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# $L^\infty$-ERROR ESTIMATES FOR VARIATIONAL INEQUALITIES WITH HÖLDER CONTINUOUS OBSTACLE (*)

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Abstract — *An error estimate is derived, using a linear finite element method, for the $L^\infty$-approximation of the solution of variational inequalities with Holder continuous obstacle If the obstacle is in $C^{0,\alpha}(\bar{\Omega})$ $(0 < \alpha \leqslant 1)$, then the $L^\infty$-error for the linear element solution is in the order of $h^{\alpha-\varepsilon}$ $(\forall \varepsilon > 0)$.*

Resume. — *On démontre que l'erreur d'approximation dans la norme $L^\infty$ de la solution d'une inéquation variationnelle, avec obstacle $\alpha$-holdérien* $(0 < \alpha \leqslant 1)$, *par la méthode des éléments finis linéaires, est de l'ordre $h^{\alpha-\varepsilon}$, pour tout $\varepsilon > 0$*

## 1. INTRODUCTION

The interest for the study of variational inequalities (V.I.) with « irregular » obstacles has recently increased. Regularity properties of solutions have been proved for V.I. with Hölder continuous ([4], [7], [8], [12]), continuous [12], or one-sided Hölder continuous [13] obstacles.

The importance of such results lies in particular in their application to the theory of quasi-variational inequalities (Q.V.I.), namely V.I. with the obstacle depending on the solution itself. Such an implicit obstacle, in fact, is in general " fairly irregular " (see [3] for some examples connected to stochastic control theory).

From a numerical point of view, some recents results are known concerning the approximation of solutions of Q.V.I. connected to some stochastic impulse control problems (see [11], [15]), by means of finite element methods.

The aim of this paper is to show an error estimate in the $L^\infty$ norm, for the approximation, by means of linear finite elements, of the solution of variational

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inequalities with Hölder continuous obstacle. If the obstacle is in $C^{0,\alpha}(\bar{\Omega})$, $0 < \alpha \leqslant 1$ (so that, according to the mentioned regularity results, the solution itself is in $C^{0,\alpha}(\bar{\Omega})$), then, under reasonable hypotheses on the triangulation, the $L^{\infty}$-error of such an approximation is in the order of $h^{\alpha-\varepsilon}$ (for each $\varepsilon > 0$), that is the expected order of convergence.

In § 2 we introduce some notations and we recall the regularity of solutions. In § 3 the discretization is studied, and we state our principal result (theorem 3.2) together with some remarks and corollaries. In § 4 we indicate some useful results which are needed, in § 5, to prove theorem 3.2.

## 2. FORMULATION OF THE PROBLEM

Let $\Omega$ be a convex bounded domain of $\mathbb{R}^N$, with sufficiently smooth boundary $\Gamma$ (we suppose for example $\Gamma \in C^2$).

With classical notations, $C^{0,\alpha}(\bar{\Omega})$, $0 < \alpha < 1$ [$\alpha = 1$], is the space of all the Hölder [Lipschitz] continuous functions of exponent $\alpha$ over $\Omega$, with the semi-norm

$$[v]_\alpha = \sup_{\substack{x,y \in \bar{\Omega} \\ x \neq y}} \frac{|v(x) - v(y)|}{|x - y|^\alpha}.$$

For $p \geqslant 1$, we let $L^p(\Omega)$ denote the classical Banach space consisting of measurable functions on $\Omega$ that are $p$-integrable, with the norm

$$\| v \|_p = \left( \int_\Omega |v|^p \, dx \right)^{1/p} \quad \text{if} \quad 1 \leqslant p < +\infty,$$

$$\| v \|_\infty = \operatorname{ess.\,sup}_\Omega |v| \quad \text{if} \quad p = \infty.$$

Then for $p \geqslant 1$, $m \in \mathbb{N}$, $W^{m,p}(\Omega)$ is the classical Sobolev space defined by

$$W^{m,p}(\Omega) = \{ v : D^\gamma v \in L^p(\Omega), \text{ for all } |\gamma| \leqslant m \};$$

in $W^{m,p}(\Omega)$ we introduce the norm

$$\| v \|_{m,p} = \sum_{|\gamma| \leqslant m} \| D^\gamma v \|_p,$$

and we set $H^m(\Omega) = W^{m,2}(\Omega)$; then $H_0^1(\Omega)$ is the closure, in the norm of $W^{1,2}(\Omega)$, of $C_0^1(\Omega)$, the space of all continuous functions with compact support in $\Omega$, having all first derivatives continuous in $\Omega$.

