RAIRO. ANALYSE NUMÉRIQUE

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RAIRO. Analyse numérique, tome 16, nº 2 (1982), p. 161-191

<http://www.numdam.org/item?id=M2AN_1982__16_2_161_0>

R A I R O Analyse numerique/Numerical Analysis
(vol 16, n° 2, 1982, p 161 a 191)

FINITE ELEMENT SOLUTION OF QUASISTATIONARY NONLINEAR MAGNETIC FIELD (*)

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Communicated by P G CIARLET

Abstract — *The computation of quasistationary nonlinear two-dimensional magnetic field leads to the following problem There is given a bounded domain Ω and an open nonempty set $R \subset \Omega$ We are looking for the magnetic vector potential $u(x_1, x_2, t)$ which satisfies*

1) *a certain nonlinear parabolic equation and an initial condition in R,*
2) *a nonlinear elliptic equation in $S = \Omega - \bar{R}$ which is the stationary case of the above mentioned parabolic equation,*
3) *a boundary condition on $\partial\Omega$,*
4) *u as well as its conormal derivative are continuous accross the common boundary of R and S*

This problem is formulated in two equivalent abstract ways There is constructed an approximate solution completely discretized in space by a generalized Galerkin method (straight finite elements are a special case) and by backward A stable differentiation methods in time Existence and uniqueness of a weak solution is proved as well as a weak and strong convergence of the approximate solution to this solution There are also derived error bounds for the solution of the two-dimensional nonlinear magnetic field equations under the assumption that the exact solution is sufficiently smooth

Resume — *Le calcul d un champ magnetique quasi stationnaire non lineaire en dimension deux conduit au probleme suivant Etant donne un domaine borne Ω et un ensemble ouvert non vide $R \subset \Omega$ on cherche le potentiel vecteur magnetique $u(x_1\ x_2, t)$ qui satisfait*

1) *une certaine equation parabolique non lineaire et une condition initiale dans R,*
2) *une equation elliptique non lineaire dans $S = \Omega - \bar{R}$ qui est le cas stationnaire de l equation parabolique ci-dessus,*
3) *une condition aux limites sur $\partial\Omega$,*
4) *u de meme que sa derivee conormale sont continus a travers la frontiere commune a R et S*

Ce probleme est enonce de deux façons abstraites differentes On construit une solution approchee completement discretisee en espace par une methode de Galerkin generalisee (les elements finis droits sont un cas particulier) et par des methodes A-stables de derivation « arriere » en temps L existence et l unicite d une solution faible sont etablies ainsi que les convergences faible et forte de la solution approchee vers cette solution On obtient egalement des majorations d erreur pour la solution des equations du champ magnetique non lineaire a deux dimensions sous l hypothese de la solution exacte est suffisamment reguliere

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1. INTRODUCTION

In recent years attention has been paid in electrical engineering journals to the computation of quasistationary non-linear magnetic field. This problem occurs, e.g., in designing the magnet systems for fusion reactors and in rotating machinery. In two dimensions it can be formulated in the following model way. There is given a two-dimensional bounded domain Ω and an open nonempty set $R \subset \Omega$. We are looking for a function $u = u(x_1, x_2, t)$ (magnetic vector potential) such that

1)
$$\sigma \frac{\partial u}{\partial t} = \frac{\partial}{\partial x_1}\left(\nu \frac{\partial u}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\nu \frac{\partial u}{\partial x_2}\right) + J \quad \text{in } R, \tag{1.1}$$

$$u(x_1, x_2, 0) = u_0(x_1, x_2) \quad \text{in } R, \tag{1.2}$$

2)
$$0 = \frac{\partial}{\partial x_1}\left(\nu \frac{\partial u}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\nu \frac{\partial u}{\partial x_2}\right) + J \quad \text{in } S,\ S = \Omega - \overline{R}, \tag{1.3}$$

3) u satisfies a boundary condition on $\partial\Omega$,

4) u satisfies the conditions

$$[u]_R^S = \left[\nu \frac{\partial u}{\partial n}\right]_R^S = 0 \quad \text{on} \quad \Gamma = \partial R \cap \partial S. \tag{1.4}$$

Here the conductivity $\sigma = \sigma(x_1, x_2)$ is a positive function on R, the reluctivity $\nu = \nu(x_1, x_2, \| \operatorname{grad} u \|)$, $\| \operatorname{grad} u \|^2 = \left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2$, is a positive function on $\Omega x[0, \infty)$. $J = J(x_1, x_2, t)$ is a given current density, $u_0(x_1, x_2)$ is a given function defined on R and n is the normal oriented in a unique way.

The problem (1.1)-(1.4) can be easily formulated in a variational form. Let us, for simplicity, consider the Dirichlet boundary condition

$$u = 0 \quad \text{on} \quad \partial\Omega. \tag{1.5}$$

Multiply (1.1) and (1.3) by a function $v \in H_0^1(\Omega)$, integrate, use Green's formula and (1.4) and sum. We get

$$\left(\sigma \frac{\partial u}{\partial t}, v\right)_{L^2(R)} + a(u, v) = (J, v)_{L^2(\Omega)} \quad \forall v \in H_0^1(\Omega) \tag{1.6}$$

and

$$a(u, v) = \int_\Omega \sum_{i=1}^{2} \nu \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i}\, dx. \tag{1.7}$$

(1.6) is taken in Melkes + Zlámal [8] as the starting point for the construction of the approximate solution.

In this paper we give two equivalent abstract formulations of the above problem. One of them is a variational formulation generalizing the special case (1.6). Under certain conditions we prove existence and uniqueness of a weak solution. A problem to find a function satisfying a linear parabolic equation in a part of the given domain and a linear elliptic equation in the remaining part was already investigated by Ladyženskaja and Stupjalis [5].

The proof of existence has a constructive nature. We define a completely discretized approximate solution. The discretization in space is carried out by a generalized Galerkin method (the finite element method with straight elements is a special case). In time we use for the discretization the only two members of the backward differentiation schemes (see Lambert [6], p. 242) which are A-stable. Written for the equation $\dot{y} = f(t, y)$ these are

$$y^i - y^{i-1} = \Delta t f^i , \tag{1.8}$$

$$\frac{3}{2} y^i - 2 y^{i-1} + \frac{1}{2} y^{i-2} = \Delta t f^i . \tag{1.9}$$

The first, the Euler backward method, is of order one, the other of order two. A weak and strong convergence of the approximate solution U^δ (extended to the whole interval $[0, T]$) to the exact solution u is proved. In case of the problem (1.1)-(1.5) the result is that U_R^δ, the restriction of U^δ to R, converges strongly to u_R in $C([0, T]; L^2(R))$ and U^δ converges strongly to u in $L^2(0, T; H_0^1(\Omega))$. We also derive error estimates in case that the solution u is smooth.

2. SOME SPACES OF FUNCTIONS VALUED IN A BANACH SPACE

Let Ω be a nonempty open subset of R^N and $k = 0, 1, \ldots, 1 \leqq p \leqq \infty$. $H^{k,p}(\Omega)$ denotes the usual Sobolev space,

$$H^{k,p}(\Omega) = \{ v \in L^p(\Omega) ; D^\alpha v \in L^p(\Omega) \quad \forall \mid \alpha \mid \leqq k \} ,$$

provided with the norm

$$\| v \|_{k,p(\Omega)} = \sum_{|\alpha| \leqq k} \| D^\alpha v \|_{L^p(\Omega)} .$$

$H_0^{k,p}(\Omega)$ is the closure of $\mathscr{D}(\Omega)$ in the norm $\| \cdot \|_{H^{k,p}(\Omega)}$, $H^{-k,p}(\Omega) = [H_0^{k,p'}(\Omega)]'$ where $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, provided with the dual norm. If $p = 2$ we write briefly $H^k(\Omega)$, $H_0^k(\Omega)$ and $H^{-k}(\Omega)$, respectively.

Let X be a Banach space normed by $\| \cdot \|_X$ and let

$$0 < T < \infty .$$

For $p \geqq 1$ we denote by $L^p(0, T; X)$ the space of strongly measurable functions $f : (0, T) \to X$ such that

$$\| f \|_{L^p(0,T;X)} = \left[\int_0^T \| f(t) \|_X^p \, dt \right]^{1/p} < \infty$$

with the usual $p = \infty$ modification. By $C([0, T]; X)$ we denote the space of continuous functions $f : [0, T] \to X$ normed by

$$\| f \|_{C([0,T];X)} = \max_{t \in [0,T]} \| f(t) \|_X .$$

If $u \in L^1(0, T; X)$ we denote by u' the weak or generalized derivative of u (see Temam [13], lemma 1.1, p. 250).

Let H be a Hilbert space with a scalar product $(.,.)$ and V a reflexive Banach space, dense and continuously imbedded in H. We identify H with its dual space. Then H can be identified with a subspace of V' so that $V \subset H \subset V'$. Here each space is dense in the following one and the injections are continuous. The following lemma will be needed in the sequel.

LEMMA 1 : *Let W be the Banach space*

$$W = \{ v \mid v \in L^p(0, T; V); v' \in L^{p'}(0, T; V') \}, \quad 1 < p < \infty ,$$

normed by $\| v \|_W = \| v \|_{L^p(0,T;V)} + \| v' \|_{L^{p'}(0,T,V')}$. Then $W \subset C([0, T]; H)$ and the imbedding is continuous. Furthermore, for any u, $v \in W$ it holds the formula of integration by parts

$$\int_0^t \{ \langle u', v \rangle_V + \langle v', u \rangle_V \} \, d\tau = (u(t), v(t)) - (u(0), v(0)) , \quad 0 \leqq t \leqq T . \tag{2.1}$$

The lemma is true even in a somewhat more general form and the proof can be found in Gajewski, Gröger and Zacharias [4], p. 147.

3. THEOREM ON EXISTENCE, UNIQUENESS AND CONVERGENCE

To formulate the problem (1.1)-(1.5) in a general way we introduce several notations and hypotheses.

