial^2 u_1^{-1}}{\partial t^2}(t) - \Delta u_1^{-1}(t) + \int_{-\infty}^{t} g(t-\tau)\, \Delta u_1^{-1}(\tau)\, d\tau = 0 \quad \text{in} \quad \Omega_1 \tag{1.11}$$

$$u_1^{-1}|_{\Gamma_0} = 0 , \quad \frac{\partial u_1^{-1}}{\partial n}(t) - \int_{-\infty}^{t} g(t-\tau) \frac{\partial u_1^{-1}}{\partial n}(\tau)\, d\tau = 0 \quad \text{on} \quad \Gamma_1$$

$$u_1^{-1}(t) = 0 \quad \forall t \in (-\infty, 0] .$$

Therefore, $u_1^{-1} = 0$.

On the other hand, u_2^{-1} is the solution of :

$$\frac{\partial^2 u_2^{-1}}{\partial t^2} - \Delta u_2^{-1} = f_2 \quad \text{in} \quad \Omega_2 \tag{1.12}$$

$$u_2^{-1}|_{\Gamma_1} = 0 , \quad u_2^{-1}|_{\Gamma_2} = 0 .$$

Case $k = 0$: u_1^0 is the solution of :

$$\frac{\partial^2 u_1^0}{\partial t^2}(t) - \Delta u_1^0(t) + \int_{-\infty}^{t} g(t-\tau)\, \Delta u_1^0(\tau)\, d\tau = f_1(t) \quad \text{in} \quad \Omega_1 \qquad (1.13)$$

$$u_1^0(t) = \phi(t) \quad \forall t \in (-\infty, 0]$$

$$u_1^0\,|_{\Gamma_0} = 0\,, \quad \frac{\partial u_1^0}{\partial n}(t) - \int_{-\infty}^{t} g(t-\tau)\frac{\partial u_1^0}{\partial n}(\tau)\, d\tau = \frac{\partial u_2^{-1}}{\partial n}(t) \quad \text{on} \quad \Gamma_1$$

As a consequence, u_2^0 is the solution of :

$$\frac{\partial^2 u_2^0}{\partial t^2} - \Delta u_2^0 = 0 \quad \text{in} \quad \Omega_2$$

$$u_2^0(0) = \phi_0\,, \quad (u_2^0(0))' = \phi_1 \qquad (1.14)$$

$$u_2^0\,|_{\Gamma_1} = u_1^0\,|_{\Gamma_1}\,, \quad u_2^0\,|_{\Gamma_2} = 0\,.$$

General case $k = 1, 2, \dots$: The functions u_1^k, restriction of u^k to Ω_1, are the solutions of the following problems :

$$\frac{\partial^2 u_1^k}{\partial t^2}(t) - \Delta u_1^k(t) + \int_{-\infty}^{t} g(t-\tau)\, \Delta u_1^k(\tau)\, d\tau = 0 \quad \text{in} \quad \Omega_1 \qquad (1.15)$$

$$u_1^k(t) = 0 \quad \forall t \in (-\infty, 0]$$

$$\frac{\partial u_1^k}{\partial n}(t) - \int_{-\infty}^{t} g(t-\tau)\frac{\partial u_1^k}{\partial n}(\tau)\, d\tau = \frac{\partial u_2^{k-1}}{\partial n}(t) \quad \text{on} \quad \Gamma_1\,, \quad u_1^k\,|_{\Gamma_0} = 0\,.$$

On the other hand, the restrictions to Ω_2 are the solutions of :

$$\frac{\partial^2 u_2^k}{\partial t^2} - \Delta u_2^k = 0 \quad \text{in} \quad \Omega_2 \quad \forall k = 1, 2, \dots \qquad (1.16)$$

$$u_2^k(0) = 0\,, \quad (u_2^k(0))' = 0$$

$$u_2^k\,|_{\Gamma_1} = u_1^k\,|_{\Gamma_1}\,, \quad u_2^k\,|_{\Gamma_2} = 0\,.$$

Remark 1.2 : As we progress in the actual calculation of the terms of the expansion it becomes, as it is usual, increasingly difficult to establish the sufficient conditions to justify the calculations. In fact, it is only possible to calculate the term u^{-1} and the restriction to Ω_1 of the term u^0. The lack of regularity results prevents us from going on.

It can be shown that, under convenient regularity hypotheses, this asymptotic expansion is rigorous. Through similar procedures to those used by Lions [5], we can prove the following proposition.