In the following $c$ will be the notation for positive constants involved in calculation, and the terms on which $c$ depends will be clarified each time.

Let $A$ be the second order linear elliptic operator defined by

$$A = - \sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial}{\partial x_i}\right) + \sum_{i=1}^{N} b_i(x)\frac{\partial}{\partial x_i} + c_0(x),$$

with the following assumptions :

i) $a_{ij} \in C^1(\overline{\Omega})$, $b_i$, $c_0 \in L^\infty(\Omega)$, $i, j = 1, 2, ..., N$ ;

ii) There is a constant $\nu > 0$ such that (uniform ellipticity) :

$$\sum_{i,j=1}^{N} a_{ij}(x)\,\xi_i\,\xi_j \geqslant \nu\,|\,\xi\,|^2, \text{ a.e. in } \Omega, \forall\xi \in \mathbb{R}^N - \{\,0\,\}\,;$$

iii) $c_0(x) \geqslant \tilde{c} > 0, \forall x \in \Omega$, with $\tilde{c}$ sufficiently large (such that $A$ is a coercive operator on the space $H_0^1(\Omega)$).

Let $a(.,.) : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ be the continuous and coercive bilinear form on $H_0^1(\Omega)$ associated with the operator $A$, namely, $\forall u, v \in H_0^1(\Omega)$,

$$a(u, v) = \sum_{i,j=1}^{N} \int_\Omega a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\,dx + \sum_{i=1}^{N}\int_\Omega b_i\frac{\partial u}{\partial x_i}\,v\,dx + \int_\Omega c_0\,uv\,dx\,.$$

Let us now consider an " obstacle problem " for the operator $A$, i.e. the following V.I. with homogeneous boundary conditions :

$$\begin{aligned} a(u, v - u) \geqslant (f, v - u)\,, \quad \forall v \in \mathbb{K} \\ u \in \mathbb{K} \end{aligned} \tag{2.1}$$

where $\mathbb{K} = \{\,v \in H_0^1(\Omega) : v \geqslant \psi \text{ in } \Omega\,\}$ is a closed convex subset of $H_0^1(\Omega)$, and

$$f \in L^\infty(\Omega)\,, \tag{2.2}$$

$$\psi \in C^{0,\alpha}(\overline{\Omega})\,, \quad 0 < \alpha \leqslant 1\,, \tag{2.3}$$

are two given functions. We assume $\psi\,|_\Gamma \leqslant 0$, in order to avoid $\mathbb{K}$ being empty. Then the following regularity result is known :

THEOREM 2.1 : *Under the assumptions* (2.2) *and* (2.3), *the unique solution $u$ of problem* (2.1) *is in* $C^{0,\alpha}(\overline{\Omega})$.

The proof in the interior of $\Omega$ can be deduced for example from Caffarelli-Kinderlehrer [7], where it is shown that the solution of problem (2.1) has the same modulus of continuity of the obstacle. For a general proof we refer to Frehse [12], where the nonlinear case has been considered. For the case $\alpha = 1$, see also Chipot [8]. Lastly we mention the result of Biroli [4] : $u \in C^{0,\alpha'}(\overline{\Omega})$, $\alpha' < \alpha$, if more general boundary conditions are involved.