1) Let H_M, $M = R, S$ be two (real) Hilbert spaces with scalar products $(.,.)_M$ (the induced norms are denoted by $|\,.\,|_M$) and let the Hilbert space $H = H_R \times H_S$ (with elements $[v_R, v_S]$, $v_R \in H_R$, $v_S \in H_S$) have the scalar product $(.,.)$ such that the norm $|v| = (v, v)^{1/2}$ satisfies

$$c^{-1} |v| \leqq |v_R|_R + |v_S|_S \leqq c|v| \quad \forall v \in H \tag{3.1}$$

(c here and in the sequel denotes a positive constant not necessarily the same at any two places). Further, let $V \subset H$ be a separable reflexive Banach space normed by $\|\,.\,\|$. Finally, the vector spaces $V_M = \{\omega \mid \omega = v_M, v \in V\}$ ($M = R, S$) and $\mathring{V}_R = \{\omega \mid \omega = v_R, v \in V, v_S = 0\}$ should posses the following properties : V_M are subspaces of reflexive Banach spaces $B_M \subset H_M$ normed by $\|\,.\,\|_M$, it holds

$$c^{-1} \|v\| \leqq \|v_R\|_R + \|v_S\|_S \leqq c \|v\| \quad \forall v \in V, \tag{3.2}$$

$\overline{V}_R$, the closure of V_R in B_R, is continuously imbedded in H_R, i.e.

$$|\omega|_R \leqq c \|\omega\|_R \quad \forall \omega \in \overline{V}_R, \tag{3.3}$$

and $\mathring{V}_R$ is dense in H_R.

Example : Let Ω, R and S be domains from section 1 with Lipschitz boundaries. We choose $H_M = L^2(M)$, $(u, v)_R = (\sigma u, v)_{L^2(R)}$ where $\sigma \in L^\infty(R)$, $\sigma \geqq \sigma_0 > 0$, $(u, v)_S = (u, v)_{L^2(S)}$, $H = L^2(\Omega)$ (u_M is the restriction of u to M),

$$V = H_0^1(\Omega), \quad \mathring{V}_R = H_0^1(R), \quad \overline{V}_R = \{\omega \mid \omega \in H^1(R), \omega|_{\partial\Omega \cap \partial R} = 0\},$$

$$B_R = H^1(R), \quad \|\,.\,\|_R = \|\,.\,\|_{H^1(R)}, \quad \overline{V}_S = \{\omega \mid \omega \in H^1(S), \omega|_{\partial\Omega \cap \partial S} = 0\},$$

$$B_S = H^1(S), \quad \|\,.\,\|_S = \|\,.\,\|_{H^1(S)}.$$

Remark 1 : We set $H = H_R$ if $H_S = \phi$. The assumption 1) is to be understood as follows : There is a separable reflexive Banach space V normed by $\|\,.\,\|$ which is dense and continuously imbedded in H.

Remark 2 : It is easy to see that $\mathring{V}_R$ is a closed subspace of B_R. Further, $\mathring{V}_R$, $\overline{V}_R$ and $\overline{V}_S$, being closed subspaces of reflexive Banach spaces B_R and B_S, respectively, are reflexive Banach spaces, and $\overline{V}_R$ is dense in H_R because $\overset{\circ}{V}_R \subset V_R$.

We identify H_R with its dual by means of its scalar product $(.,.)_R$. Then H_R can be identified with subspaces of $\overline{V}'_R$ and $\overset{\circ}{V}'_R$ and we have inclusions

$$\overline{V}_R \subset H_R \subset \overline{V}'_R, \qquad \overset{\circ}{V}_R \subset H_R \subset \overset{\circ}{V}'_R \tag{3.4}$$

where each space is dense in the following one and the injections are continuous. Furthermore, the scalar product $\langle .,. \rangle_R$ in the duality between $\overline{V}'_R$ and $\overline{V}_R$ is an extension of $(.,.)_R$, i.e.

$$\langle u, v \rangle_R = (u, v)_R \quad \text{if} \quad u \in H_R, \quad v \in \overline{V}_R.$$

We denote the scalar product between V' and V by

$$\langle .,. \rangle$$

and between V'_S and V_S by

$$\langle .,. \rangle_S.$$

Let $A^M(u)$, $M = R, S$, be two, in general, nonlinear operators from $\overline{V}_M$ to $\overline{V}'_M$ with the following properties :

2) $A^M(u)$ are hemicontinuous, i.e. $\lambda \to \langle A^M(u + \lambda v), w \rangle_M$ are continuous functions on the interval $(-\infty, \infty)$ $\forall u, v, w \in \overline{V}_M$.

3) It holds

$$\| A^M(u) \|_* \leqq c \| u \|_M^{p-1} \quad \forall u \in \overline{V}_M \tag{3.5}$$

where

$$1 < p < \infty.$$

4) $A^M(u)$ are monotone, i.e.

$$\langle A^M(u) - A^M(v), u - v \rangle_M \geqq 0 \quad \forall u, v \in \overline{V}_M \tag{3.6}$$

and $A^s(u)$ is strictly monotone in the following sense :

$$\langle A^S(u) - A^S(v), u - v \rangle_S > 0 \quad \forall u, v \in \overline{V}_S, \quad u \neq v, \quad u - v \in \overset{\circ}{V}_S \tag{3.7}$$

where $\overset{\circ}{V}_S = \{ \omega \mid \omega = v_S, v \in V, v_R = 0 \}$.

The first of the above mentioned formulations is the following :

Problem P : Given

$$f^M \in L^{p'}(0, T; \overline{V}'_M); \quad M = R, S, \quad \text{and} \quad u_0 \in H_R \tag{3.8}$$

find $u \in W_R = \{ u \mid u \in L^p(0, T; V); u'_R \in L^{p'}(0, T; \overline{V}'_R) \}$ such that

$$\frac{du_R}{dt} + A^R(u_R) = f^R, \quad u(0)_R = u_0, \tag{3.9}$$

$$A^S(u_S) = f^S. \tag{3.10}$$

Remark 3 : If $H = H_R$ then we denote $A^R(u)$ by $A(u)$ and the assumptions 2, 3, 4, are to be understood as follows : $A(u)$ is hemicontinuous, monotone and bounded, i.e. $\| A(u) \|_* \leqq c \| u \|^{p-1}$. The formulation of the problem P reads : Given $f \in L^{p'}(0, T; V')$ and $u_0 \in H$ find

$$u \in W = \{ u \mid u \in L^p(0, T; V); u' \in L^{p'}(0, T; V') \}$$

such that

$$\frac{du}{dt} + A(u) = f, \qquad u(0) = u_0.$$

Remark 4 : We could leave the requirement $u'_R \in L^{p'}(0, T; \overline{V}'_R)$ because due to (3.9) it is automaticaly satisfied. From $u \in W_R$ it follows

$$u_R \in \{ \omega \mid \omega \in L^p(0, T; \overline{V}_R); \omega' \in L^{p'}(0, T; \overline{V}'_R) \}.$$

By lemma 1 $u_R \in C([0, T]; H_R)$ and the initial condition $u(0)_R = u_0$ makes sense.

We introduce an equivalent variational formulation of problem P. To this end we define a form $a(u, v)$ on $V \times V$ which is linear in v and, in general, nonlinear in u and a functional f from $L^{p'}(0, T; V')$:

$$a(u, v) = \langle A^R(u_R), v_R \rangle_R + \langle A^S(u_S), v_S \rangle_S \quad \forall u, v \in V, \tag{3.11}$$

$$\langle f, v \rangle = \langle f^R, v_R \rangle_R + \langle f^S, v_S \rangle_S \quad \forall v \in V. \tag{3.12}$$

The form $a(u, v)$ possesses the following properties :

a) it is hemicontinuous on $V \times V$, i.e. $\lambda \to a(u + \lambda v, w)$ is a continuous function on the interval $(-\infty, \infty)$ $\forall u, v, w \in V$.

b)

$$| a(u, v) | \leqq c \| u \|^{p-1} \| v \| \quad \forall u, v \in V, \tag{3.13}$$

c) $a(u, v)$ is monotone on $V \times V$, i.e.

$$a(u, u - v) - a(v, u - v) \geqq 0 \quad \forall u, v \in V. \tag{3.14}$$

At this place we add the last assumption which we shall later need :

5)

$$a(v, v) \geqq \alpha \| v \|^p \quad \text{or} \quad a(v, v) \geqq \alpha [v]^p \quad \forall v \in V , \quad \alpha = \text{const.} > 0 . \quad (3.15)$$

Here $[.]$ is a seminorm on V such that

$$[v] + \lambda | v_R |_R \geqq \beta \| v \| \quad \forall v \in V , \quad \lambda, \beta = \text{const.} > 0 . \quad (3.16)$$

Problem P' : Given $f^M \in L^{p'}(0, T ; \overline{V}'_M)$, $M = R, S$, and $u_0 \in H_R$ find $u \in W_R$ such that

$$\frac{d}{dt}(u_R, z_R)_R + a(u, z) = \langle f, z \rangle \quad \text{in} \quad \mathscr{D}'((0, T)) \quad \forall z \in V , \quad (3.17)$$

$$u(0)_R = u_0 . \quad (3.18)$$

Here $a(u, v)$ and f are defined by (3.11) and (3.12), respectively.

Remark 5 : If $H = H_R$ then the problem P' reads : Given $f \in L^{p'}(0, T ; V')$ and $u_0 \in H$ find $u \in W$ such that in $\mathscr{D}'((0, T))$

$$\frac{d}{dt}(u, z) + a(u, z) = \langle f, z \rangle \quad \forall z \in V , \quad u(0) = u_0 .$$

THEOREM 1 : *Let the assumptions* 1) *and* 3) *be satisfied. Then the problems P and P' are equivalent.*

Proof : If u is a solution of problem P then (3.9), (3.10), (3.11) and (3.12) imply

$$\left\langle \frac{du_R}{dt}, z_R \right\rangle_R + a(u, z) = \langle f, z \rangle \quad \forall z \in V . \quad (3.19)$$

All terms in (3.19) belong to $L^{p'}(0, T)$ and for $h(t) \in \mathscr{D}((0, T))$ we have

$$\int_0^T \left\langle \frac{du_R}{dt}, z_R \right\rangle_R h \, dt = - \int_0^T (u_R, z_R)_R \, h' \, dt$$

by (2.1) as $z_R h' \in L^{p'}(0, T ; \overline{V}'_R)$. Therefore, it holds (3.17).