PROPOSITION 1.2 : *If* $u^k \in C^0([0, T]; H_0^1(\Omega))$ *and* $(u^k)' \in C^0([0, T]; L^2(\Omega))$ $\forall k = -1, 0, 1, \ldots, n+1$ *and if also* $(u^{n+1})' \in L^2(0, T; H_0^1(\Omega))$ *and*

$$(u^{n+1})'' \in L^2(0, T; L^2(\Omega)) ;$$

then :

$$\left\| u_\varepsilon - \left(\frac{1}{\varepsilon} u^{-1} + u^0 + \cdots + \varepsilon^n u^n \right) \right\|_{L^2(0,T;H_0^1(\Omega))} \leqslant C\varepsilon^{n+1}$$

$$\left\| u_\varepsilon' - \left(\frac{1}{\varepsilon} u^{-1} + u^0 + \cdots + \varepsilon^n u^n \right)' \right\|_{L^2(0,T;L^2(\Omega))} \leqslant C\varepsilon^{n+1} .$$

2. SPECTRAL STUDY

The study of the convergence of the restrictions of u_ε to each domain Ω_i $i = 1, 2$ may be described in terms of two associated operators A_1 and A_2, closely related to $\mathscr{A}_\varepsilon$.

2.1. Location of the spectra of A_1, A_2 and $\mathscr{A}_\varepsilon$

a) *The operator* A_1

The operator A_1 is defined from the Hilbert space

$$H = H_0^1(\Omega_1, \Gamma_0) \times L^2(\Omega_1) \times L_g^2(0, \infty; H_0^1(\Omega_1, \Gamma_0))$$

into itself and it is given by :

$$A_1 U = \begin{bmatrix} -v \\ cAu - \displaystyle\int_0^\infty g(\xi)\, Aw(\xi)\, d\xi \, . \\ w' + v \end{bmatrix} \tag{2.1}$$

Being A a linear bounded operator from $H_0^1(\Omega_1, \Gamma_0)$ into its dual, associated with the form :

$$a : (u, v) \in H_0^1(\Omega_1, \Gamma_0) \times H_0^1(\Omega_1, \Gamma_0) \to a(u, v) = \int_{\Omega_1} \nabla u \, \nabla \bar{v} \, dx \in \mathbb{C} . \tag{2.2}$$

Finally, the domain of A_1 is :

$$D(A_1) = \left\{ (u, v, w) \in H / v \in H_0^1(\Omega_1, \Gamma_0), \, cAu - \int_0^\infty g(\xi) \, Aw(\xi) \, d\xi \in L^2(\Omega_1) , \right.$$
$$\left. w' \in L_g^2(0, \infty ; H_0^1(\Omega_1, \Gamma_0)) , \quad w(0) = 0 \right\} .$$

The operator A_1, studied by Lobo [6], has a resolvent set $\rho(A_1)$ containing the whole semiplane $\operatorname{Re} \eta < 0$ and the points of the strip $0 \leqslant \operatorname{Re} \eta < \mu/2$ which satisfy :

$$- \frac{\eta^2}{1 - \int_0^\infty g(\xi) \, e^{\eta \xi} \, d\xi} \neq b_k^2 \quad \forall k \in N \tag{2.3}$$

where the b_k^2, which belong to the A spectrum, form an unbounded sequence of real positive numbers. On the other hand, A_1 has a point spectrum $\sigma_p(A_1)$ formed by the points of the above mentioned strip which do not satisfy the inequality (2.3). Its residual spectrum $\sigma_R(A_1)$ is the semiplane $\operatorname{Re} \eta \geqslant \mu/2$.

b) *The operator* A_2

The operator A_2 from $H_0^1(\Omega_2) \times L^2(\Omega_2)$ into itself is defined by :

$$A_2 \, U = \begin{bmatrix} - v \\ Au \end{bmatrix} \tag{2.4}$$

where $A \in L(H_0^1(\Omega_2), H^{-1}(\Omega_2))$ is associated with the form :

$$a : (u, v) \in H_0^1(\Omega_2) \times H_0^1(\Omega_2) \to a(u, v) = \int_{\Omega_2} \nabla u \, \nabla \bar{v} \, dx \in \mathbb{C} .$$

As it is well known, A_2 is a skew selfadjoint operator whose spectrum consists of the points $\pm \, id_k$ such that $\{ d_k^2 \}$ form the spectrum of A.

c) *The operator* $\mathcal{A}_\varepsilon$

Now, let us study the location of the spectrum of the operator $\mathcal{A}_\varepsilon$.