### 3. DISCRETIZATION AND PRINCIPAL RESULT

Let $\Omega_h$ denote a polyhedral domain inscribed in $\Omega$, such that the diameter of every " face " of $\Gamma_h = \partial\Omega_h$ has length less than $h$. Let us consider that over $\Omega_h$ a " triangulation " $\mathcal{T}_h$ is defined (in the usual way, see [9]), regular, in the sense that, setting $\forall T \in \mathcal{T}_h$ :

$$h_T = \operatorname{diam}(T),$$

$$\rho_T = \sup \{ \operatorname{diam}(B) : B \subset T \text{ is a ball in } \mathbb{R}^N \},$$

then :

i) there is a constant $\sigma$ such that, $\forall T \in \mathcal{T}_h, \dfrac{h_T}{\rho_T} \leqslant \sigma$ ;

ii) $h \geqslant \max\limits_{T \in \mathcal{T}_h} h_T$.

A piecewise linear subspace $V_h$ can be defined on $\overline{\Omega}$ in the following way

$$V_h = \{ v \in C^0(\overline{\Omega}) : v|_T \text{ is a linear function}, \forall T \in \mathcal{T}_h ; v \equiv 0 \text{ in } \overline{\Omega} - \Omega_h \}.$$

Let us denote by $\{ P_i \}_{i=1}^{r(h)}$ the internal nodes of $\mathcal{T}_h$. Then the functions $\{ \phi_i \}_{i=1}^{r(h)}$ of $V_h$ such that

$$\phi_i(P_j) = \delta_{ij}, \quad i, j = 1, 2, \dots, r(h),$$

form a basis of $V_h$ ; in particular for every $v \in C^0(\overline{\Omega}) \cap H_0^1(\Omega)$ the function

$$v_I(x) = \sum_{i=1}^{r(h)} v(P_i)\, \phi_i(x) \tag{3.1}$$

represents the interpolate of $v$ over $\mathcal{T}_h$.

Furthermore, from the definition of $\mathcal{T}_h$,

$$P_i \in \partial T \Rightarrow T \subset B(P_i, h), \quad i = 1, 2, \dots, r(h), \quad \forall T \in \mathcal{T}_h,$$

where $B(P_i, h)$ is the ball of $\mathbb{R}^N$ with its center in $P_i$ and radius $h$ ; then

$$\operatorname{supp} \phi_i \subset \overline{B(P_i, h)}, \quad i = 1, 2, \dots, r(h). \tag{3.2}$$

Now let us consider the discrete problem associated with (2.1) :

$$\begin{aligned} a(u_h, v_h - u_h) \geqslant (f, v_h - u_h), \quad &\forall v_h \in \mathbb{K}_h \\ &u_h \in \mathbb{K}_h \end{aligned} \tag{3.3}$$

where $\mathbb{K}_h = \{ v_h \in V_h : v_h \geqslant \psi_h \}$, and $\psi_h$ is the piecewise linear function on $\Omega$

equal to $\psi$ at the nodes of $\mathcal{T}_h$ (and defined on every connected component of $\Omega - \Omega_h$ by a constant extension in directions normal to $\Gamma_h$, see [6]).

REMARK 3.1 : *Such a choice of* $\mathbb{K}_h$ *means that the constraint* $u_h \geqslant \psi$ *is only imposed over the internal nodes of* $\mathcal{T}_h$. *It could in fact be defined in an equivalent way* :

$$\mathbb{K}_h = \{ v_h \in V_h : v_h(P_i) \geqslant \psi(P_i), i = 1, 2, \dots, r(h) \} . \quad \blacksquare$$

Let $M_h = (m_{ij})$ be the matrix of problem (3.3), i.e. the real $r(h) \times r(h)$ matrix whose generic term is

$$m_{ij} = a(\phi_j, \phi_i) , \quad i, j = 1, 2, \dots, r(h) .$$

The following assumption is needed :

$$m_{ij} \leqslant 0 \quad \text{if} \quad i \neq j , \quad i, j = 1, 2, \dots, r(h) ; \tag{3.4}$$

then, by the hypotheses on the coefficients of $A$, $M_h$ is an $M$-matrix, and the discrete problem (3.3) satisfies a discrete maximum principle, in the sense of [10] (where conditions of essentially geometric type on the triangulation $\mathcal{T}_h$ are given, under which (3.4) holds).

