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Iterative refinement of finite element approximations for elliptic problems


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ITERATIVE REFINEMENT OF FINITE ELEMENT APPROXIMATIONS FOR ELLIPTIC PROBLEMS (*)

by Lin Qun (1)

Communicé par J A Nitsche

Résumé — On présente une extrapolation itérative d’approximations de problèmes elliptiques par des éléments fins de bas degré

Abstract — An iterative refinement of low-degree finite element approximations for elliptic problems is presented

1. We will consider the boundary value problem

\[ \Delta u + \sum a_i \frac{\partial u}{\partial x_i} + bu = -f \quad \text{in} \quad \Omega, \]

\[ u = 0 \quad \text{on} \quad \partial \Omega. \]  \hspace{1cm} (1)

Here \( \Omega \subset \mathbb{R}^N \) is a bounded domain with boundary \( \partial \Omega \) sufficiently smooth. We will adopt the standard notations (cf. Gilbarg-Trudinger, 1977). Especially \( \langle ., . \rangle \) respective \( \langle ., . \rangle_1 \) denote the \( L_2(\Omega) \)-inner-product respective the Dirichlet integral and \( \| . \|_k \) the norm in \( H^k_0(\Omega) \).

The weak formulation of problem (1) is \( u \in H^1 \) and

\[ (u, v)_1 = (\sum a_i u |_{\partial} + bu + f, v) \quad \text{for} \quad v \in H^1. \]  \hspace{1cm} (2)

Our basic assumption is : problem (1) resp. (2) has a unique solution \( u \) to \( f \in H^0 \) with \( u \in H^1 \cap H^2 \) and \( \| u \|_2 \leq c \| f \| \). Now let \( S_h \) be the space of linear finite

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éléments with isoparametric modifications in the boundary éléments such that $S_h \subset \bar{H}_1$ holds true. Due to an argument of Schatz (1974) for $h$ sufficiently small the Galerkin-approximation $u^0 = u_h \in S_h$ defined by

$$(u^0, \chi)_1 = \left( \sum a_i u^0 |_i + b u^0 + f, \chi \right) \text{ for } \chi \in S_h$$

(3)

is uniquely defined. The error estimate

$$\| u - u^0 \| + h \| u - u^0 \|_1 \leq c h^2 \| u \|_2$$

(4)

is well known.

In Lin Qun (1978), (1980) we introduced a refinement of $u^0$ on the basis of the additional assumption: to $F \in H_0$ given the solution of

$$- \Delta U = F \quad \text{in } \Omega,$$

$$U = 0 \quad \text{on } \partial \Omega$$

(5)

resp. $U \in \bar{H}_1$ and

$$(U, v)_1 = (F, v) \quad \text{for } v \in \bar{H}_1$$

(6)

is computable. Then given $u^0$ we can compute $\overline{u}^0$ defined by $\overline{u}^0 \in \bar{H}_1$ and

$$(\overline{u}^0, v)_1 = \left( \sum a_i u^0 |_i + b u^0 + f, v \right) \quad \text{for } v \in \bar{H}_1.$$ 

(7)

This leads to a higher accuracy in the $H_1$-norm:

$$\| u - \overline{u}^0 \|_1 \leq c h^2 \| u \|_2.$$ 

(8)

Of course $\overline{u}^0$ is not an element of $S_h$.

Following a suggestion of Nitsche (private communication) we construct starting with the pair $(u^0, \overline{u}^0)$ iterates $(u^{\nu+1}, \overline{u}^{\nu+1})$ for $\nu \geq 0$ defined

$$u^{\nu+1} = \overline{u}^\nu + \varphi^\nu$$

(9)

with $\varphi^\nu \in S_h$ and

$$(\varphi^\nu, \chi)_1 - \left( \sum a_i \varphi^\nu |_i + b \varphi^\nu, \chi \right) =$$

$$= \left( \sum a_i(\overline{u}^\nu - u^\nu) |_i + b(\overline{u}^\nu - u^\nu), \chi \right) \text{ for } \chi \in S_h$$

(10)

and on the other hand by ($\nu \geq 0$)

$$(\overline{u}^\nu, v)_1 = \left( \sum a_i u^\nu |_i + b u^\nu + f, v \right) \quad \text{for } v \in \bar{H}_1.$$ 

(11)
In Section 3 we give the proof of:

**THEOREM 1:** Let \((v, w)\) be defined as above. Then

\[
\|u - u^v\| + \|u - w^v\|_1 \leq (ch)^{v+2} \|u\|_2
\]

is valid.

