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ERROR ESTIMATES FOR A MIXED FINITE ELEMENT APPROXIMATION OF THE STOKES EQUATIONS (*)

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Abstract — *We consider a mixed finite element method for the Stokes problem in a polygonal domain $\Omega \subset \mathbb{R}^2$ An* inf-sup *condition is established which fits into the abstract framework of Babuška and Brezzi Using linear finite elements, we obtain $O(h^\alpha)$-error estimates for the energy norm of the velocity and the L^2-norm of the pressure The exponent α only depends on the greatest interior angle at a corner of Ω and equals 1 if Ω is convex The analysis can be extended to the use of quadratic finite elements for the velocity*

Résumé — *Nous considérons une méthode d'éléments finis mixtes pour les équations de Stokes dans un polygone $\Omega \subset \mathbb{R}^2$ On établit une condition du type* inf-sup *qui permet l'application des résultats abstraits de Babuška et Brezzi Pour les éléments finis linéaires, nous démontrons une majoration d'erreur d'ordre $O(h^\alpha)$ pour la norme H^1 du champ de vitesse et pour la norme L^2 de la pression Le nombre α dépend seulement du plus grand angle intérieur aux sommets de Ω, et $\alpha = 1$ si Ω est convexe L'analyse s'étend aux éléments finis quadratiques pour la vitesse*

1. INTRODUCTION

We give a refined analysis of a mixed finite element method proposed by Bercovier, Pironneau [2] for the Stokes equations

$$\begin{aligned} -\nu\,\Delta\mathbf{u} + \nabla p &= \mathbf{f} && \text{in } \Omega \\ \operatorname{div}\mathbf{u} &= 0 && \text{in } \Omega \\ \mathbf{u} &= 0 && \text{on } \partial\Omega \end{aligned} \tag{1.1}$$

in a simply connected bounded polygonal domain $\Omega \subset \mathbb{R}^2$ where $\nu > 0$ denotes the viscosity. The velocity $\mathbf{u} = (u_1, u_2)$ and the pressure p are approximated by conforming linear finite elements with mesh size $h/2$ and h resp.

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Bercovier, Pironneau [2] use an inf-sup condition (see Lemma 3.1 below) which does not fit into the abstract framework of Babuška and Brezzi. Therefore, a crucial point of our analysis is the discrete analogue of the Brezzi-type condition

$$\inf_{\substack{p \in L^2(\Omega)/\mathbb{R} \\ p \neq 0}} \sup_{\substack{\mathbf{u} \in H_0^1(\Omega)^2 \\ \mathbf{u} \neq 0}} \frac{(p, \operatorname{div} \mathbf{u})}{\| p \|_{0/\mathbb{R}} \, | \mathbf{u} |_1} > 0 \tag{1.2}$$

(*cf.* Theorem 3.7 in [9]) where

$$| \mathbf{u} |_1 := \int_\Omega | \nabla \mathbf{u} |^2 \, dx ,$$

$$\| p \|_{0/\mathbb{R}} := \inf_{c \in \mathbb{R}} \| p + c \|_{L^2(\Omega)} .$$

We obtain the discrete version of (1.2) for the appropriate function spaces by combining the inf-sup condition of Bercovier and Pironneau [2] and an approximation argument.

The discrete inf-sup condition allows us to apply the abstract error analysis of Brezzi [4] for mixed problems. Combining it with the regularity results of [3, 11] we immediately get an $O(h^\alpha)$ error estimate for the energy norm of the velocity and the L^2-norm of the pressure. The exponent α only depends on the greatest interior angle at a corner of Ω and equals 1 when Ω is convex. A standard duality argument yields an $O(h^{2\alpha})$ error estimate for the L^2-norm of the velocity.

For convenience, we perform the analysis only for piecewise linear functions. The results can easily be extended to quadratic finite elements. When using piecewise quadratic functions for the velocity we get improved error estimates provided $\mathbf{f} \in H^1(\Omega)^2$ and Ω is convex.