Let u be a solution of problem P'. Choose $z = [\omega, 0]$, $\omega \in \overset{\circ}{V}_R$ in (3.17). Then by (3.11) and (3.12)

$$\frac{d}{dt}(u_R, \omega)_R = \langle f^R - A^R(u_R), \omega \rangle_R \quad \text{in} \quad \mathscr{D}'((0, T)) \quad \forall \omega \in \overset{\circ}{V}_R .$$

The function $G(t)=(u(t)_R, \omega)_R$ is continuous on $[0, T]$ because $u_R \in C([0, T]; H_R)$ and the function $g(t) = \langle f^R - A^R(u_R), \omega \rangle_R$ belongs to $L^{p'}(0, T)$ (due to $f^R, A^R(u_R) \in L^{p'}(0, T; \overline{V}'_R)$). Hence,

$$F(t) = \int_0^t g(\tau)\, d\tau$$

is an absolutely continuous function on $[0, T]$, consequently $F' = g$ a.e. and the distributional derivative of $G - F$ is equal to zero (due to the above equation). Thus $G(t) = c_0 + \int_0^t g(\tau)\, d\tau$ and evidently $c_0 = G(0) = (u_0, \omega)_R$. We have proved that

$$(u(t)_R, \omega)_R = (u_0, \omega)_R + \left\langle \int_0^t [f^R - A^R(u_R)], \omega \right\rangle_R d\tau \quad \forall \omega \in \mathring{V}_R .$$

As $u(t)_R \in H_R\ \forall t \in [0, T]$, $u_0 \in H_R$ and $\mathring{V}_R$ is dense and continuously imbedded in H_R it follows

$$u(t)_R = u_0 + \int_0^t [f^R - A^R(u_R)]\, d\tau$$

taken as elements of H_R.

Further, $f^R - A^R(u_R) \in \overline{V}'_R$ and H_R is dense and continuously imbedded in $\overline{V}'_R$. Hence

$$u(t)_R = u_0 + \int_0^t [f^R - A^R(u_R)]\, d\tau \quad \text{taken as elements of } \overline{V}'_R ,$$

and by i) of lemma by Temam [13] (p. 250) it follows (3.9). Finally, as

$$d/dt(u_R, z_R)_R = \langle u'_R, z_R \rangle_R ,$$

(3.17), (3.11) and (3.12) imply (3.10).

Now we define a completely discretized approximate solution of problem P'. The discretization in space is carried out by means of a generalized Galerkin method (see Nečas [9], p. 47), in time we use the schemes (1.8) and (1.9). They are written in a common form

$$\sum_{j=0}^{k} \alpha_{k-j}\, y^{i-j} = \Delta t f^i \tag{3.20}$$

where

$$\left.\begin{array}{llll} \alpha_1 = 1\,, & \alpha_0 = -1 & & \text{if } k = 1 \\ \alpha_2 = \frac{3}{2}, & \alpha_1 = -2\,, & \alpha_0 = \frac{1}{2} & \text{if } k = 2 \end{array}\right\} . \tag{3.21}$$

We assume that there exists a family $\{ V^h \}$, $h \in (0, h^*)$, $h^* > 0$, of finite dimensional subspaces of V, such that

$$\lim_{h \to 0_+} \operatorname{dist}(V^h, v) = 0 \quad \forall v \in V\,. \tag{3.22}$$

We have three important remarks :

1) If a family $\{ V^{h_n} \}$, $n = 1, 2, \ldots, h_1 > h_2 > \cdots$, $\lim\limits_{n \to \infty} h_n = 0$, with $\lim\limits_{n \to \infty} \operatorname{dist}(V^{h_n}, v) = 0$ $\forall v \in V$ exists, then defining $V^h = V^{h_n}$ for $h \in (h_{n+1}, h_n]$ we have a family with the above property.

2) A family V^h with the property (3.22) always exists under the assumption that V is a separable Banach space. In this case there exists a sequence $\{ \varphi_i \}_{i=1}^{\infty}$, $\varphi_i \in V$, such that for all $n = 1, 2, \ldots$ the elements $\varphi_1, \varphi_2, \ldots, \varphi_n$ are linearly independent and the finite linear combinations of φ_i's are dense in V. We take for V^{h_n}, $h_n = 1/n$, the space of all linear combinations of $\varphi_1, \varphi_2, \ldots, \varphi_n$.

3) In case that V is a Hilbert space, $H_0^1(\Omega) \subset V \subset H^1(\Omega)$, and Ω is a polyhedron, all in practice used finite element spaces have the property (3.22). We consider the boundary value problem : find $z \in V$ such that

$$a_0(z, \varphi) = a_0(v, \varphi) \quad \forall \varphi \in V$$

where

$$a_0(u, \varphi) = \int_\Omega \left[\sum_{i=1}^{N} \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_i} + u\varphi \right] dx$$

and v is a given element of V (of course, $z = v$). If v_h is the finite element approximate solution and the finite element spaces satisfy certain requirements then $\lim\limits_{h \to 0^+} \| v - v_h \|_{H^1(\Omega)} = 0$ (see Ciarlet [2], theorem 3.23, p. 134); h is the maximum diameter of all elements.

We introduce $\Delta t = T/r$, r being a natural number and consider the partition of the interval $[0, T]$ with nodes

$$t_i = i\,\Delta t\,, \quad i = 0, \ldots, r\,.$$

We set

$$f^i = \frac{1}{\Delta t}\int_{t_{i-1}}^{t_i} f(\tau)\,d\tau \in V', \quad i = 1, \dots, r \tag{3.23}$$

and define $U^i \in V^h$, $i = 1, \dots, r$ by

$$\left(\sum_{j=0}^{k} \alpha_{k-j}\, U_R^{i-j}, z_R\right)_R + \Delta t a(U^i, z) = \Delta t \langle f^i, z \rangle \quad \forall z \in V^h, \tag{3.24}$$

$$U_R^{-1} = U_R^0 = u_0 .$$

Remark 6 : Instead of u_0 we can take any approximation u_0^h of u_0 such that $| u_0 - u_0^h |_R \to 0$.

We shall later prove that for each i (3.24) is equivalent to a nonlinear system $\mathbf{F}(\boldsymbol{\alpha}) = \mathbf{0}$. Here $\mathbf{F} : R^{d_h} \to R^{d_h}$ (where d_h is the dimension of V^h) is continuous, coercive and strictly monotone from which existence and uniqueness of U^i follows (see Ortega and Rheinbold [10], 6.4.2, 6.4.3). We extend the approximate solution on the interval (0, T]. The extended approximate solution U^δ, $\delta = (h, \Delta t)$, is the step function

$$U^\delta = U^i \text{ in } (t_{i-1}, t_i]; \quad i = 1, \dots, r, \quad \delta = (h, \Delta t) \tag{3.25}$$

THEOREM 2 : *Let the assumptions* 1)-5) *be fulfilled, let* $f^M \in L^{p'}(0, T; V'_M)$, $M = R, S$, $1 < p < \infty$, $1/p + 1/p' = 1$, *and* $u_0 \in H_R$. *Then there exists a unique function* $u \in W_R = \{ u \mid u \in L^p(0, T; V);\ u'_R \in L^{p'}(0, T; \overline{V'_R}) \}$ *satisfying* (3.17) *and* (3.18). *Further, the approximate solution* U^δ *defined by* (3.24) *and* (3.25) *exists, is unique and*

$$U^\delta \to u \quad \textit{in} \quad L^p(0, T; V) \quad \textit{weakly if} \quad \delta \to 0 . \tag{3.26}$$

If $u \in C([0, T]; V)$ and the form $a(u, v)$ is uniformly monotone, i.e.

$$a(u, u - v) - a(v, u - v) \geqq \rho(\| u - v \|) \quad \forall u, v \in V \tag{3.27}$$

where $\boldsymbol{\rho}$ is a strictly increasing function on the interval $[0, \infty)$ with $\boldsymbol{\rho}(0) = 0$, then

$$\lim_{\delta \to 0} \| u_R - U_R^\delta \|_{C([0,T];H_R)} = 0, \quad \lim_{\delta \to 0} \int_0^T \rho(\| u - U^\delta \|)\, dt = 0 . \tag{3.28}$$

Remark 7 : If $H = H_R$ then the assumptions 1)-5) are the same as those of theorem 1.2 and 1.2bis in Lions [7], p. 162-163.

Proof of uniqueness : Let u^1, $u^2 \in W_R$ satisfy (3.17) and (3.18). Then they satisfy (3.9) and (3.10). From (3.9) we get

$$\left\langle \frac{d}{dt}(u_R^1 - u_R^2), u_R^1 - u_R^2 \right\rangle_R + \langle A^R(u_R^1) - A^R(u_R^2), u_R^1 - u_R^2 \rangle_R = 0$$

and $u^1(0)_R - u^2(0)_R = 0$. Integrating in $(0, t)$ we obtain by means of (2.1) and (3.6) $| u^1(t)_R - u^2(t)_R |_R^2 \leqq 0$. Hence $u_R^1 = u_R^2$. It follows $u_S^1 - u_S^2 \in \mathring{V}_S$. From (3.10) we get

$$\langle A^S(u_S^1) - A^S(u_S^2), u_S^1 - u_S^2 \rangle_S = 0$$

and by (3.7) $u_S^1 = u_s^2$, i.e. $u^1 = u^2$.