PROPOSITION 2.1 : *The eigenvalues of $\mathscr{A}_\varepsilon$ lie on the strip of the complex plane* $0 \leqslant \operatorname{Re} \eta < \mu/2$.

Proof : First, because of proposition 1.1, the whole semiplane $\operatorname{Re} \eta < 0$ is contained in the resolvent set.

On the other hand, the discussion on the eigenvalue problem allows us to assert that the third component w of the eigenvector U must have the form :

$$w(\xi) = u(e^{\eta\xi} - 1) \tag{2.5}$$

where u is the first component of U. A necessary condition for w to belong to $L_g^2(0, \infty ; H_0^1(\Omega_1, \Gamma_0))$ is that $\operatorname{Re} \eta < \mu/2$, thus the point spectrum is contained in the strip $0 \leqslant \operatorname{Re} \eta < \mu/2$.

Finally, in a similar way, it can easily be shown that if $\operatorname{Re} \eta \geqslant \mu/2$ then η belongs to the residual spectrum.

2.2. Existence of eigenvalues of the operator $\mathscr{A}_\varepsilon$

Now, let us discuss an equivalent formulation of the eigenvalue problem that allow us to proof the existence of eigenvalues of the operator $\mathscr{A}_\varepsilon$ when ε is sufficiently small.

Moreover, we proof that the eigenvalues of A_1 and A_2 are accumulation points of the eigenvalues of $\mathscr{A}_\varepsilon$ i.e. it is possible to obtain a sequence $\{ \zeta_\varepsilon \}$, so that ζ_ε is an eigenvalue of $\mathscr{A}_\varepsilon$ and the sequence $\{ \zeta_\varepsilon \}$ converges to ζ when $\varepsilon \to 0^+$, where ζ is an eigenvalue of A_1 or A_2.

Let us write the eigenvalue problem :

Find $\eta \in \mathbb{C}$ and $U = (u, v, w) \in D(\mathscr{A}_\varepsilon)$, $U \neq 0$ such that :

$$- v - \eta u = 0 \tag{2.6}$$

$$A_\varepsilon u - \int_0^\infty g(\xi) A w(\xi) \, d\xi - \eta v = 0 \tag{2.7}$$

$$w' + v_1 - \eta w = 0 \,. \tag{2.8}$$

If we obtain v from (2.6) and substitute into (2.8), we can assert that w is the solution of :

$$\begin{aligned} w' - \eta w &= \eta u \\ w(0) &= 0 \,. \end{aligned} \tag{2.9}$$

As a consequence, w has the following form :

$$w(\xi) = u(e^{\eta\xi} - 1) \,. \tag{2.10}$$

Now, the substitution of v and w into (2.7) allow us to obtain the following equivalent eigenvalue problem :

Find $\eta \in \mathbb{C}$ and $u \in H_0^1(\Omega)$, $u \neq 0$ such that :

$$a_1(\eta, u, v) + \varepsilon a_2(u, v) + \eta^2(b_1(u, v) + \varepsilon b_2(u, v)) = 0 \quad \forall v \in H_0^1(\Omega) \tag{2.11}$$

where :

$$a_1(\eta, u, v) = \left(1 - \int_0^\infty g(\xi)\, e^{\eta\xi}\, d\xi\right) \int_{\Omega_1} \nabla u\, \nabla \bar{v}\, dx \tag{2.12}$$

$$a_2(u, v) = \int_{\Omega_2} \nabla u\, \nabla \bar{v}\, dx \tag{2.13}$$

$$b_1(u, v) = \int_{\Omega_1} u\bar{v}\, dx \tag{2.14}$$

$$b_2(u, v) = \int_{\Omega_2} u\bar{v}\, dx\,. \tag{2.15}$$

In the sequel, we denote $1 - \int_0^\infty g(\xi)\, e^{\eta\xi}\, d\xi$ by $G(\eta)$.

In order to solve the problem (2.11), we use a modified version of the method of expansion in powers of ε used by Lobo-Sanchez [7].