The advantage when compared with Bercovier and Pironneau [2] are error estimates for the pressure under weaker regularity assumptions for the solution. Naturally, when $p \in H^1(\Omega)$ (i.e. Ω convex) our results coincide with those obtained by Glowinski, Pironneau via a duality argument [10]. In this context we note that using other finite element spaces, Crouzieux, Raviart [7] obtain optimal error estimates for convex domains. Using the stream function formulation of (1.1) an $O(h^{1/2} \, | \ln h |)$ L^2-error estimate for the velocity and pressure has been established in [3, 8].

Moreover, the discrete analogue of (1.2) is interesting in itself. It is crucial for the analysis of a multigrid method for the numerical solution of the mixed finite element approximation (2.6) of (1.1).

2. MIXED FORMULATION OF THE STOKES EQUATIONS

We denote by $H^m(\Omega)$, $m \geqslant 0$, $H_0^1(\Omega)$ and $L^2(\Omega) = H^0(\Omega)$ the usual Sobolev and Lebesgue spaces, equipped with the norm

$$\| u \|_m := \left\{ \int_\Omega \sum_{|\beta| \leqslant m} | D^\beta u(x) |^2 \, dx \right\}^{1/2} \tag{2.1}$$

and the semi-norm

$$| u |_m := \left\{ \int_\Omega \sum_{|\beta| = m} | D^\beta u(x) |^2 \, dx \right\}^{1/2} . \tag{2.2}$$

Since no confusion can arise, we use the same notation for the corresponding product norm and semi-norm of $H^m(\Omega)^2$. For ease of notation put

$$\begin{aligned} X &:= H_0^1(\Omega)^2 , \\ M &:= \left\{ p \in L^2(\Omega) : \int_\Omega p(x) \, dx = 0 \right\} . \end{aligned} \tag{2.3}$$

Given $\mathbf{f} \in L^2(\Omega)^2$ we consider the following weak formulation of (1.1) :

Find $(\mathbf{u}, p) \in X \times M$ such that

$$\begin{aligned} \nu(\nabla \mathbf{u}, \nabla \mathbf{v}) - (\operatorname{div} \mathbf{v}, p) &= (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in X \\ (\operatorname{div} \mathbf{u}, q) &= 0 \qquad \forall q \in M , \end{aligned} \tag{2.4}$$

where $(\mathbf{a}, \mathbf{b}) := \int_\Omega \mathbf{a}(x) . \mathbf{b}(x) \, dx$ denotes the scalar product of $L^2(\Omega)^n$, $n = 1, 2, 4$.
According to [9, 12, 13] problem (2.4) has a unique solution in $X \times M$.

Let $\mathcal{T}_h$, $h > 0$, be a sequence of triangulations of Ω which satisfy the usual regularity assumptions :

(i) Any two triangles in $\mathcal{T}_h$ may meet at most in whole common sides or in vertices.

(ii) Each triangle has at least one vertex in the interior of Ω.

(iii) Each triangle contains a circle with radius $c_0 \, h$ and is contained in a circle with radius $c_0^{-1} \, h$.

Here and in the sequel, c_0, c_1, ... denote generic constants which do not depend on h. The triangulation $\mathcal{T}_{h/2}$ is obtained from $\mathcal{T}_h$ by dividing each T of $\mathcal{T}_h$ into 4 equal triangles, the vertices of which are the vertices and midpoints of T.

Let S_h denote the space of continuous, piecewise linear finite elements defined by nodal values at the vertices in the triangulation $\mathcal{T}_h$. Put

$$X_h := (S_{h/2})^2 \cap X\,, \quad M_h := S_h \cap M\,. \tag{2.5}$$

The finite element approximation of (2.4) then leads to the following mixed problem :

Find $(\mathbf{u}_h, p_h) \in X_h \times M_h$ such that

$$\begin{aligned} \nu(\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) - (\operatorname{div} \mathbf{v}_h, p_h) &= (\mathbf{f}, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in X_h \\ (\operatorname{div} \mathbf{u}_h, q_h) &= 0 \qquad\quad \forall q_h \in M_h\,. \end{aligned} \tag{2.6}$$

3. THE INF-SUP CONDITION

Recall that $\Omega \subset \mathbb{R}^2$ is a simply connected bounded polygon. First we provide two prepository lemmata.

