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## THE COMBINATION OF APPROXIMATE SOLUTIONS FOR ACCELERATING THE CONVERGENCE (\*)

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Communiqué par A. BENSOUSSAN

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Résumé. — *D'abord nous mentionnons le principe de combinaison des solutions approchées pour l'accélération de la convergence. Et puis nous donnons deux types de combinaisons, le premier est l'interpolation, l'autre est l'extrapolation de décomposition, accélérant la convergence des solutions numériques des équations intégrales et des équations aux dérivées partielles.*

Abstract. — *A principle for the combination of approximate solutions for accelerating the convergence is first proposed, and then two types of combination, one of interpolation and the other of the splitting extrapolation, are given for accelerating the numerical solution of integral equations and partial differential equations.*

### 1. COMBINATION PRINCIPLE

Let us consider the linear operator equation

$$Lu = f. \quad (1)$$

The operator  $L$  will map a subspace of a Banach space  $U$  into a Banach space  $F$ . Problem (1) will be assumed to have a unique solution  $u$ .

Usually this problem cannot be solved in a closed form and it is replaced by some associate, simpler problems depending on a small parameter  $h$  :

$$L_i^h u_i^h = f_i^h, \quad 0 \leq i \leq M. \quad (2)$$

For ease of understanding we first consider a simple case in which the operators  $L_i^h$  map the Banach space  $U$  into the Banach space  $F$  just as  $L$  does and  $f_i^h = f$ .

The operators  $L_i^h$ ,  $0 \leq i \leq M$ , in (2) will be assumed to have the following properties :

$$(A) \quad \|(L_i^h)^{-1}\| \leq c_p,$$

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(B) for  $u$ , the solution of (1), there exists  $l_i(u) \in F$  such that

$$L_i^h u - Lu = h^p(l_i(u) + r_i^h) \rightarrow 0 \quad (h \rightarrow 0),$$

(C) the problem

$$Lw_i = l_i(u)$$

has a solution  $w_i$  and the associate approximate problem

$$L_i^h w_i^h = l_i(u)$$

has a solution  $w_i^h$  which converges to  $w_i$  :

$$e_i^h = w_i^h - w_i \rightarrow 0 \quad (h \rightarrow 0)$$

LEMMA 1 (Combination principle) : Under the assumptions (A), (B), (C), if there exist constants  $a_i, 0 \leq i \leq M$ , such that

$$(D) \quad \sum_0^M a_i l_i(u) = 0, \quad \text{where} \quad \sum_0^M a_i \neq 0$$

then

$$u - \left(\sum_0^M a_i\right)^{-1} \sum_0^M a_i u_i^h = h^p \left(\sum_0^M a_i\right)^{-1} \sum_0^M a_i (e_i^h + (L_i^h)^{-1} r_i^h) \quad (3)$$

Proof : By assumption (B),

$$L_i^h(u - u_i^h - h^p w_i) = L_i^h u - Lu - h^p L_i^h w_i = h^p(l_i(u) - L_i^h w_i + r_i^h).$$

This leads to, by assumption (C),

$$u - u_i^h - h^p w_i = h^p(w_i^h - w_i + (L_i^h)^{-1} r_i^h) = h^p(e_i^h + (L_i^h)^{-1} r_i^h).$$

Hence (3) follows from assumption (D) :

$$\sum a_i w_i = L^{-1} \sum a_i l_i(u) = 0.$$

In order to include the finite difference method we have to describe the spaces and operators in (1), (2) in detail. The spaces  $U$  and  $F$  will consist of functions defined on the domain  $\Omega$ , and  $Lu \in F$  for  $u \in U$ . We define  $U_i^h$  and  $F_i^h$ , the spaces of the mesh functions defined on the mesh  $\Omega_i^h \subset \Omega$  and  $\Omega_i^h \subset \Omega$  respectively ; and the operator  $L_i^h$  will map  $U_i^h$  into  $F_i^h$ .

If  $u \in U, f \in F$ , then  $u, f, Lu, l_i(u), r_i^h, w_i$  can be considered as a mesh function defined on the mesh  $\Omega_i^h$  (or  $\Omega_i^h$ ), and  $L_i^h u, L_i^h u - Lu$  is defined, and (3) will hold true on the mesh points  $x \in \bigcap_0^M \Omega_i^h$ .

We note that the combination method stated above seems appropriate for the parallel algorithm since the approximations  $u_i^h$ ,  $0 \leq i \leq M$ , are independent.

## 2. INTERPOLATION OF APPROXIMATE SOLUTIONS

We want to solve the problem

$$u(x) - \int_0^1 K(x, y) u(y) dy = f(x), \quad (4)$$

where the kernel  $K(x, y)$  and the given function  $f$  are assumed to be sufficiently smooth and 1 is not an eigenvalue of (4).