In this way, the following problem is considered :

Let be $F \in L^2(\Omega)$. Find $u_\varepsilon \in H_0^1(\Omega)$, depending on η and F, such that :

$$a_1(\eta, u, v) + \varepsilon a_2(u, v) + \eta^2(b_1(u, v) + \varepsilon b_2(u, v)) = \int_\Omega F\bar{v}\, dx\,, \quad \forall v \in H_0^1(\Omega) \tag{2.16}$$

First, let us find and asymptotic expansion of the kind :

$$\frac{1}{\varepsilon} u^{-1} + u^0 + \varepsilon u^1 + \cdots + \varepsilon^k u^k + \cdots, \quad u^k \in H_0^1(\Omega) \quad \forall k = -1, 0, \ldots \tag{2.17}$$

By formal substitution in (2.11), we obtain the equations that allow us to calculate the terms of the above expansion :

$$a_1(\eta, u^{-1}, v) + \eta^2\, b_1(u^{-1}, v) = 0 \quad \forall v \in H_0^1(\Omega) \tag{2.18}$$

$$a_2(u^{-1}, v) + \eta^2\, b_2(u^{-1}, v) + a_1(\eta, u^0, v) + \eta^2\, b_1(u^0, v) = \int_\Omega F\bar{v}\, dx \tag{2.19}$$

$$a_2(u^{k-1}, v) + \eta^2\, b_2(u^{k-1}, v) + a_1(\eta, u^k, v) + \eta^2\, b_1(u^k, v) = 0\,, \quad k = 1, 2, \dots \tag{2.20}$$

PROPOSITION 2.2 : *Let be* $F \in L^2(\Omega)$. *Let* V *be an open set of* $\mathbb{C}$ *contained in* $\rho(A_1) \cap \rho(A_2)$. *Then, the functions* $u^k : \eta \in V \subset \mathbb{C} \to u^k(\eta) \in H_0^1(\Omega)$ $k = -1, 0, 1, 2, \dots$ *exist and are holomorphic.*

Proof : First, taking into account the equation (2.18), u_1^{-1} is the solution of :

$$- G(\eta)\, \Delta u_1^{-1} + \eta^2\, u_1^{-1} = 0 \quad \text{in} \quad \Omega_1 \tag{2.21}$$

$$u_1^{-1}\,|_{\Gamma_0} = 0\,, \quad G(\eta)\, \frac{\partial u_1^{-1}}{\partial n} = 0 \quad \text{on} \quad \Gamma_1\,.$$

Since $\eta \in \rho(A_1)$, $-\eta^2$ belongs to the resolvent set of the above problem, therefore $u_1^{-1} = 0$.

On the other hand, if in the equation (2.19) we take $v \in H_0^1(\Omega)$ such that $v = 0$ in Ω_1, u_2^{-1} is the solution of the following problem :

$$\begin{aligned} - \Delta u_2^{-1} + \eta^2\, u_2^{-1} &= F_2 \quad \text{in} \quad \Omega_2 \\ u_2^{-1}\,|_{\Gamma_1} = 0\,, \quad u_2^{-1}\,|_{\Gamma_2} &= 0\,. \end{aligned} \tag{2.22}$$

Since $\eta \in \rho(A_2)$, $-\eta^2$ belongs to the resolvent set of the Laplace operator with homogeneous boundary conditions, therefore $u_2^{-1} \in H_0^1(\Omega_2) \cap H^2(\Omega_2)$ and is holomorphic. Moreover, u_2^{-1} satisfies :

$$\| u_2^{-1}(\eta) \|_{H^2(\Omega_2)} \leqslant C_{-1}(\eta) \tag{2.23}$$

where C_{-1} is a continuous function of η.

In short, u^{-1} is holomorphic and satisfies :

$$\| u^{-1}(\eta) \|_{H_0^1(\Omega)} \leqslant C_{-1}(\eta)\,. \tag{2.24}$$

Now, using the equation (2.19) we obtain that u_1^0 is the solution of :

$$\begin{aligned} - G(\eta)\, \Delta u_1^0 + \eta^2\, u_1^0 &= F_1 \quad \text{in} \quad \Omega_1 \\ u_1^0\,|_{\Gamma_0} = 0\,, \quad G(\eta)\, \frac{\partial u_1^0}{\partial n} &= \frac{\partial u_2^{-1}}{\partial n} \quad \text{on} \quad \Gamma_1\,. \end{aligned} \tag{2.25}$$

As a consequence of the previous calculation of $u_2^{-1}(\eta)$, $\partial u_2^{-1}/\partial n$ is an holomorphic function of η into $H^{1/2}(\Gamma_1)$. Thus, u_1^0 belongs to $H_0^1(\Omega_1, \Gamma_0) \cap H^2(\Omega_1)$ and is holomorphic.