Proof of existence and of convergence : It will be carried out under the assumption $a(v, v) \geqq \alpha[v]^p$. The case $a(v, v) \geqq \alpha \| v \|^p$ (see (3.15)) can be treated similarly. We are using the compactness method (see Raviart [11], [12], Lions [7] and the references given there).

a) First, we consider U^δ defined by means of the scheme (1.8). In this case U^i is defined by

$$(U_R^i - U_R^{i-1}, z_R)_R + \Delta t a(U^i, z) = \Delta t \langle f^i, z \rangle \quad \forall z \in V^h, \tag{3.29}$$
$$U_R^0 = u_0 .$$

Let $\{ \varphi^j \}_{j=1}^d$ be the basis of V_h (for the sake of simple notation we write φ^j and d instead of $\varphi^{h,j}$ and d^h). Setting $U^i = \sum_{j=1}^{d} \alpha_j \varphi^j$, denoting

$$g_p = (U_R^{i-1}, \varphi_R^p)_R + \Delta t \langle f^i, \varphi^p \rangle, \quad \boldsymbol{g} = (g_1, \dots, g_d)^T$$

(T written as a superscript means transposition of a vector),

$$\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d)^T, F_p(\boldsymbol{\alpha}) = \left(\sum_{j=1}^{d} \alpha_j \varphi_R^j, \varphi_R^p \right)_R + $$
$$+ \Delta t a\left(\sum_{j=1}^{d} \alpha_j \varphi^j, \varphi^p \right), \quad \mathbf{F}(\boldsymbol{\alpha}) = (F_1, \dots, F_d)^T$$

we see that (3.29) is equivalent to the nonlinear system of d equations

$$\mathbf{F}(\boldsymbol{\alpha}) = \mathbf{g} . \tag{3.30}$$

The mapping **F** is continuous because of hemicontinuity of $a(u, v)$. It is strictly monotone, i.e.

$$(\boldsymbol{\alpha} - \boldsymbol{\beta})^T (\mathbf{F}(\boldsymbol{\alpha}) - \mathbf{F}(\boldsymbol{\beta})) > 0 \quad \forall \boldsymbol{\alpha}, \boldsymbol{\beta} \in R^d, \quad \boldsymbol{\alpha} \neq \boldsymbol{\beta}. \tag{3.31}$$

If $v = \sum_{j=1}^{d} \beta_j \varphi^j$ then the left-hand side of (3.31) is namely equal to

$$| u_R - v_R |^2 + \Delta t [a(u, u - v) - a(v, u - v)] .$$

Either is $u_R \neq v_R$, then (3.31) is true. Or $u_R = v_R$. Then the left-hand side of (3.31) is equal to $\Delta t [a^s(u_s, u_s - v_s) - a^s(v^s, u_s - v_s)]$. As u_s must be different from v_s and (due to $u_R = v_R$) $u_S - v_s \in \overset{\circ}{V}_s$ the inequality (3.31) follows from (3.7).

Finally, we show that the mapping $\mathbf{F}(\boldsymbol{\alpha})$ is coercive, i.e.

$$\lim_{\|\boldsymbol{\alpha}\| \to \infty} \frac{\boldsymbol{\alpha}^T \mathbf{F}(\boldsymbol{\alpha})}{\| \boldsymbol{\alpha} \|} = + \infty \tag{3.32}$$

where $\| \boldsymbol{\alpha} \|$ denotes for the moment the Euclidean norm of $\boldsymbol{\alpha}$. Because R^d is a finite dimensional space and $\{ \varphi^j \}_{j=1}^d$ are linearly independent $\| \boldsymbol{\alpha} \|$ is equivalent to $\| U \| = \left\| \sum_{j=1}^{d} \alpha_j \varphi^j \right\|$.

To prove (3.32) we estimate

$$\frac{\boldsymbol{\alpha}^T \mathbf{F}(\boldsymbol{\alpha})}{\| \boldsymbol{\alpha} \|} \geqq c(h) \frac{| U_R |_R^2 + \Delta t a(U, U)}{\| U \|} \geqq c(h) \frac{| U_R |_R^2 + \alpha \Delta t [U]^p}{\| U \|}$$

$$\geqq c(h, \Delta t) \frac{| U_R |_R^2 + [U]^p}{\| U \|} \geqq c(h, \Delta t) \frac{| U_R |_R^2 + [U]^p}{| U_R |_R + [U]}$$

(the last estimate is true due to (3.16)). We set $| U_R |_R = a, [U] = b, a + b = x$. If $\| U \| \to \infty$ then $x \to \infty$ so that it is sufficient to prove $\lim_{x \to \infty} \frac{a^2 + b^p}{x} = \infty$. Let first $p \geqq 2$. If $b \geqq 1$ then

$$\frac{a^2 + b^p}{x} \geqq \frac{a^2 + b^2}{x} \geqq \frac{1}{2} x .$$

If $b < 1$ then

$$\frac{a^2 + b^p}{x} \geqq \frac{(x - b)^2}{x} \geqq \frac{(x - 1)^2}{x} .$$

Let $1 < p < 2$. If $a \geqq 1$ then

$$\frac{a^2 + b^p}{x} \geqq \frac{a^p + (x-a)^p}{x} \geqq \frac{c(a + x - a)^p}{x} = c x^{p-1} ;$$

if $a < 1$ then

$$\frac{a^2 + (x-a)^p}{x} \geqq \frac{(x-1)^p}{x} \quad \text{for} \quad x \geqq 1 .$$

Evidently

$$\lim_{x \to \infty} \frac{a^2 + b^p}{x} = \infty .$$

b) We derive some estimates of U^i. We choose $z = U^i$ in (3.29) and sum. We get using (3.23)

$$| U_R^J |_R^2 + 2 \alpha \Delta t \sum_{i=1}^{J} [U^i]^p \leqq | U_R^J |_R^2 + 2 \Delta t \sum_{i=1}^{J} a(U^i, U^i)$$

$$\leqq 2 \sum_{i=1}^{J} \int_{t_{i-1}}^{t_i} \langle f, U^i \rangle \, dt + | u_0 |_R^2 .$$

As

$$\sum_{i=1}^{J} \int_{t_{i-1}}^{t_i} \langle f, U^i \rangle \, dt \leqq c \sum_{i=1}^{J} \int_{t_{i-1}}^{t_i} \| f \|_* \{ [U^i] + | U_R^i |_R \} \, dt$$

$$\leqq c \sum_{i=1}^{J} \varepsilon \Delta t^{1/p} [U^i] \frac{1}{\varepsilon} \left\{ \int_{t_{i-1}}^{t_i} \| f \|_*^{p'} \, dt \right\}^{1/p'} +$$

$$+ c \sum_{i=1}^{J} \Delta t^{1/p} | U_R^i |_R \left\{ \int_{t_{i-1}}^{t_i} \| f \|_*^p \, dt \right\}^{1/p'}$$

$$\leqq c \varepsilon^p \Delta t \sum_{i=1}^{J} [U^i]^p + \frac{c}{\varepsilon^p} \| f \|_{L^{p'}(0,T,V')}^p +$$

$$+ c \left\{ \Delta t \sum_{i=1}^{J} | U_R^i |_R^p \right\}^{1/p} \| f \|_{L^{p'}(0,T,V')}$$

$$\leqq \alpha \Delta t \sum_{i=1}^{J} [U^i]^p + c \left[\left\{ \Delta t \sum_{i=1}^{J} | U_R^i |_R^p \right\}^{2/p} + 1 \right]$$

is true if we choose $c\varepsilon^p = \alpha$ (the inequality $a.b \leqq a^p/p + b^{p'}/p'$, a, $b \geqq 0$, Hölder's inequality and the inequality $a \leqq (a^2 + 1)$ are used), it holds

$$| U_R^j |_R^2 + \alpha \, \Delta t \sum_{i=1}^{j} [U^i]^p \leqq c \left[\left\{ \Delta t \sum_{i=1}^{j} | U_R^i |_R^p \right\}^{2/p} + 1 \right], \qquad (3.33)$$

thus

$$| U_R^j |_R^p \leqq c \, \Delta t \sum_{i=1}^{j} | U_R^i |_R^p + c \,, \quad j = 1, ..., r \,. \qquad (3.34)$$

The discrete Gronwall inequality gives

$$| U_R^i |_R \leqq c \,, \quad i = 1, ..., r \qquad (3.35)$$

where, what we want to stress, c depends neither on h nor on Δt. From (3.33) it follows (due to $\| U \| \leqq c \{ [U] + | U_R |_R \}$)

$$\Delta t \sum_{i=1}^{r} \| U^i \|^p \leqq c \,. \qquad (3.36)$$

With respect to the definition (3.25) of U^δ the inequality (3.36) is equivalent to

$$\| U^\delta \|_{L^p(0,T;V)} \leqq c \,. \qquad (3.37)$$

From (3.35) it follows

$$\| U^\delta \|_{L_\infty(0,T;H_R)} \leqq c \,, \qquad (3.38)$$

$$| U^\delta(T)_R |_R \leqq c \,. \qquad (3.39)$$

c) Let h_n, $\Delta t^n > 0$ for $n = 1, 2, ...$ and

$$h_n \to 0 \,, \quad \Delta t^n \to 0 \quad \text{if} \quad n \to \infty$$

and consider the sequence $\{ U^{\delta_n} \}_{n=1}^{\infty}$ with $\delta_n = (h_n, \Delta t^n)$. For simplicity, we leave out the subscript n and write δ, h and Δt instead of δ_n, h_n and Δt^n. Then (3.37), (3.38), (3.39) and well known compactness theorems (see, e.g., Cea [1], p. 24, 26) imply : There exists a subsequence, denoted here again by U^δ, such that

$$U^\delta \to u \quad \text{in } L^p(0, T; V) \quad \text{weakly}, \qquad (3.40)$$

$$U_R^\delta \to \xi \quad \text{in } L^\infty(0, T; H_R) \quad \text{weakly*}, \qquad (3.41)$$

$$U^\delta(T)_R \to \xi \quad \text{in } H_R \quad \text{weakly}. \qquad (3.42)$$

It is easy to see that $\xi = u_R$. (3.41) means that

$$\int_0^T (\omega, U_R^\delta - \xi)_R \, dt \to 0 \quad \forall \omega \in L^1(0, T; H_R).$$

Choose $\omega \in L^{p'}(0, T; H_R)$. Then on one hand $\omega \in L^1(0, T; H_R)$, on the other hand $(\omega, v_R)_R \in L^{p'}(0, T; V')$. From (3.40) it follows

$$\int_0^T (\omega, U_R^\delta - u_R)_R \, dt \to 0 \quad \forall \omega \in L^{p'}(0, T; H_R),$$

consequently $\xi = u_R$ and

$$U_R^\delta \to u_R \quad \text{in } L^\infty(0, T; H_R) \quad \text{weakly*}. \tag{3.43}$$

In addition, $a(U^\delta, v) \in V'$ and if we denote it by $\langle \chi^\delta, v \rangle$ then

$$\| \chi^\delta \|_* \leqq c \, \| U^\delta \|^{p-1}.$$

Hence $\chi^\delta \in L^{p'}(0, T; V')$ and $\| \chi^\delta \|_{L_{p'}(0,T;V')} \leqq c$. Further, $a^M(U_M^\delta, \omega) \in \overline{V}'_M$ and denoting it by $\langle \chi^{M,\delta}, \omega \rangle_M$ we find $\| \chi^{M,\delta} \|_{L^{p'}(0,T;\overline{V}'_M)} \leqq c$. We can extract a subsequence of U^δ (denoted here again by U^δ) such that

$$\chi^\delta \to \chi \quad \text{in } L^{p'}(0, T; V') \text{ weakly},$$
$$\chi^{M,\delta} \to \chi^M \quad \text{in } L^{p'}(0, T; \overline{V}'_M) \text{ weakly}, \; M = R, S.$$