LEMMA 3.1 (Bercovier, Pironneau [2]) : *There is a constant $c_1 > 0$ independent of h such that*

$$\inf_{q_h \in M_h \setminus \{0\}} \sup_{\mathbf{v}_h \in X_h \setminus \{0\}} \frac{(\operatorname{div} \mathbf{v}_h, q_h)}{\| \mathbf{v}_h \|_0 \, | q_h |_1} \geqslant c_1\,. \quad \square \tag{3.1}$$

Lemma 3.1 immediately follows from Proposition 1 in [2] by partial integration. In the framework of Brezzi [4] it is not appropriate for the mixed problem (2.6) since the wrong norms occur in the denominator of (3.1).

The next Lemma is an immediate consequence of Lemma 3.2 in [9].

LEMMA 3.2 : *For every $p \in M$ there is a $\mathbf{u} \in X$ such that*

$$\operatorname{div} \mathbf{u} = p\,, \tag{3.2}$$

$$| \mathbf{u} |_1 \leqslant c_2 \, \| p \|_0\,, \tag{3.3}$$

where $c_2 > 0$ is independent of $\mathbf{u}$ *and p.* $\square$

Now we can establish an inf-sup condition which is well suited to (2.6).

PROPOSITION 3.3 : *There is a constant $c_3 > 0$ independent of h such that*

$$\inf_{q_h \in M_h \setminus \{0\}} \sup_{\mathbf{v}_h \in X_h \setminus \{0\}} \frac{(\operatorname{div} \mathbf{v}_h, q_h)}{| \mathbf{v}_h |_1 \, \| q_h \|_0} \geqslant c_3\,. \tag{3.4}$$

Proof : Let $q_h \in M_h$, $\| q_h \|_0 = 1$, and put for abbreviation

$$\eta := h \, | q_h |_1\,. \tag{3.5}$$

Since

$$|\mathbf{v}_h|_1 \leqslant c_4 h^{-1} \|\mathbf{v}_h\|_0$$

for every $\mathbf{v}_h \in X_h$, Lemma 3.1 implies

$$\sup_{v_h \in X_h \setminus \{0\}} \frac{(\operatorname{div} \mathbf{v}_h, q_h)}{|\mathbf{v}_h|_1} \geqslant c_1 c_4^{-1} h |q_h|_1 = c_5 \eta \tag{3.6}$$

where $c_5 := c_1 c_4^{-1}$.

According to Lemma 3.2 there is a $\mathbf{w} \in X$ satisfying

$$\begin{aligned} &\operatorname{div} \mathbf{w} = q_h, \\ &|\mathbf{w}|_1 \leqslant c_2 \|q_h\|_0 = c_2. \end{aligned} \tag{3.7}$$

From [6] we know that there is an operator $R_h : X \to X_h$ such that

$$|\mathbf{v} - R_h \mathbf{v}|_k \leqslant c_6 h^{1-k} |\mathbf{v}|_1, \quad k = 0, 1, \tag{3.8}$$

for all $\mathbf{v} \in X$. Put $w_h := R_h \mathbf{w}$. Then we have

$$\begin{aligned} \sup_{\mathbf{v}_h \in X_h \setminus \{0\}} \frac{(\operatorname{div} \mathbf{v}_h, q_h)}{|\mathbf{v}_h|_1} &\geqslant \frac{|(\operatorname{div} \mathbf{w}_h, q_h)|}{|\mathbf{w}_h|_1} \geqslant \\ &\geqslant \frac{|(\operatorname{div} \mathbf{w}, q_h) - (\operatorname{div}(\mathbf{w} - \mathbf{w}_h), q_h)|}{|\mathbf{w}|_1 + |\mathbf{w} - \mathbf{w}_h|_1} \\ &\geqslant \frac{1}{c_2(1 + c_6)} \{ \|q_h\|_0^2 - \|\mathbf{w} - \mathbf{w}_h\|_0 \|\nabla q_h\|_0 \} \\ &\geqslant \frac{1}{c_2(1 + c_6)} \{ 1 - c_2 c_6 h |q_h|_1 \} = c_7 - c_8 \eta \end{aligned} \tag{3.9}$$

where $c_7 := \dfrac{1}{c_2(1 + c_6)}$ and $c_8 := \dfrac{c_6}{1 + c_6}$.