In a similar way, we obtain that u_2^0 is the solution of the problem :

$$\begin{aligned} - \Delta u_2^0 + \eta^2 u_2^0 = 0 \quad \text{in} \quad \Omega_2 \\ u_2^0 \,|_{\Gamma_2} = 0 , \quad u_2^0 \,|_{\Gamma_1} = u_1^0 \,|_{\Gamma_1} \end{aligned} \tag{2.26}$$

and, as a consequence, $u_2^0 \in H_0^1(\Omega_2, \Gamma_2) \cap H^2(\Omega_2)$ and is holomorphic.

In short, we deduce that u^0 is holomorphic and satisfies :

$$\| u^0(\eta) \|_{H_0^1(\Omega)} \leqslant C_0(\eta) \tag{2.27}$$

where C_0 is a continuous function of η.

In an analogous way, we obtain that u^k is holomorphic from $V \subset \mathbb{C}$ into $H_0^1(\Omega)$ and satisfies :

$$\| u^k(\eta) \|_{H_0^1(\Omega)} \leqslant [C_1(\eta)]^k \quad \forall k \geqslant 1 . \tag{2.28}$$

Proposition 2.3 : *Let K be a compact set contained in $\rho(A_1) \cap \rho(A_2)$. Let be $F \in L^2(\Omega)$. Then, for ε sufficiently small, the asymptotic expansion :*

$$\frac{1}{\varepsilon} u^{-1}(\eta) + u^0(\eta) + \varepsilon u^1(\eta) + \cdots$$

is uniformly convergent in $H_0^1(\Omega)$ to $u_\varepsilon(\eta)$, i.e. to the solution of the problem (2.11).

Proof : Since K is a compact set and the functions C_{-1}, C_0 and C_1 are continuous, they are bounded.

Thus :

$$\left\| \sum_{n=-1}^{N} \varepsilon^n u^n(\eta) \right\|_{H_0^1(\Omega)} \leqslant \frac{1}{\varepsilon} k + k + \sum_{n=1}^{N} \varepsilon^n k^n .$$

As a consequence, if $\varepsilon < 1/k$ the conclusion follows.

Now the principal result of this paper is stated.

Théorème 2.1 : *Let be $\zeta \in \sigma_p(A_1) \cup \sigma(A_2)$. If ε is sufficiently small, then there exist eigenvalues of $\mathcal{A}_\varepsilon$ near to ζ.*

Proof : We consider a simple closed curve γ enclosing ζ. It is possible to choose γ such that γ is contained in $\rho(A_1) \cap \rho(A_2)$ and γ does not enclose any other point of $\sigma_p(A_1) \cup \sigma(A_2)$ because $\sigma(A_2)$ is contained in the complex axis and if $g \not\equiv 0$ then the eigenvalues of A_1 are not purely imaginary numbers.

First, we assume that $\zeta \in \sigma_p(A_1)$. Let F be an element of $L^2(\Omega)$ such that F_1 is an eigenvector of A_1 associated with ζ and $F_2 = 0$. The existence of a solution of the problem (2.11) can be easily deduced from the previous results when $\eta \in \gamma$. Then $u_\varepsilon(\eta)$ becomes :

$$u_\varepsilon(\eta) = \sum_{n=-1}^{\infty} \varepsilon^n u^n(\eta) . \tag{2.29}$$

Integrating along γ, we obtain :

$$\int_\gamma u_\varepsilon(\eta)\, d\eta = \frac{1}{\varepsilon}\int_\gamma u^{-1}(\eta)\, d\eta + \int_\gamma u^0(\eta)\, d\eta + \cdots . \tag{2.30}$$

Taking into account (2.22), u^{-1} has no singularity in the region enclosed by γ and using the Cauchy's integral theorem we can assert :

$$\int_\gamma u^{-1}(\eta)\, d\eta = 0 . \tag{2.31}$$

In a similar way, considering (2.25), we obtain :

$$\int_\gamma u^0(\eta)\, d\eta \neq 0 . \tag{2.32}$$

Then, by taking the limit value as $\varepsilon \to 0^+$ in (2.30), we get :

$$\operatorname*{Lim}_{\varepsilon \to 0^+} \int_\gamma u_\varepsilon(\eta)\, d\eta \neq 0 . \tag{2.33}$$

u_ε is holomorphic because it is the uniform limit of holomorphic functions, hence we can easily verify that, for sufficiently small ε, u_ε has a singularity in the region enclosed by γ and consequently, this singularity is an eigenvalue.