2. Our proof is based on the following operator framework (cf. Chatelin, 1981, Hackbusch, 1981). Let us consider the equation

\[ u = Ku + y \]  

in a Banach-space \(X\) with \(K\) being a linear compact operator. Further let \(S\) be an approximating subspace and \(P : X \rightarrow S\) a bounded projection onto \(S\). The standard Galerkin solution is defined by

\[ u^0 = PKu^0 + Py. \]  

Now we construct iterates \(\bar{u}^v\) and \(u^{v+1}\) in the way

\[ \bar{u}^v = Ku^v + y, \]  
\[ u^{v+1} = \bar{u}^v + r^v \]

with \(r^v\) defined by

\[ r^v = PKr^v + PK(\bar{u}^v - u^v). \]

**Remark 1:** \(d^v = \bar{u}^v - u^v = Ku^v - u^v + y\) is the defect of the \(v\)-th iterate. Therefore \(r^v\) may be interpreted as the Galerkin-solution to the right hand side \(Kd^v\).

**Remark 2:** The approximations \(\bar{u}^0\) are also considered in Sloan (1976), but the higher iterates introduced there differ from ours.

**LEMMA 1:** Suppose that \(K\) is compact, 1 is not an eigenvalue of \(K\) and \(\kappa = \| (I - P) K \|\) is sufficiently small.

Then \((I - PK)^{-1}\) exists as a bounded operator in \(X\) and the Galerkin solutions are well defined. Moreover

\[ u - u^v = (I - PK)^{-1} (I - P) K (u - u^{v-1}). \]

**Proof:** Since \((I - K)^{-1}\) is bounded for \(\kappa\) small enough also \((I - PK)^{-1}\) is bounded. As a consequence the Galerkin solution is uniquely defined. The identity

\[ (I - K)^{-1} = (I - PK)^{-1} + (I - PK)^{-1} (I - P) K (I - K)^{-1} \]  

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will be useful. The solution \( u \) of (12) may be written in the form
\[
  u = (I - PK)^{-1} y + (I - PK)^{-1} (I - P) Ku .
\] (19)

Because of our construction we have
\[
  u^{r+1} = Ku^r + y + (I - PK)^{-1} PK(Ku^r + y - u^r)
  = (I - PK)^{-1} y + (I - PK)^{-1} (I - P) Ku^r .
\] (20)

Subtraction of (20) from (19) gives (17).

**Remark 3**: We mention that under our assumptions also \( (I - KP)^{-1} \) exists and the recurrence relation
\[
  u - \overline{u}^r = (I - KP)^{-1} K(I - P) (u - \overline{u}^{-1})
\] (21)
is valid. The proof is omitted.

By our assumptions \( \| u^0 \| \) is bounded by a multiple of \( \| y \| \). Because of
\[
  \| (I - PK)^{-1} \| \leq \frac{\gamma}{1 - \kappa \gamma}
\] (22)
with \( \gamma \) being the norm of \( \| (I - K)^{-1} \| \) we conclude from lemma 1:

**Corollary 1**: Let \( \kappa = \| (I - P) K \| \) be less than the half of
\[
  \gamma^{-1} = \| (I - K)^{-1} \|^{-1} .
\]

Then error-estimates of the type
\[
  \| u - u^r \| \leq c \left\{ \frac{\kappa \gamma}{1 - \kappa \gamma} \right\}^r \| y \|
\] (23)
hold true.