The solution $u_h$ of (3.3) represents the approximation of the solution $u$ of (2.1) in the linear finite element discretization. Under the previous assumptions we are able to obtain an error estimate, in $L^{\infty}$ norm, for such an approximation.

Namely, our principal result is :

THEOREM 3.2 : *If* (2.2), (2.3), (3.4) *hold, then* $\forall p > 1$

$$\| u - u_h \|_{\infty} \leqslant c h^{\alpha - N/p} \, | \log h \, | , \tag{3.5}$$

*where* $c$ *depends on* $\Omega$, $\psi$, $p$, *and* $\alpha$, *not on* $h$.

Estimate (3.5) is quasi-optimal. In fact the interpolation error in $L^{\infty}$ for Hölder continuous functions in $C^{0,\alpha}(\overline{\Omega})$ is a $0(h^{\alpha})$. Here this result is shown under the hypotheses :

$$u \,|_{\Gamma} = 0 ; \tag{3.6}$$

$$\operatorname{dist}(\Gamma, \Gamma_h) \leqslant c h^2 . \tag{3.7}$$

Condition (3.6) can be easily eliminated. It should also be noted that, under the assumptions made on $\Omega$(convex, with $\Gamma \in C^2$), it is always possible to construct $\Omega_h$ such that (3.7) holds. (We remark that, in the non-convex case, assuming condition (3.7) as an hypothesis, we still obtain an estimate such as (3.5).)

LEMMA 3.3 : *If* $u \in C^{0,\alpha}(\overline{\Omega})$, $0 < \alpha \leqslant 1$, *and conditions* (3.6), (3.7) *are satisfied, then*

$$\| u - u_I \|_\infty \leqslant ch^\alpha ,$$

*where c depends only on u,* $\alpha$ *and* $\Omega$.

*Proof.* — From the definition (3.1) $\left(\text{since } \sum_{i=1}^{r(h)} \phi_i(x) \leqslant 1, \forall x \in \overline{\Omega}\right)$ :

$$| u(x) - u_I(x) | \leqslant \left(1 - \sum_{i=1}^{r(h)} \phi_i(x)\right) | u(x) | + \sum_{i=1}^{r(h)} \phi_i(x) | u(x) - u(P_i) | ; \quad (3.8)$$

the first term in the right hand side of (3.8) is either equal to zero (when $x$ belongs to the convex envelope of the internal nodes, $\sum_{i=1}^{r(h)} \phi_i(x) = 1\Big)$, or, in the other case, it is less than $ch^{2\alpha}$ (from (3.7)). For the second term we have

$$\sum_{i=1}^{r(h)} \phi_i(x) | u(x) - u(P_i) | \leqslant [u]_\alpha \sum_{i=1}^{r(h)} \phi_i(x) | x - P_i |^\alpha$$
$$\leqslant [u]_\alpha h^\alpha ,$$

since, from (3.2), $\phi_i(x) \neq 0$ implies $| x - P_i | < h$. ∎

As a corollary of theorem 3.2 we have an approximation result for the set $D = \{ x \in \Omega : u(x) > \psi(x) \}$, where the solution does not touch the obstacle. The boundary of $D$ is the so-called free boundary, and it is in many cases the real unknown of problems such as (2.1). Usually the convergence of $u_h$ to $u$ is not enough to ensure the convergence to $D$ (in set theoretical sense) of sets $D_h = \{ x \in \Omega : u_h(x) \geqslant \psi(x) \}$. However, theorem 3.2 implies :

COROLLARY 3.4 : *Under the same assumptions of theorem* 3.2, *the sequence* $\{ D_{h,\varepsilon} \}$, *where*

$$D_{h,\varepsilon} = \{ x \in \Omega : u_h(x) > \psi(x) + h^{\alpha-\varepsilon} \} ,$$

" *converges from the interior* " *to* $D$, $\forall \varepsilon > 0$, *in the sense that* :

a) $\lim_{h \to 0^+} D_{h,\varepsilon} = D$ *(in set theoretical sense)*;

b) $D_{h,\varepsilon} \subset D$, *if h is sufficiently small.*

(See [2] for the proof.)