This means that (due to reflexivity of V and $\overline{V}_M$)

$$\left.\begin{aligned} &\int_0^T \langle \chi^\delta, v \rangle \, dt = \int_0^T a(U^\delta, v) \, dt \to \int_0^T \langle \chi, v \rangle \, dt \quad \forall v \in L^p(0, T; V) \\ &\int_0^T \langle \chi^{M,\delta}, \omega \rangle_M \, dt = \int_0^T a^M(U_M^\delta, \omega) \, dt \to \\ &\qquad \to \int_0^T \langle \chi^M, \omega \rangle_M \, dt \quad \forall \omega \in L^p(0, T; \overline{V}_M) \end{aligned}\right\} \tag{3.44}$$

and

$$\int_0^T \langle \chi, v \rangle \, dt = \int_0^T [\langle \chi^R, v_R \rangle_R + \langle \chi^S, v_S \rangle_S] \, dt \quad \forall v \in L^p(0, T; V). \tag{3.45}$$

d) Consider a function $h(t) \in C^\infty([0, T])$. Let

$$h^i = h(t_i), \quad i = 0, \ldots, r, \quad h^{r+1} = h^r = h(T)$$

and let us define two functions $h_{\Delta t}$, $\tilde{h}_{\Delta t}$ (see, e.g., Lions [7], p. 435-436) :

$$h_{\Delta t} = h^{i+1} \quad \text{in } (t_i, t_{i+1}], \quad i = 0, \ldots, r-1,$$

$$\tilde{h}_{\Delta t} = h^{i+1} + \frac{t - t_i}{\Delta t}(h^{i+2} - h^{i+1}) \quad \text{in } [t_i, t_{i+1}], \quad i = 0, \ldots, r-1. \tag{3.46}$$

We also define $f_{\Delta t} \in L^{p'}(0, T; V')$:

$$f_{\Delta t} = f^i \quad \text{in } (t_i, t_{i+1}], \quad i = 0, \ldots, r-1. \tag{3.47}$$

We set $z = h^i z^h$, $z^h \in V^h$, in (3.29) and sum. As

$$\sum_{i=1}^{r} (U_R^i - U_R^{i-1}, z_R^h)_R h^i = -\sum_{i=1}^{r-1} (U_R^i, z_R^h)_R (h^{i+1} - h^i) - (u^0, z_R^h)_R h^1 + \\ + (U^\delta(T)_R, z_R^h)_R h^r$$

we get (due to (3.25)) that

$$-\int_0^T (U_R^\delta, z_R^h)_R \tilde{h}'_{\Delta t}\, dt + \int_0^T a(U^\delta, z^h)\, h_{\Delta t}\, dt = \int_0^T \langle f_{\Delta t}, z^h \rangle h_{\Delta t}\, dt + \\ + (u_0, z_R^h)_R h(\Delta t) - (U^\delta(T)_R, z_R^h)_R h(T). \tag{3.48}$$

Now, let $z \in V$ be given. We choose $z^h \in V^h$ such that $\| z^h - z \| \to 0$ and we pass to the limit in (3.48). We get, which is easy to prove,

$$-\int_0^T (u_R, z_R)_R h'\, dt + \int_0^T \langle \chi, z \rangle h\, dt = \int_0^T \langle f, z \rangle h\, dt + \\ + (u_0, z_R)_R h(0) - (\zeta, z_R)_R h(T) \quad \forall h \in C^\infty([0, T]), \quad \forall z \in V. \tag{3.49}$$

Restricting h to $\mathscr{D}((0, T))$ we see that (3.49) gives

$$\frac{d}{dt}(u_R, z_R)_R + \langle \chi, z \rangle = \langle f, z \rangle \quad \text{in } \mathscr{D}'((0, T)) \quad \forall z \in V. \tag{3.50}$$

Now, let $\omega \in \mathring{V}_R$, $z_R = \omega$, $z_S = 0$ so that $z \in V$. (3.50) and (3.45) imply

$$\frac{d}{dt}(u_R, \omega)_R + \langle \chi^R, \omega \rangle_R = \langle f^R, \omega \rangle_R \quad \text{in } \mathscr{D}'((0, T)) \quad \forall \omega \in \mathring{V}_R.$$

Using the notation

$$G(t) = (u(t)_R, \omega)_R, \quad g(t) = \langle f^R(t) - \chi^R(t), \omega \rangle_R \tag{3.51}$$

we easily see that $G(t) \in L^p(0, T)$, $g(t) \in L^{p'}(0, T)$. The reasoning used in the proof of theorem 1 gives again $G(t) = c_0 + \int_0^T g(\tau)\, d\tau$. To determine c_0 we choose in (3.49) $h(t) \in C^\infty([0, T])$ with $h(0) = 1$, $h(T) = 0$ and obtain

$$-\int_0^T Gh'\, dt = \int_0^T gh\, dt + (u_0, \omega)_R .$$

Integrating by parts the left-hand side and taking into account that $G' = g$ a.e. in $(0, T)$ we come to $c_0 = (u_0, \omega)_R$. Therefore

$$(u(t)_R, \omega)_R = (u_0, \omega)_R + \left\langle \int_0^t [f^R - \chi^R]\, d\tau, \omega \right\rangle_R \quad \forall \omega \in \mathring{V}_R .$$

It follows as before

$$u(t)_R = u_0 + \int_0^T [f^R - \chi^R]\, d\tau \quad \text{taken as elements of } \overline{V}'_R ,$$

thus $u'_R \in \overline{V}'_R$,

$$u'_R + \chi^R = f^R , \tag{3.52}$$

$u \in W_R$, the initial condition makes sense and it is fulfilled. Further, (3.50), (3.45) and (3.52) imply

$$\chi^S = f^S . \tag{3.53}$$

From (3.52), (3.53), (3.45) and (2.1) we get

$$\int_0^T \langle \chi, u \rangle\, dt = \frac{1}{2} | u_0 |_R^2 - \frac{1}{2} | u(T)_R |_R^2 + \int_0^T \langle f, u \rangle\, dt . \tag{3.54}$$

Now we return to (3.49). Integrating the first term by parts and using (3.50) we obtain

$$-(u(T)_R, z_R)_R\, h(T) + (u_0, z_R)_R\, h(0) = (u_0, z_R)_R\, h(0) - (\zeta, z_R)_R\, h(T) .$$

Hence

$$\zeta = u(T)_R . \tag{3.55}$$

e) We prove the existence of a solution if we show that $\langle \chi, v \rangle = a(u, v)$. We use an argument from Lions [7], p. 160-161. From monotonicity of $a(u, v)$ it follows

$$X^{\delta} = \int_0^T [a(U^{\delta}, U^{\delta} - v) - a(v, U^{\delta} - v)]\, dt \geqq 0 \quad \forall v \in L^p(0, T; V). \tag{3.56}$$

Putting $z = U^{\iota}$ in (3.29) and summing one gets

$$\int_0^T a(U^{\delta}, U^{\delta})\, dt = \int_0^T \langle f_{\Delta t}, U^{\delta} \rangle\, dt + \frac{1}{2} | u_0 |_R^2 - \frac{1}{2} | U^{\delta}(T)_R |_R^2 - \\ - \frac{1}{2} \sum_{i=1}^{r} | U_R^i - U_R^{i-1} |_R^2 .$$

Hence

$$X^{\delta} \leqq \int_0^T \langle f_{\Delta t}, U^{\delta} \rangle\, dt + \frac{1}{2} | u_0 |_R^2 - \frac{1}{2} | U^{\delta}(T)_R |_R^2 - \int_0^T a(U^{\delta}, v)\, dt - \\ - \int_0^T a(v, U^{\delta} - v)\, dt ,$$

from which ($\limsup - | U^{\delta}(T)_R |_R^2 \leqq - | u(T)_R |_R^2$ due to (3.42) and (3.55)) and from (3.40), (3.44), (3.54)

$$\limsup X^{\delta} \leqq \int_0^T \langle \chi, u - v \rangle\, dt - \int_0^T a(v, u - v)\, dt .$$

Therefore

$$\int_0^T \langle \chi, u - v \rangle\, dt - \int_0^T a(v, u - v)\, dt \geqq 0 \quad \forall v \in L^p(0, T; V).$$

Consequently (see Lions [7], p. 161) $\langle \chi, v \rangle = a(u, v)$.

f) We have proved that if there exists a family $\{ V^h \}$ with the property (3.22) then a subsequence of $\{ U^{\delta} \}$ converges weakly in $L^p(0, T; V)$ and its limit u belongs to W_R and satisfies (3.17) and (3.18). From the proof and from uniqueness it is obvious that from any sequence $\{ U^{\delta_j} \}$ with $\delta_j \to 0$ we can extract a subsequence converging weakly to u. Therefore $U^{\delta} \to u$ if $\delta \to 0$ weakly in $L^p(0, T; V)$. From separability of V it follows that a family $\{ V^h \}$ with the property (3.22) exists. Therefore besides the uniqueness and existence we have proved (3.26) in case that U^δ is constructed by means of the scheme (1.8). We now prove (3.28) in a way similar to that used by Gajewski, Gröger and Zacharias [4] to prove the strong convergence of a semidiscrete Galerkin solution of a parabolic equation (p. 209-210).

g) Let

$$Z_R = \{ u \mid u \in C([0, T]; V); \quad u'_R \in L^{p'}(0, T; \overline{V}'_R) \},$$
$$\| u \|_{Z_R} = \| u \|_{C([0,T];V)} + \| u'_R \|_{L^{p'}(0,T;\overline{V}'_R)} .$$

We shall make use of the following

LEMMA 2 : *If the family* $\{ V^h \}$ *has the property* (3.22) *and* $h_n \to 0$ *then*

$$\bigcup_{n=1}^{\infty} C^1([0, T]; V^{h_n})$$

is dense in Z_R.