Equations (3.6), (3.9) imply

$$\begin{aligned} \sup_{\mathbf{v}_h \in X_h \setminus \{0\}} \frac{(\operatorname{div} \mathbf{v}_h, q_h)}{|\mathbf{v}_h|_1} &\geqslant \max \{ c_5 \eta, c_7 - c_8 \eta \} \geqslant \\ &\geqslant \min_{t \geqslant 0} \max \{ c_5 t, c_7 - c_8 t \} = \frac{c_5 c_7}{c_5 + c_8}. \end{aligned} \tag{3.10}$$

Now a simple homogeneity argument completes the proof. □

4. ERROR ESTIMATES FOR THE VELOCITY AND PRESSURE

The spaces X, X_h equipped with $| \cdot |_1$ and M, M_h equipped with $\| \cdot \|_{L^2/\mathbb{R}}$ are Hilbert spaces. The bilinear form $(\nabla \mathbf{u}, \nabla \mathbf{v})$ is elliptic, and the bilinear form $(\operatorname{div} \mathbf{v}, p)$ satisfies by (1.2) and Proposition 3.3 an inf-sup condition. Hence the abstract results of Babuška and Brezzi [1, 4] yield the existence of unique solutions $(\mathbf{u}, p) \in X \times M$ and $(\mathbf{u}_h, p_h) \in X_h \times M_h$ of problems (2.4) and (2.6), resp. and the error estimate

$$| \mathbf{u} - \mathbf{u}_h |_1 + \| p - p_h \|_0 \leqslant c_9 \left\{ \inf_{\mathbf{v}_h \in X_h} | \mathbf{u} - \mathbf{v}_h |_1 + \inf_{q_h \in M_h} \| p - q_h \|_0 \right\}. \tag{4.1}$$

To estimate $\| \mathbf{u} - \mathbf{u}_h \|_0$ we use a duality argument due to Aubin-Nitsche. Denote by $\mathscr{L} : L^2(\Omega)^2 \to X \times M$ the solution operator of the Stokes problem (2.4). Let $\mathbf{v} \in L^2(\Omega)^2$ and $(\mathbf{z}, r) = \mathscr{L}(v)$. From (2.4), (2.6) we then obtain for any $\mathbf{z}_h \in X_h$, $r_h \in M_h$

$$\begin{aligned}(\mathbf{v}, \mathbf{u} - \mathbf{u}_h) &= \nu(\nabla \mathbf{z}, \nabla(\mathbf{u} - \mathbf{u}_h)) - (\operatorname{div}(\mathbf{u} - \mathbf{u}_h), r) \\ &= \nu(\nabla(\mathbf{z} - \mathbf{z}_h), \nabla(\mathbf{u} - \mathbf{u}_h)) - (\operatorname{div}(\mathbf{u} - \mathbf{u}_h), r - r_h) - \\ &\qquad - (\operatorname{div}(\mathbf{z} - \mathbf{z}_h), p - p_h).\end{aligned}$$

Hence we have

$$\| \mathbf{u} - \mathbf{u}_h \|_0 \leqslant c_{10} \gamma_h \{ | \mathbf{u} - \mathbf{u}_h |_1 + \| p - p_h \|_0 \} \tag{4.2}$$

with

$$\gamma_h = \sup_{\substack{\| \mathbf{v} \|_0 = 1 \\ (\mathbf{z}, r) = \mathscr{L}(\mathbf{v})}} \inf_{\substack{\mathbf{z}_h \in X_h \\ r_h \in M_h}} \{ | \mathbf{z} - \mathbf{z}_h |_1 + \| r - r_h \|_0 \}. \tag{4.3}$$

In order to formulate the regularity results for the Stokes problem, which together with (4.1)-(4.3) yield the final error estimates, we use an additional notation.