3. Now we come back to the situation discussed in section 1. We identify \( X \) with the Hilbertspace \( H_0 = L_2(\Omega) \). Since we want to work with the Ritz-method we have to impose the condition \( S \subseteq \overline{H}_1 \). For simplicity we focuss our attention to the case : \( S = S_h \) is the space of linear finite elements with isoparametric modifications along the boundary. Further let \( P = R_h \) be the standard Ritz-projection defined by \( Pu \in S_h \) and
\[
  (Pu, \chi)_1 = (u, \chi)_1 \quad \text{for} \quad \chi \in S_h .
\] (24)

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The operator $K$ is defined by

$$w = Kw \Rightarrow w \in \tilde{H}_1 \quad \text{and} \quad (w, g)_1 = (v, -\sum (a_i g)_i + bg) \quad \text{for} \quad g \in \tilde{H}_1.$$  \hspace{1cm} (25)

Under suitable conditions concerning the regularity of $a$, $b$ and since the original problem (1) resp. (2) is assumed to be uniquely solvable $K$ is a bounded operator from $H_0$ into $\tilde{H}_1$ and hence compact as mapping of $H_0$ into itself.

By duality the error-estimate

$$\| u - Pu \| \leq ch \| u \|_1$$  \hspace{1cm} (26)

is a consequence of (4). Because of

$$\| (I - P) Kv \| \leq ch \| Kv \|_1 \leq c' h \| v \|$$  \hspace{1cm} (27)

we find

$$\kappa = \kappa_h = \| (I - P) K \| \leq ch$$  \hspace{1cm} (28)

with some constant $c$.

The estimates derived in section 2 lead to

$$\| u - u^r \| \leq (ch)^r \| u - u^0 \|$$  \hspace{1cm} (29)

and because of (4) to

$$\| u - u^r \| \leq (ch)^{r+2} \| u \|_2.$$  \hspace{1cm} (30)

Finally the terms $\| u - \bar{u}^r \|_1$ are bounded in the same way since by definition

$$u - \bar{u}^r = K(u - u^r).$$  \hspace{1cm} (31)

This completes the proof of theorem 1.

4. In this section we consider the model problem

$$- \Delta u = f(\cdot, u) \quad \text{in} \quad \Omega$$
$$u = 0 \quad \text{on} \quad \partial \Omega$$  \hspace{1cm} (32)

in two or three space dimensions. The weak formulation of (32) is : Find $u \in \tilde{H}_1$ such that

$$(u, v)_1 = (f(u), v) \quad \text{for} \quad v \in \tilde{H}_1.$$  \hspace{1cm} (33)
Our assumptions are:

(i) $f(x, z)$ is twice continuously differentiable with respect to $z \in \mathbb{R}$ and

$$|f_{zz}(x, z)|$$

is uniformly bounded.

(ii) For $z = u(x) \in C^0(\Omega)$ the functions $f(x, u(x))$, $f_z(x, u(x))$ and $f_{zz}(x, u(x))$ are in $C^0(\Omega)$.

(iii) $u$ is an isolated solution of (32), i.e. the linear problem

$$(w, g)_1 = (f(u)w, g) \quad \text{for} \quad g \in \overset{\circ}{H}_1$$

admits only $w = 0$ in $\overset{\circ}{H}_1$.

Now let $u^0 = u_h \in S_h$ be the solution of the corresponding Galerkin-problem

$$(u^0, \chi)_1 = (f(u^0), \chi) \quad \text{for} \quad \chi \in S_h.$$ 

Corresponding to section 1 we define the iterates $\bar{u}^\nu$ for $\nu \geq 0$ by

$$(\bar{u}^\nu, g)_1 = (f(\bar{u}^\nu), g) \quad \text{for} \quad g \in \overset{\circ}{H}_1,$$ 

and

$$u^{\nu+1} = \bar{u}^\nu + \varphi^\nu$$

with $\varphi^\nu \in S_h$ and

$$(\varphi^\nu, \chi)_1 = (f(u^0)(\varphi^\nu + \bar{u}^\nu - u^\nu), \chi) \quad \text{for} \quad \chi \in S_h.$$ 

The counterpart of theorem 1 is:

**Theorem 2**: Let $(u^\nu, \bar{u}^\nu)$ be defined as above. Then

$$\| u - u_h \| + \| u - \bar{u}^\nu \|_2 \leq c_1(c_2 h^2)^{\nu+1}$$

is valid. The constants $c_1$, $c_2$ depend on $u$ and bounds of $f_z, f_{zz}$ but are independent of $h$ and $\nu$.