On the other hand, let be $\zeta \in \sigma(A_2)$. If we choose $F \in L^2(\Omega)$ such that $F_1 = 0$ and F_2 is an eigenvector of A_2 associated with ζ, we have that $u_\varepsilon(\eta)$ exists and it is defined by the series (2.29). Integrating along γ, we have that :

$$\int_\gamma u_\varepsilon(\eta)\, d\eta = \frac{1}{\varepsilon}\int_\gamma u^{-1}(\eta)\, d\eta + \int_\gamma u^0(\eta)\, d\eta + \cdots . \tag{2.34}$$

Now, taking into account (2.22), we can verify that :

$$\int_\gamma u^{-1}(\eta)\, d\eta \neq 0 . \tag{2.35}$$

As a consequence

$$\lim_{\varepsilon \to 0^+} \varepsilon \int_\gamma u_\varepsilon(\eta)\, d\eta \neq 0 \tag{2.36}$$

and, following the procedure above, the conclusion follows.

COROLLARY 2.1 : *Let be $\zeta \in \sigma_p(A_1) \cup \sigma(A_2)$. There exist a sequence $\{\zeta_n\}$, where ζ_n is an eigenvalue of $\mathscr{A}_{\varepsilon_n}$, such that ζ_n converges to ζ, when $\varepsilon_n \to 0^+$.*

3. COMPLEMENTS

Similar results are obtained by considering some problems closely related to the one here discussed. For example, if the roles of the domains Ω_1 and Ω_2 are interchanged (i.e. the parameter ε acts on the outer domain) or if the integro-differential operator is defined on the domain Ω_2.

The problem discussed in this paper may be taken as a model applicable to several situations appearing in other fields of science. In this way, it may be applied to the study of the elastic vibrations of a non-homogeneous bounded body, with a viscoelastic layer immersed in other merely elastic, layer having a very small density and rigidity with respect to the first one.

An adequate dimensional analysis leads us to the following problem, where the displacement vector u and the strain tensor σ_{ij} are used.

$$\frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \sigma_{ij}}{\partial x_j}$$

where σ_{ij} is defined in Ω_1 by :

$$\sigma_{ij} = C^1_{ijkl}(x)\frac{\partial u_k}{\partial x_1} - \int_{-\infty}^{t} G_{ijkl}(x, t-\tau)\frac{\partial u_k}{\partial x_1}\, d\tau$$

and in Ω_2 it is given by :

$$\sigma_{ij} = C^2_{ijkl}(x)\frac{\partial u_k}{\partial x_1}.$$

The transmission conditions are :

$$u^1_i = u^2_i \quad \text{on } \Gamma_1 \quad i = 1, 2, 3$$

$$\sigma^1_{ij}.n_j = \varepsilon\sigma^2_{ij}.n_j \quad \text{on } \Gamma_1 .$$

The numerical study of several particular cases, such as the acoustic waves interaction with layered elastic or viscoelastic obstacles in water can be see in Peterson-Varadan-Varadan [8].

Acknowledgement : I am grateful to Ldo. E. Patiño for his advice in the preparation of the manuscript in English.

REFERENCES

[1] H. BREZIS, *Opérateurs maximaux monotones,* North-Holland, Amsterdam (1973).

[2] C. M. DAFERMOS, *An abstract Volterra equation with application to linear viscoelasticity,* J. Diff. Équations, 7 (1970), pp. 554-569.

[3] C. M. DAFFERMOS, *Asymptotic stability in viscoelasticity,* Arch. Rat. Mech. Anal., 37 (1970), pp. 297-308.

[4] C. M. DAFERMOS, *Contraction semigroups and trend to equilibrium in continuum mechanics,* Lec. Notes Math., 503, Springer, Berlin (1975), pp. 295-306.

[5] J. L. LIONS, *Perturbations singulières dans les problèmes aux limites et en contrôle optimal,* Lec. Notes Math., 323, Springer, Berlin (1973).

[6] M. LOBO-HIDALGO, *Propriétés spectrales de certaines équations différentielles intervenant en viscoélasticité,* Rend. Sem. Mat. Univ. Polit. Torino, 39 (1981), pp. 33-51.

[7] M. LOBO-HIDALGO and E. SANCHEZ-PALENCIA, *Perturbation of spectral properties for a class of stiff problems,* Proc. Fourth Inter. Symp. Comp. Method. in Science and Engineering, North-Holland, Amsterdam (1980).

[8] V. PETERSON, V. V. VARADAN and V. K. VARADAN, *Scattering of acoustic waves by layered elastic and viscoelastic obstacles in water,* J. Acoust. Soc. Am., 68 (1980), pp. 673-685.