The integral operator

$$Ku = \int_0^1 K(x, y) u(y) dy$$

in (4) will be replaced by the rectangle cubature operator  $K_0^h$  defined by

$$K_0^h u = h \sum_0^{n-1} K\left(x, \left(i + \frac{1}{2}\right)h\right) u\left(\left(i + \frac{1}{2}\right)h\right)$$

and by the trapezoid cubature operator  $K_1^h$  defined by

$$K_1^h u = \frac{h}{2} \sum_0^{n-1} (K(x, ih) u(ih) + K(x, (i+1)h) u((i+1)h))$$

respectively, where  $h = 1/n$ . Correspondingly problem (4) will be replaced by the rectangle Nyström solution  $u_0^h$  defined by

$$u_0^h - K_0^h u_0^h = f$$

and by the trapezoid Nyström solution  $u_1^h$  defined by

$$u_1^h - K_1^h u_1^h = f$$

respectively.

We now explain how this problem can be embedded in the framework of Section 1. Set

$$\begin{aligned} U &= F = C[0, 1], & L &= I - K, \\ L_i^h &= I - K_i^h, & i &= 0, 1. \end{aligned}$$

From the collectively compact operator theory of Anselone [1],

$$\| (K_i^h - K) K_i^h \| \rightarrow 0$$

which will lead to assumption (A). Assumptions (B), (C) can be derived by Taylor expressions :

$$K_i^h u - Ku = h^2 l_i(u) + 0(h^4), \quad i = 0, 1$$

with

$$l_0(u) = -\frac{1}{24} \int_0^1 \frac{\partial^2}{\partial y^2} K(x, y) u(y) dy, \quad l_1(u) = \frac{1}{12} \int_0^1 \frac{\partial^2}{\partial y^2} K(x, y) u(y) dy.$$

And the combination condition (D) is satisfied by

$$l_1(u) + 2 l_0(u) = 0.$$

Hence, by (3),

$$u - \frac{1}{3}(u_1^h + 2 u_0^h) = 0(h^4) \tag{5}$$

on  $[0, 1]$ . Compare it with the approximations  $u_1^h, u_0^h$  which are convergent of order  $h^2$ .

The combination principle also applies for the Poisson equation

$$Lu = \sum_{i=1}^3 u_{x_i} = f \quad \text{in } \Omega \tag{6}$$

$$u = g \quad \text{on } \partial\Omega$$

with domain  $\Omega = (-1, 1)^3$  for simplicity and the solution  $u$  being sufficiently smooth.

The Laplacian operator  $L$  in (6) will be replaced by the 7-point difference operator  $L_7^h$  defined by

$$L_7^h u(x_1, x_2, x_3) = (u(x_1 + h, x_2, x_3) + u(x_1 - h, x_2, x_3) + u(x_1, x_2 + h, x_3) + u(x_1, x_2 - h, x_3) + u(x_1, x_2, x_3 + h) + u(x_1, x_2, x_3 - h) - 6u(x_1, x_2, x_3))/h^2$$

and by the 9-point difference operator  $L_9^h$  defined by

$$L_9^h u(x_1, x_2, x_3) = (\sum u(x_1 \pm h, x_2 \pm h, x_3 \pm h) - 8u(x_1, x_2, x_3))/4h^2, \\ \sum u(x_1 \pm h, x_2 \pm h, x_3 \pm h) = u(x_1 + h, x_2 + h, x_3 + h) + u(x_1 - h, x_2 + h, x_3 + h) +$$

$$+u(x_1+h, x_2-h, x_3+h) + u(x_1-h, x_2-h, x_3+h) + u(x_1+h, x_2+h, x_3-h) \\ + u(x_1-h, x_2+h, x_3-h) + u(x_1+h, x_2-h, x_3-h) + u(x_1-h, x_2-h, x_3-h).$$

It is easy to see that the truncation error will be

$$L_i^h u - Lu = h^2 l_i(u) + O(h^4), \quad i = 7, 9$$

with

$$l_7(u) = \frac{1}{12} \sum u_{x_i x_i x_i x_i},$$

$$l_9(u) = \frac{1}{12} \left( \sum u_{x_i x_i x_i x_i} + 6 \sum_{i < j} u_{x_i x_i x_j x_j} \right)$$

and

$$\frac{2}{3} l_7(u) + \frac{1}{3} l_9(u) = \frac{1}{12} L^2 u = L \left( \frac{1}{12} f \right). \quad (7)$$

Correspondingly the problem (6) will be replaced by the 7-point difference solution  $u_7^h$  defined by

$$L_7^h u_7^h = f \quad \text{in } \overset{\circ}{\Omega} \\ u_7^h = g - \frac{h^2}{12} f \quad \text{on } \partial\Omega_h \quad (8)$$

and by the 9-point difference solution  $u_9^h$  defined by

$$L_9^h u_9^h = f \quad \text{in } \overset{\circ}{\Omega}_h \\ u_9^h = g - \frac{h^2}{12} f \quad \text{on } \partial\Omega_h$$

with lattice domain

$$\overset{\circ}{\Omega}_h = \{ (x_1, x_2, x_3), x_i = m_i h, m_i = 0, \pm 1, \dots, \pm n-1, nh = 1 \}.$$