**Proof**: Let $K : H_0 \to \overset{\circ}{H}_1 \cap H_2$ be the inverse of the Laplacian defined by

$$w = Kv \iff (w, g)_1 = (v, g) \quad \text{for} \quad g \in \overset{\circ}{H}_1,$$ 

and let $P = R_h$ be the Ritz operator defined by

$$\Phi = Pv \iff \Phi \in S_h$$

and $(\Phi, \chi)_1 = (v, \chi)_1$ for $\chi \in S_h$. 

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Problem (32) is equivalent to $u = Kf(u)$. We may rewrite this in the form

$$ (I - PKf'(u^0)) u = Kf(u) - PKf'(u^0) u . $$  (43)

In terms of $K$ and $P$ the iterates $\bar{u}^r$ and $\varphi^r$ have the representation

$$ \bar{u}^r = Kf(u^r) , $$  (44)

$$ (I - PKf'(u^0)) \varphi^r = PKf(u^0) (\bar{u}^r - u^r) . $$  (45)

This leads to

$$ (I - PKf'(u^0)) u^{r+1} = Kf(u^r) - PKf'(u^0) u^r . $$  (46)

By comparison of (43) and (46) and by adding and subtracting appropriate terms we come to

$$ (I - PKf'(u^0)) (u^{r+1} - u) = (I - P) Kf'(u^0) (u^r - u) + $$

$$ + K \{ f(u^r) - f(u) - f'(u) (u^r - u) + f'(u) - f'(u^0) \} (u^r - u) . $$  (47)

The Ritz operator $P$ is the orthogonal projection in $H_0$ onto $S = S_h$. For $v, w \in H_0$ arbitrary we get

$$ ((I - P) K, v, w) = ((I - P) K, v, K w)_1 $$

$$ = ((I - P) K, (I - P) K w)_1 $$

$$ \leq ch^2 \| K v \|_2 \| K w \|_2 \leq ch^2 \| v \| \| w \| . $$  (48)

This implies that the norm of $(I - P) K$ as mapping of $H_0$ into $H_0$ is bounded by $ch^2$. Next let $a$ be a continuous function and $v, w \in H_0$. Then also $K(avw)$ is in $H_0$ and

$$ \| K(avw) \| \leq c \| v \| \| w \| . $$  (49)

This follows from

$$ \| K(avw) \|_0 = \sup \{ (Kavw, g) | \| g \| = 1 \} $$

(50)

and

$$ (K(avw), g) = (v, \{ aKg \} w) $$

(51)

in combination with Sobolev's embedding lemma.

For $h$ small enough the initial Galerkin solution $u^0$ is "near" to $u$. Because of our assumption (iii) then the operator $I - PKf'(u^0)$ will have a bounded inverse.

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By the aid of these arguments we derive from the recurrence relation (47) the corresponding error bound
\[ \| u^{\nu+1} - u \| \leq c_3 h^2 \| u^\nu - u \| + c_4 \| u^\nu - u \|^2 + \\ + c_5 \| u^0 - u \| \| u^\nu - u \|. \] (52)

For the sake of clarity we have numbered the constants. Since an estimate of the type
\[ \| u^0 - u \| \leq ch^2 \] (53)
holds true anyway we derive from (52)
\[ \| u^{\nu+1} - u \| \leq c_6 h^2 \| u^\nu - u \| + c_4 \| u^\nu - u \|^2 . \] (54)

Because of (53) by complete inductions there is a constant \( c_7 \) such that for \( h < h_0 \) with \( h_0 \) chosen appropriate the relation
\[ \| u^{\nu+1} - u \| \leq c_7 h^2 \| u^\nu - u \| \] (55)
holds true (55) together with (53) lead to the error bound stated in theorem 2 for \( u^\nu - u \).

Because of
\[ \overline{u}^\nu - u = K(f(u^\nu) - f(u)) \] (56)
we come to
\[ \| \overline{u}^\nu - u \|_2 \leq c \| f(u^\nu) - f(u) \| \\
\leq c \| u^\nu - u \|. \] (57)

Remark 3: Whereas assumption (iii) is essential the two preceding ones can be reduced.

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