Proof : First we show that $C^\infty([0, T]; V)$, and consequently also

$$C^1([0, T]; V),$$

are dense in Z_R. Let $u \in Z_R$. We extend $u(t)$ to the interval $(-\infty, \infty)$ setting $\tilde{u}(t) = u(t)$ in $[0, T]$, $\tilde{u}(t) = u(0)$ for $t < 0$, $\tilde{u}(t) = u(T)$ for $t > T$. Then

$$\frac{d}{dt}\tilde{u}(t)_R = u'_R \quad \text{in } (0, T) \quad \text{and} \quad \frac{d}{dt}\tilde{u}(t)_R = 0$$

for $t < 0$ and $t > T$. Further, let $\tilde{u}^\varepsilon(t)$ be the mollifier of $\tilde{u}(t)$, i.e.

$$\tilde{u}^\varepsilon(t) = \varepsilon^{-1} \int_{-\infty}^{\infty} \tilde{u}(\tau)\, \rho\left(\frac{t-\tau}{\varepsilon}\right) d\tau$$

where

$$\rho \in C_0^\infty(R^1), \quad \rho \geqq 0, \quad \operatorname{supp} \rho = [-1, 1] \quad \text{and} \quad \int_{-1}^{1} \rho(y)\, dy = 1 .$$

Then

$$\| u - \tilde{u}^\varepsilon \|_{C([0,T];V)} = \left\| \int_{-1}^{1} [\tilde{u}(t) - \tilde{u}(t - \varepsilon y)]\, \rho(y)\, dy \right\|_{C([0,T];V)} \leqq$$

$$\leqq \max_{0 \leqq t \leqq T} \max_{-1 \leqq y \leqq 1} \| \tilde{u}(t) - \tilde{u}(t - \varepsilon y) \|_V \to 0 \quad \text{if} \quad \varepsilon \to 0^+$$

because $\tilde{u}(t)$ is uniformly continuous on $[-1, T+1]$. As

$$\frac{d}{dt}\tilde{u}_R^\varepsilon(t) = \int_{-1}^{1} \tilde{u}'(t-\varepsilon y)_R\, \rho(y)\, dy = \varepsilon^{-1}\int_{-\infty}^{\infty} \tilde{u}'(\tau)_R\, \rho\left(\frac{t-\tau}{\varepsilon}\right) d\tau$$

also $\left\| u_R' - \frac{d}{dt}\tilde{u}_R^\varepsilon \right\|_{L^{p'}(0,T;\bar{V}_R)} \to 0$, hence $\| u - \tilde{u}^\varepsilon \|_{Z_R} \to 0$.

Now

$$\| v \|_{Z_R} \leqq c \| v \|_{C^1([0,T];V)} \quad \forall v \in C^1([0, T]; V)$$

and $\bigcup_{n=1}^{\infty} C^1([0, T]; V^{h_n})$ is dense in $C^1([0, T]; V)$ (the proof is the same as the proof of Lemma 1.5 in Gajewski, Gröger and Zacharias [4], p. 209). Hence $\bigcup_{n=1}^{\infty} C^1([0, T]; V^{h_n})$ is dense in Z_R.

Let $v \in C^1([0, T]; V^h)$. One can prove that for v close to u, say $\| u-v \|_{Z_R} < 1$, it holds

$$Y^\delta \equiv \max_{1 \leqq j \leqq r} | u_R^j - U_R^j |_R^2 + \int_0^T \rho(\| u - U^\delta \|)\, dt \leqq$$

$$\leqq c\left\{ \| u - v \|_{Z_R} + \left[\sum_{i=1}^{r} \int_{t_{i-1}}^{t_i} \| u - u^i \|^p\, dt \right]^{1/p} \right\}. \quad (3.57)$$

We want now to show that

$$\lim_{\delta \to 0} Y^\delta = 0. \quad (3.58)$$

Assume that Y^δ does not converge to zero as $\delta \to 0$. Then there exists an $\varepsilon > 0$ and $\{ \delta_n \}_{n=1}^{\infty}$ with $\delta_n = (h_n, \Delta t^n) \to 0$ such that $Y^{\delta_n} \geqq \varepsilon$. From Lemma 2 it follows that there exists a sequence $\{ v^j \}_{j=1}^{\infty}$, $v^j \in V^{h_{n_j}}$ where $\{ h_{n_j} \}_{j=1}^{\infty}$ is a subsequence of $\{ h_n \}_{n=1}^{\infty}$ such that $\lim_{j\to\infty} \| u - v^j \|_{Z_R} = 0$ (we choose v^1 such that $\| u - v^1 \|_{Z_R} < 1$, $v^1 \in C^1([0, T]; V^{h_{n_1}})$; as $\{ h_n \}_{n > n_1}$ is a subsequence of $\{ h_n \}_{n=1}^{\infty}$ Lemma 2 implies that $\bigcup_{n > n_1} C^1([0, T]; V^{h_n})$ is dense in Z_R, hence there exists $v^2 \in C^1([0, T]; V^{h_{n_2}})$, $n_2 > n_1$, such that $\| u - v^2 \|_{Z_R} < 1/2$, etc.). Setting $v = v^j$ in (3.57) we get

$$Y^{\delta_{n_j}} \leqq o(1) + \left\{ \sum \int_{t_{i-1}}^{t_i} \| u - u^i \|^p\, dt \right\}^{1/p}.$$

The second term on the right-hand side converges also to zero because $u(t)$ is uniformly continuous on $[0, T]$ in the norm $\| \cdot \|$. This is in contradiction with our assumption.

To prove (3.28) we remark that

$$\max_{t \in [0,T]} | u_R - U_R^\delta |_R = | u(t^*)_R - U_R^s |_R , \quad t^* \in (t_{s-1}, t_s]$$

for some s, $1 \leqq s \leqq r$. Then

$$\max_{t \in [0,T]} | u_R - U_R^\delta | \leqq | u(t^*)_R - u_R^s |_R + \max_{1 \leqq \iota \leqq r} | u_R^\iota - U_R^\iota |_R \to 0$$

because u_R is uniformly continuous on $[0, T]$ in the norm $| \cdot |_R$.

h) It remains to prove (3.26) and (3.28) in case that U^ι is defined by

$$\left. \begin{array}{r} \left(\frac{3}{2} U_R^\iota - 2\, U_R^{\iota-1} + \frac{1}{2} U_R^{\iota-2}, z_R\right)_R + \Delta t a(U^\iota, z) = \Delta t \langle f^\iota, z \rangle \\ U_R^0 = U_R^{-1} = u_0 \end{array} \right\} \forall z \in V_h . \tag{3.59}$$

We briefly mention some changes in the proof.

Ad *b*) We set $z = U^\iota$ in (3.59). Because

$$\left(\frac{3}{2} U_R^\iota - 2\, U_R^{\iota-1} + \frac{1}{2} U_R^{\iota-2}, U_R^\iota\right)_R = \frac{5}{4} | U_R^\iota |_R^2 - | U_R^{\iota-1} |_R^2 - \frac{1}{4} | U_R^{\iota-2} |_R^2 -$$
$$- (U_R^\iota, U_R^{\iota-1})_R + (U_R^{\iota-1}, U_R^{\iota-2})_R + \frac{1}{4} | U_R^\iota - 2\, U_R^{\iota-1} + U_R^{\iota-2} |_R^2 \tag{3.60}$$

we get

$$\sum_{\iota=1}^{J} \left(\frac{3}{2} U_R^\iota - 2\, U_R^{\iota-1} + \frac{1}{2} U_R^{\iota-2}, U_R^\iota\right)_R \geqq \frac{5}{4} | U_R^J |_R^2 + \frac{1}{4} | U_R^{J-1} |_R^2 -$$
$$- (U_R^J, U_R^{J-1})_R - \frac{1}{2} | u_0 |_R^2 \tag{3.61}$$

(this inequality is a special case of the inequality (2.16) by Zlámal [14] which is true for any A-stable linear two-step method of the second order). From (3.61) we get (3.37), (3.38) and (3.39).

From (3.61) it also follows another inequality :

$$\sum_{\iota=1}^{r} \left(\frac{3}{2} U_R^\iota - 2\, U_R^{\iota-1} + \frac{1}{2} U_R^{\iota-2}, U_R^\iota\right)_R \geqq - \frac{1}{2} | u_0 |_R^2 + \frac{1}{2} \left| \frac{3}{2} U_R^r - \frac{1}{2} U_R^{r-1} \right|_R^2 . \tag{3.62}$$

(3.62) and (3.39) give easily $| U_R^{r-1} |_R \leqq C$. Hence, we can extract a subsequence of U^δ, still denoted by U^δ, such that it holds (3.40), (3.43), (3.42) and

$$U_R^{r-1} \to \eta \quad \text{in } H_R \text{ weakly}. \tag{3.63}$$

Ad *d*) We extend $\tilde{h}_{\Delta t}(t)$ setting $\tilde{h}_{\Delta t}(t_{r+1}) = h^r$ in $(t_r, t_{r+1}]$. As

$$\sum_{i=1}^{r} \left(\frac{3}{2} U_R^i - 2 U_R^{i-1} + \frac{1}{2} U_R^{i-2}, z_R^h\right)_R h^i = -\frac{3}{2} \sum_{i=1}^{r} (U_R^i, z_R^h)_R [\tilde{h}_{\Delta t}(t_i) - \tilde{h}_{\Delta t}(t_{i-1})]$$
$$+ \frac{1}{2} \sum_{i=1}^{r} (U_R^i, z_R^h)_R [\tilde{h}_{\Delta t}(t_{i+1}) - \tilde{h}_{\Delta t}(t_i)] + (U_R^{r-1}, z_R^h)_R \left(\frac{3}{2} h^{r-1} - 2 h^r\right)$$
$$+ \frac{3}{2} (U_R^r, z_R^h)_R h^r + \frac{1}{2} (u_0, z_R^h)_R (h^2 - 3 h^1)$$

it holds

$$-\frac{3}{2} \int_0^T (U_R^\delta, z_R^h)_R \tilde{h}'_{\Delta t}(t)\, dt + \frac{1}{2} \int_0^T (U_R^\delta, z_R^h)_R \tilde{h}'_{\Delta t}(t + \Delta t)\, dt +$$
$$+ \int_0^T a(U^\delta, z^h)\, h_{\Delta t}(t)\, dt = \int_0^T \langle f_{\Delta t}, z^h \rangle\, h_{\Delta t}(t)\, dt + \frac{1}{2} (u_0, z_R^h)_R (3 h^1 - h^2)$$
$$- (U_R^{r-1}, z_R^h)_R \left(\frac{3}{2} h^{r-1} - 2 h^r\right) - \frac{3}{2} (U_R^r, z_R^h)_R h^r .$$