Then the combination principle will lead to

$$u - \frac{1}{3} (2 u_7^h + u_9^h) - \frac{h^2}{12} f = O(h^4) \quad \text{in } \overset{\circ}{\Omega}_h. \quad (10)$$

In fact letting  $w_i$  be the solution of an auxiliary problem

$$Lw_i = l_i(u) \quad \text{in } \Omega \\ w_i = \frac{1}{12} f \quad \text{on } \partial\Omega$$

for  $i = 7, 9$ , we have

$$\begin{aligned} L_i^h(u - u_i^h - h^2 w_i) &= L_i^h u - Lu - h^2 L_i^h w_i \\ &= h^2 l_i(u) - h^2 L w_i + h^2(L w_i - L_i^h w_i) + O(h^4) = O(h^4) \text{ in } \overset{\circ}{\Omega}_h \\ u - u_i^h - h^2 w_i &= o \quad \text{on } \partial\Omega_h \end{aligned}$$

and, by the maximum principle,

$$u - u_i^h - h^2 w_i = O(h^4) \text{ in } \overset{\circ}{\Omega}_h.$$

Hence

$$u - \frac{1}{3}(2u_7^h + u_9^h) - \frac{h^2}{3}(2w_7 + w_9) = O(h^4) \text{ in } \overset{\circ}{\Omega}_h$$

where  $w = \frac{1}{3}(2w_7 + w_9)$  satisfies, by (7),

$$\begin{aligned} Lw &= \frac{2}{3}l_7(u) + \frac{1}{3}l_9(u) = L\left(\frac{1}{12}f\right) \text{ in } \Omega \\ w &= \frac{1}{12}f \quad \text{on } \partial\Omega \end{aligned}$$

i.e.  $w = \frac{1}{12}f$  and (10) is proved.

*Remark* : Bramble has proposed a 19-point difference operator  $L_{19}^h$  and a difference solution  $u_{19}^h$  defined by

$$\begin{aligned} L_{19}^h u_{19}^h &= f + \frac{h^2}{12} \Delta f \text{ in } \overset{\circ}{\Omega} \\ u_{19}^h &= g \quad \text{on } \partial\Omega_h \end{aligned} \tag{11}$$

and proved that

$$u - u_{19}^h = O(h^4) \text{ in } \overset{\circ}{\Omega}_h.$$

It seems that (8) with (9) is easier to solve than the Bramble scheme (11) not only because of the 19-point operator  $L_{19}^h$  but also because of the  $\Delta f$  appearing in (11).

We would like to mention that the combination principle can also be used to deal with the eigenvalue problems and the mildly nonlinear elliptic problems by combining with the correction procedure (see Fox, Pereyra). We will discuss it in detail in a separate paper.

### 3. SPLITTING EXTRAPOLATION

The combination principle stated above is very similar to the classical extrapolation method due to Richardson (see Laurent).

The disadvantage of the extrapolation method is the computation of another approximation with a small parameter, say  $h/2$ , which involves computing once again an approximation of a much larger size than the one corresponding to the original  $h$  for multidimensional problems. To remove this imperfection, we present in the following a splitting extrapolation procedure which will save computational work and storage.

It is known (see Lin Qun and Lu Tao) that there exists an asymptotic expansion in powers of  $h = (h_1, \dots, h_s)$  for the numerical integral  $u(h)$  or the difference solution  $u(h)$  :

$$u(h) = u + \sum_{1 \leq |\alpha| \leq m} c_\alpha h^{2\alpha} + O(|h|^{2m+1}) \quad (12)$$

where

$$\alpha = (\alpha_1, \dots, \alpha_s), \quad |\alpha| = \alpha_1 + \dots + \alpha_s, \quad h^\alpha = h_1^{\alpha_1} \dots h_s^{\alpha_s}.$$

Let  $N_m$  be the number of the elements in the index set

$$S_m = \{ \alpha : 1 \leq |\alpha| \leq m \}.$$

The left side  $u(h)$  of (12) is known while we take  $u$  and  $c_\alpha$  ( $\alpha \in S_m$ ) as unknowns. Let us now make up the following  $N_m$  equations

$$u(h/2^\beta) = u + \sum_{1 \leq |\alpha| \leq m} c_\alpha h^{2\alpha} / 2^{2(\beta, \alpha)} \quad \forall \beta \in S_m$$

where

$$h/2^\beta = (h_1/2^{\beta_1}, \dots, h_s/2^{\beta_s}), \quad (\beta, \alpha) = \sum_{i=1}^s \beta_i \alpha_i.$$

Then we can solve  $u$  by using the general extrapolation algorithm presented by Brezinski. Note that the number of the points for numerical integral is  $\binom{m+s}{m} 2^m$ .

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