Let ω denote the greatest interior angle at a vertex of Ω, $0 < \omega < 2\pi$, $\omega \neq \pi$. Then $\omega < \pi$ if and only if Ω is convex. Put

$$\eta_0(\omega) := \inf \{ \eta \in \mathbb{R}_+^* : z = \zeta + i\eta \text{ is a solution of } \sinh^2(\omega z) = z^2 \sin^2 \omega,\ z \neq i \}. \tag{4.4}$$

According to [3] the function $\eta_0(\omega)$ is decreasing and satisfies

$$\begin{aligned} &\eta_0(\omega) > \frac{\pi}{\omega} > 1\,, && \text{if} \quad 0 < \omega < \pi\,, \\ &1 - \frac{\pi}{\omega} < \frac{1}{2} < \eta_0(\omega) < \frac{\pi}{\omega} < 1\,, && \text{if} \quad \pi < \omega < 2\pi\,. \end{aligned} \tag{4.5}$$

The values of $\eta_0(0.05\, k\pi)$, $1 \leqslant k \leqslant 40$, are listed in [3].

Finally put

$$\beta := \max\{\,0, 1 - \eta_0(\omega) + \varepsilon\,\} \tag{4.6}$$

with an arbitrary $\varepsilon > 0$.

From [11] and Theorem 2.1 in [3] we then know that $\mathscr{L}$ is a continuous operator from $L^2(\Omega)^2$ to the weighted Sobolev space $H_\beta^2(\Omega)^2 \times H^1(\Omega)$. (For an exact definition of $H_\beta^m(\Omega)$ see for example [3]. If $\beta = 0$, $H_\beta^m(\Omega)$ coincides with $H^m(\Omega)$.) Combining this with (4.1)-(4.3), the standard approximation theorems for finite elements and Lemma III.3, III.4 in [3], we obtain the final error estimate.

THEOREM 4.1 : *Let* $(\mathbf{u}, p)$ *and* $(\mathbf{u}_h, p_h)$ *denote the solution of problem* (2.4) *and* (2.6), *resp. Then :*

$$\|\mathbf{u} - \mathbf{u}_h\|_0 + h^\alpha\,|\mathbf{u} - \mathbf{u}_h|_1 + h^\alpha\,\|p - p_h\|_0 \leqslant C_{11}\,h^{2\alpha}\,\|f\|_0 \tag{4.7}$$

where C_{11} *is independent of* h, $\mathbf{u}$ *and* p *and*

$$\begin{aligned} &\alpha = 1\,, && \text{if} \quad \omega < \pi\,, \\ &\alpha = \eta_0(\omega) - \varepsilon \geqslant \frac{1}{2}\,, && \text{if} \quad \pi < \omega < 2\pi\,, \end{aligned} \tag{4.8}$$

with $\varepsilon > 0$ *arbitrarily small.* □

5. REMARK ON QUADRATIC FINITE ELEMENTS

Let $\tilde{S}_h$ denote the space of continuous, piecewise quadratic finite elements defined by nodal values at the vertices and at the midpoints in the triangulation $\mathcal{T}_h$. There are only a few changes necessary when the discrete problem (2.6) is solved with $\tilde{S}_h$ instead of $S_{h/2}$.

Since Bercovier and Pironneau [2] have established their results also for this choice of X_h, the proof of Proposition 3.3 remains unchanged. Therefore, the abstract error estimates (4.1)-(4.3) still hold. Obviously, the approximation of $\mathbf{u}$

by piecewise quadratic functions only gives an improvement of (4.7), if $(\mathbf{u}, p)$ have a higher regularity.

We therefore assume that in addition Ω is convex and $\mathbf{f} \in H^1(\Omega)^2$. From Theorem 2.2 in [3] we then know that $\mathscr{L}$ is a continuous operator from $H^1(\Omega)^2$ to $H_\gamma^3(\Omega)^2 \times H_\gamma^2(\Omega)$ where

$$\gamma := \max \{ 0, 2 - \eta_0(\omega) + \varepsilon \} \tag{5.1}$$

with an arbitrary $\varepsilon > 0$. If $\omega \leqslant 0.7\,\pi$, we have $\gamma = 0$ (*cf.* [3]). Together with (4.1)-(4.3) and Lemma 3.3, 3.4 in [3] this implies the error estimate

$$\| \mathbf{u} - \mathbf{u}_h \|_0 + h \,| \mathbf{u} - \mathbf{u}_h |_1 + h \,\| p - p_h \|_0 \leqslant C_{12}\, h^{3-\gamma} \| \mathbf{f} \|_1 . \tag{5.2}$$

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