Passing to the limit we obtain

$$-\int_0^T (u_R, z_R)_R h'\, dt + \int_0^T \langle \chi, z \rangle h\, dt = \int_0^T \langle f, z \rangle h\, dt +$$
$$+ (u_0, z_R)_R h(0) - \left(\frac{3}{2} \zeta - \frac{1}{2} \eta, z_R\right)_R h(T).$$

Instead of (3.55) we get

$$\frac{3}{2} \zeta - \frac{1}{2} \eta = u(T)_R . \tag{3.64}$$

Ad *e*) Setting $z = U^i$ in (3.59), summing and using (3.62) one derives

$$\int_0^T a(U^\delta, U^\delta)\, dt \leqq \int_0^T \langle f_{\Delta t}, U^\delta \rangle\, dt + \frac{1}{2} | u_0 |_R^2 - \frac{1}{2} \left| \frac{3}{2} U_R^r - \frac{1}{2} U_R^{r-1} \right|_R^2$$

and due to (3.64) again

$$\limsup X^\delta \leqq \int_0^T \langle \chi, u - v \rangle - \int_0^T a(v, u - v)\, dt \quad \forall v \in L^p(0, T; V).$$

Ad *g*) We use (3.61) and the inequality

$$\frac{1}{4} | \omega_R^j |_R^2 + \int_0^{t_j} [a(u, u - U^\delta) - a(U^\delta, u - U^\delta)]\, dt \leqq \int_0^{t_j} [a(u, u - U^\delta) - a(U^\delta, u - U^\delta)]\, dt + \frac{1}{2} | \omega_R^0 |_R^2 + \sum_{i=1}^{j} \left(\frac{3}{2} \omega_R^i - 2\, \omega_R^{i-1} + \frac{1}{2} \omega_R^{i-2}, \omega_R^i \right)_R .$$

4. THE TWO-DIMENSIONAL NONLINEAR MAGNETIC FIELD

We apply Theorem 2 to the problem (1.1)-(1.5). Let

$$\sigma \in L^\infty(R), \qquad \sigma \geqq \sigma_0 > 0 \tag{4.1}$$

and let $\partial\Omega$, ∂R be polygons. We choose the spaces H_R, H_S, etc. as in the example introduced at the beginning of section 3. Then the assumption 1) is satisfied. We consider a regular family of triangulations $\mathcal{T}_h$ (see Ciarlet [2], p. 132) covering Ω and satisfying the assumptions of theorem 3.2.3 from [2]. Then the family $\{ V^h \}$ satisfies the condition (3.22). The operators $A^M(u_M)$ (in the sequel the subsript $M = R, S$ means restriction to M and will be often left out) and the form $a(u, v)$ are :

$$A^M(u_M) = - \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left(\nu_M \frac{\partial u_M}{\partial x_i} \right), \quad a(u, v) = \int_\Omega \nu \sum_{i=1}^{2} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i}\, dx . \tag{4.2}$$

Concerning the function $\nu(x_1, x_2, \xi)$ we assume :

a) $\forall \xi \in [0, \infty)$ the function $(x_1, x_2) \to \nu(x_1, x_2, \xi)$ is measurable on Ω and for almost all $(x_1, x_2) \in \Omega$ the function $\xi \to \nu(x_1, x_2, \xi)$ is continuous in $[0, \infty)$ (Caratheodory's property);

b) $\forall \xi \in [0, \infty)$ and for almost all $(x_1, x_2) \in \Omega$, $\nu(x_1, x_2, \xi)$ is bounded from above and satisfies for almost all $(x_1, x_2) \in \Omega$

$$\xi \nu(x_1, x_2, \xi) - \eta \nu(x_1, x_2, \eta) \geqq \alpha(\xi - \eta) \quad \forall \xi \geqq \eta \geqq 0, \quad \alpha = \text{const.} > 0 . \tag{4.3}$$

Then the assumptions 2)-4) are satisfied with $p = 2$ (see Gajewski, Gröger and Zacharias [4], p. 68-71). (4.3) implies that $\nu(x_1, x_2, \xi) \geqq \alpha > 0$ for almost all $(x_1, x_2) \in \Omega$ and $\forall \xi \in [0, \infty)$. Therefore the assumption 5) is also satisfied with $p = 2$ and, in addition,

$$a(u, u - v) - a(v, u - v) \geqq \beta \| u - v \|^2_{H^1(\Omega)} \quad \forall u, v \in H^1_0(\Omega), \quad \beta > 0 \quad (4.4)$$

i.e. $a(u, v)$ is uniformly monotone with $\rho(\xi) = \beta\xi^2$. Concerning the data J and u_0 we require

$$J \in L^2(0, T; L^2(\Omega)), \quad u_0 \in L^2(R). \quad (4.5)$$

The equation (3.25) can be written as follows :

$$\left.\begin{array}{l} \left(\sigma \sum_{j=0}^{k} \alpha_{k-j} \ U^{i-j}, z\right)_{L^2(R)} + \Delta t a(U^i, z) = \Delta t(J^i, z)_{L^2(\Omega)} \quad \forall z \in V^h \\ U_R^{-1} = U_R^0 = u_0 \end{array}\right\} \quad (4.6)$$

where

$$J^i = \frac{1}{\Delta t} \int_{t_{i-1}}^{t_i} J(., t)\, dt .$$

THEOREM 3 : *Under the above introduced assumptions there exists a unique function $u \in W_R$ which is the solution of the problem* (1.1)-(1.5). *Further, the approximate solution U^δ, defined by* (4.6) *and* (3.25) *exists, is unique and*

$$U^\delta \to u \quad in\ L^2(0, T; H^1_0(\Omega)) \quad weakly\ if \quad \delta \to 0 . \quad (4.7)$$

If $u \in C([0, T]; H^1_0(\Omega))$ then

$$\lim_{\delta \to 0} \| u - U^\delta \|_{C([0,T];L^2(R))} = 0 , \quad \lim_{\delta \to 0} \| u - U^\delta \|_{L^2(0,T;H^1_0(\Omega))} = 0 . \quad (4.8)$$

We derive now error bounds under assumption that the solution u is smooth enough. We restrict ourselves to triangular elements and to piecewise linear trial functions which are mostly applied in practise even if the same approach gives error bounds for higher degree shape functions. We take into account only triangulations which consist of triangles belonging either to $\bar{R}$ or to $\bar{S}$ and which form a regular family.

In applications, the coefficient $\nu(x_1, x_2, \xi)$ is a piecewise continuous function of $x = (x_1, x_2)$. Every discontinuity in x along a boundary of a subdomain leads to a natural boundary condition of the form (1.4). We consider a model problem assumming ν to be continuous in R and in S for all $\xi \in [0, \infty)$ with discontinuity along $\Gamma = \partial R \cap \partial S$. We add two more assumptions :

$$| \xi \nu(x_1, x_2, \xi) - \eta \nu(x_1, x_2, \eta) | \leqq L | \xi - \eta | \quad \forall \xi, \eta \in [0, \infty), (x_1, x_2) \in R \cup S \tag{4.9}$$

$$J \in C([0, T]; L^2(\Omega)), \tag{4.10}$$

and investigate first the approximate solution constructed by means of the scheme (1.8). The righ-hand side of the defining equation will not be the same as in (4.6). U^i is now defined by

$$(\sigma \Delta U^i, z)_{L^2(R)} + \Delta t a(U^i, z) = \Delta t (J^i, z)_{L^2(\Omega)} \quad \forall z \in V^h \tag{4.11}$$

where $\Delta U^i = U^i - U^{i-1}$ and $J^i = J(., t_i)$.
The initial condition is

$$U(0)_R = u_0^h \tag{4.12}$$

where $u_0^h \in V_R^h = (\omega \mid \omega = v_R, v \in V^h)$ is any approximation of u_0 such that

$$\| u_0 - u_0^h \|_{L^2(R)} \leqq Ch \| u_0 \|_{H^1(R)} . \tag{4.13}$$

Remark 8 : If $u_0 \in H^2(R)$ we can take for u_0^h the interpolate of u_0. If u satisfies (4.14) then u^0 must belong to $H^1(R)$ and the orthogonal projection of u_0 in $L^2(R)$ onto the subspace V_R^h has the property (4.13).

THEOREM 4 : *Let the above assumptions be satisfied and let the exact solution u be so smooth that*

$$u_M \in C([0, T]; H^2(M)), \quad M = R, S, \quad u' \in L^2(0, T; H^1(\Omega)), \tag{4.14}$$

$$u_R'' \in L^2(0, T; \overline{V}_R') . \tag{4.15}$$

Then for the approximate solution defined uniquely by (4.11) *and* (4.12) *it holds*

$$\left\{ \Delta t \sum_{i=1}^{r} \| u^i - U^i \|_{H^1(\Omega)}^2 \right\}^{1/2} = O(h + \Delta t) . \tag{4.16}$$

We shall make use of a little modified approximation of Clément [3]. Keeping all notations of Clément we choose $\rho = 1$ and $\gamma_i(p) = p(Q_i)$ where Q_i is a node. The hypotheses $H_1, ..., H_4$ of Clément are satisfied (see also Ciarlet [2],

p. 145 and 130). The approximation $\hat{u}$ of a function $u \in L^2(\Omega)$ applied in the sequel is defined by

$$\hat{u} = \sum_{Q_i \in R} p_i(Q_i)\, \varphi_i + \sum_{Q_i \in S} p_i(Q_i)\, \varphi_i + \sum_{Q_i \in \Gamma^0} \frac{1}{2} [p_i^R(Q_i) + p_i^S(Q_i)]\, \varphi_i . \quad (4.17)$$

Here $\Gamma^0 = \partial R \cap \partial S - \partial\Omega$, φ_i is the basis function corresponding to the node Q_i, p_i is the linear polynomial which is the best approximation of u with respect to the norm $\| \cdot \|_{L^2(S_i)}$ and S_i is the support of φ_i (S_i is the notation of Clément and has nothing to do with the domain S). Evidently, if $Q_i \in M$ the support of φ_i lies in $\overline{M}$. If $Q_i \in \Gamma^0$ we consider the supports of φ_i either in $\overline{R}$ or in $\overline{S}$ and denote the best approximations of u by p_i^R and p_i^S, respectively instead of by p_i. We denote by $| \cdot |_{H^k(M)}$ the seminorm

$$\left\{ \int_M \sum_{|\alpha| = k} | D^\alpha u |^2 \, dx \right\}^{1/2}$$

and introduce :

LEMMA 3 : *If* $u \in H_0^1(\Omega)$ *then* $\hat{u} \in H_0^1(\Omega)$ *and*

$$\| u - \hat{u} \|_{L^2(\Omega)} \leqq Ch \, | u |_{H^1(\Omega)} . \quad (4.18)$$

If, in addition, $u_M \in H^2(M)$, $M = R, S$, *then*

$$\| u - \hat{u} \|_{H^j(\Omega)} \leqq Ch^{2-j} \sum_{M = R,S} | u |_{H^2(M)} , \quad j = 0, 1 . \quad (4.19)$$

Proof : $\hat{u} \in H_0^1(\Omega)$ is obvious because in all sums in (4.17) the nodes lie in Ω. To prove (4.18) and (4.19) we use the same technique as Clément used to prove inequalities (1) and (3) from his paper with one change. If the node Q_i belongs to Γ^0 then instead of making use of $| u |_{0,\tau_i} = 0$ (see [3], p. 83, the sixth line from above) we estimate as follows :

$$| p_i^R - p_i^S |_{0,\tau_i} = | p_i^R - u + u - p_i^S |_{0,\tau_i} \leqq | p_i^R - u |_{0,\tau_i} + | p_i^S - u |_{0,\tau_i} .$$

Proof of theorem 4 : From (4.14) and (3.17) it follows

$$(u'(t), \sigma z)_{L^2(R)} + a(u(t), z) = (J(., t), z)_{L^2(\Omega)} \quad \text{in } [0, T] \quad \forall z \in H_0^1(\Omega) . \quad (4.20)$$

First we estimate $\varepsilon^i = \hat{u}^i - U^i$ where $\hat{u}(t)$ is Clément's approximation of $u(t)$

defined for $t > 0$ by (4.17) and for $t = 0$ by $\hat{u}_0 = \pi u_0$ where πu_0 is the original Clément approximation. From (4.20) we have for all $z \in V^h$

$$(\Delta \hat{u}^i, \sigma z)_{L^2(R)} + \Delta t a(\hat{u}^i, z) = \Delta t (J^i, z)_{L^2(\Omega)} + (\Delta u^i - \Delta t u'(t_i), \sigma z)_{L^2(R)} + \\ + (\Delta(\hat{u}^i - u^i), \sigma z)_{L^2(R)} + \Delta t[a(\hat{u}^i, z) - a(u^i, z)] .$$

Subtracting (4.11) we get

$$(\Delta \varepsilon^i, \sigma z)_{L^2(R)} + \Delta t[a(\hat{u}^i, z) - a(U^i, z)] = \\ = (\Delta u^i - \Delta t u'(t_i), \sigma z)_{L^2(R)} + (\Delta(\hat{u}^i - u^i), \sigma z)_{L^2(R)} + \\ + \Delta t[a(\hat{u}^i, z) - a(u^i, z)] \quad \forall z \in V^h . \tag{4.21}$$

We estimate the terms on the righ-hand side of (4.21). By Taylor's theorem with integral remainder and by (4.1)

$$| (\Delta u^i - \Delta t u'(t_i), \sigma z)_{L^2(R)} | = | \langle \Delta u_R^i - \Delta t u'(t_i)_R, z_R \rangle_R | = \\ = \left| \int_{t_{i-1}}^{t_i} (t_{i-1} - t) \langle u''(t)_R, z_R \rangle_R \, dt \right| \leqq \\ \leqq c \, \Delta t^{3/2} \left\{ \int_{t_{i-1}}^{t_i} \| u_R'' \|_{V_R'}^2 \, dt \right\}^{1/2} \| z \|_{H^1(R)} .$$

Using (4.18) we obtain

$$| (\Delta(\hat{u}^i - u^i), \sigma z)_{L^2(R)} | \leqq Ch \, | \Delta u^i |_{H^1(\Omega)} \| z \|_{L^2(R)} \leqq \\ \leqq Ch \, \Delta t^{1/2} \left\{ \int_{t_{i-1}}^{t_i} \| u'(t) \|_{H^1(\Omega)}^2 \, dt \right\}^{1/2} \| z \|_{L^2(R)} .$$

From (4.9) it follows Lipschitz continuity of the operators $A^M(u_M)$ (see Gajewski, Gröger and Zacharias [4], p. 70, 71, assertion *e*)).

Hence

$$| a(\hat{u}^i, z) - a(u^i, z) | \leqq C \, | \hat{u}^i - u^i |_{H^1(\Omega)} \, | z |_{H^1(\Omega)} \leqq c(u) \, h \, | z |_{H^1(\Omega)} \tag{4.22}$$

where we use (4.19) ($c(u)$ denotes a constant depending on the norms of u in spaces occurring in (4.14) and (4.15)).

We choose $z = \varepsilon^i = \hat{u}^i - U^i$ in (4.21) and apply (4.4) and the inequality

$| ab | \leqq \frac{1}{2\eta} a^2 + \frac{1}{2}\eta b^2$ with suitable η's to estimate the righ-hand sides of the preceding inequalities. The result is

$$(\Delta\varepsilon^i, \sigma\varepsilon^i)_{L^2(R)} + \beta\,\Delta t \,\| \varepsilon^i \|^2_{H^1(\Omega)} \leqq \frac{1}{2}\beta\,\Delta t \,\| \varepsilon^i \|^2_{H^1(\Omega)} +$$

$$+ c\,\Delta t^2 \int_{t_{i-1}}^{t_i} \| u''_R \|^2_{\overline{V}'_R}\, dt + ch^2 \int_{t_{i-1}}^{t_i} \| u'(t) \|^2_{H^1(\Omega)}\, dt + c(u)\,\Delta t h^2 .$$

Summing we get

$$\Delta t \sum_{i=1}^{r} \| \varepsilon^i \|^2_{H_1(\Omega)} \leqq c(\sigma\varepsilon^0, \varepsilon^0)_{L^2(R)} + c(u)\,(h^2 + \Delta t^2) . \tag{4.23}$$

As $\varepsilon^0_R = \hat{u}_0 - u^h_0 = \hat{u}_0 - u_0 + u_0 - u^h_0$ it holds

$$(\sigma\varepsilon^0, \varepsilon^0)_{L^2(R)} \leqq c \,\| \varepsilon^0 \|^2_{L^2(R)} \leqq c(u)\, h^2 .$$

Further, $u^i - U^i = u^i - \hat{u}^i + \varepsilon^i$, hence from (4.19) and (4.23) it follows (4.16).

Now we define U^i by means of the scheme (1.9) :

$$\left.\begin{array}{l}\left(\sigma\left[\frac{3}{2}U^i - 2\,U^{i-1} + \frac{1}{2}U^{i-2}\right], z\right)_{L^2(R)} + \Delta t a(U^i, z) = \Delta t(J^i, z)_{L^2(\Omega)} \\ \qquad \forall z \in V^h , \quad i \geqq 2 ,\end{array}\right\} \tag{4.24}$$

$$U^0_R = \hat{u}^h_0 , \quad U^1 \quad \text{computed from (4.11)} . \tag{4.25}$$

THEOREM 5 : *Let the exact solution fulfill* (4.14) *and*

$$u''_R \in C([0, T] ; L^2(R)) , \quad u'''_R \in L^2(0, T ; \overline{V}'_R) . \tag{4.26}$$

Then for the approximate solution U^i *defined uniquely by* (4.24) *and* (4.25) *it holds*

$$\left\{ \Delta t \sum_{i=1}^{r} \| u^i - U^i \|^2_{H^1(\Omega)} \right\}^{1/2} = O(h + \Delta t^2) . \tag{4.27}$$

Proof : Instead of (4.21) we derive

$$\left(\frac{3}{2}\varepsilon^i - 2\,\varepsilon^{i-1} + \frac{1}{2}\varepsilon^{i-2}, \sigma z\right)_{L^2(R)} + \Delta t[a(\hat{u}^i, z) - a(U^i, z)] =$$

$$= \left(\frac{3}{2}u^i - 2\,u^{i-1} + \frac{1}{2}u^{i-2} - \Delta t u'(t_i), \sigma z\right)_{L^2(R)} + \left(\frac{3}{2}\Delta(\hat{u}^i - u^i\right) -$$

$$- \frac{1}{2}\Delta(\hat{u}^{i-1} - u^{i-1}, \sigma z)_{L^2(R)} + \Delta t[a(\hat{u}^i, z) - a(u^i, z)]\,, \quad i \geqq 2\,.$$

The second and the third term on the right-hand side can be estimated as before. The first term is easy to estimate if we use the equality

$$\frac{3}{2}u_R^i - 2\,u_R^{i-1} + \frac{1}{2}u_R^{i-2} - \Delta t u'(t_i)_R =$$

$$= \int_{t_{i-1}}^{t_i}\left[-(t_{i-1} - t)^2 + \frac{1}{4}(t_{i-1} - t)^2\right] u'''(t)_R\, dt +$$

$$+ \frac{1}{4}\int_{t_{i-2}}^{t_i}(t_{i-2} - t)^2\, u'''(t)_R\, dt\,.$$

Choosing again $z = \check{\varepsilon}^i$, summing and using (3.61) we obtain

$$\Delta t \sum_{i=2}^{r} \|\,\varepsilon^i\,\|_{H^1(\Omega)}^2 \leqq c\,\{(\sigma\varepsilon^0, \varepsilon^0)_{L^2(R)} + (\sigma\varepsilon^1, \varepsilon^1)_{L^2(R)}\} + c(u)\,(h^2 + \Delta t^4) \tag{4.28}$$

(of cause, $c(u)$ depends now on $\|\,u_R''\,\|_{C([0,T];L^2(R))}$ and on $\|\,u_R'''\,\|_{L^2(0,T;\overline{V}_R')}$ instead of on $\|\,u_R''\,\|_{L^2(0,T;\overline{V}_R')}$). From (4.21) one can prove that

$$\|\,\varepsilon^1\,\|_{L^2(R)}^2 + \Delta t\,\|\,\varepsilon^1\,\|_{H^1(\Omega)}^2 \leqq c(u)\,(h^2 + \Delta t^4)\,.$$

This inequality together with (4.28) gives

$$\Delta t \sum_{i=1}^{r} \|\,\varepsilon^i\,\|_{H^1(\Omega)}^2 = O(h^2 + \Delta t^4)$$

from which (4.27) follows.

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