## 4. PRELIMINARY RESULTS

Let us state some useful results in order to prove theorem 3.2.

— *A priori estimates*

The following relation between solutions and obstacles of two different V.I. is well known (see [5]) :

LEMMA 4.1 : *Let* $u$ *[resp.* $w$*]* $\in H_0^1(\Omega)$ *be the unique solution of a* V.I. *such as* (2.1), *with obstacle* $\psi$ *[resp.* $\varphi$*]* $\in L^{\infty}(\Omega)$*; then*

$$\| u - w \|_{\infty} \leqslant \| \psi - \varphi \|_{\infty} .$$

The discrete analogue of lemma 4.1 is also valid (see [11]) :

LEMMA 4.2 : *Let* $u_h$ *[resp.* $w_h$*]* $\in V_h$ *denote the approximation of* $u$ *[resp.* $w$*] given by problem* (3.3) ; *if* $M_h$ *satisfies* (3.4), *then*

$$\| u_h - w_h \|_{\infty} \leqslant \| \psi_h - \varphi_h \|_{\infty} .$$

— *V.I. with* $W^{2,p}$*-obstacle*

Let us consider a V.I. such as (2.1), with the assumption (2.2), but now let $\psi \in W^{2,p}(\Omega)$. Then it is well known [14] that the solution $u$ is in $W^{2,p}(\Omega)$. Baiocchi [1] and Nitsche [17] have already studied the approximation for the solution of this problem. In particular we have :

THEOREM 4.3 : *Let* $f \in L^p(\Omega)$, $\psi \in W^{2,p}(\Omega)$, $\forall p < +\infty$ ; *if* (3.4) *holds, then*

$$\| u - u_h \|_{\infty} \leqslant c h^{2-N/p} \,|\log h|\, \{ \| u \|_{2,p} + \| \psi \|_{2,p} \} , \quad \forall p < +\infty , \qquad (4.1)$$

*c independent of h.*

Proof of theorem 4.3 can be easily derived from [1], by means of the interpolation theory (see [9]), and of error estimates in $L^{\infty}$ for solutions of equations. Estimates such as (4.1) hold in fact for equations with solutions in $W^{2,p}(\Omega)$ : they can be stated using Nitsche's techniques of weighted norms ; when $A = -\Delta$, see also [18], where a quasi-optimality result in $L^{\infty}$ is given for the $H_0^1$-projection into finite element spaces.

## 5. PROOF OF THEOREM 3.2

Without loss of generality, let us consider $\psi|_\Gamma = 0$ (such that in problem (3.3) now $\psi_h = \psi_I$) ; it can be shown in fact that solution $u$ of (2.1) is equal to solution $\hat{u}$ of

$$\begin{aligned}&a(\hat{u}, z - \hat{u}) \geqslant (f, z - \hat{u}), \quad \forall z \in H_0^1(\Omega), \quad z \geqslant \hat{\psi}\\&\hat{u} \in H_0^1(\Omega), \quad \hat{u} \geqslant \hat{\psi}\end{aligned}$$

where $\hat{\psi} = \psi \vee u_0$, and $u_0$ is the solution of the related equation

$$\begin{aligned}&a(u_0, v) = (f, v), \quad v \in H_0^1(\Omega)\\&u_0 \in H_0^1(\Omega).\end{aligned}$$

We have $u_0 \in W^{2,p}(\Omega)$, $\forall p < +\infty$ : hence $\hat{\psi} \in C^{0,\alpha}(\overline{\Omega})$, with the same $\alpha$ of $\psi$.

The proof of theorem 3.2 is based on a regularization procedure, consisting in the " approximation " of the initial problem by means of " more regular " V.I. (namely with $W^{2,p}$-obstacle, $\forall p < +\infty$), for which we can apply theorem 4.3. We then conveniently " go back " to problem (2.1), through continuity results. This procedure can be divided into four steps.

*Step* 1 : *Regularization by convolution.*

LEMMA 5.1 : *There is a sequence* $\{\psi^n\}$ *converging to* $\psi$ *in* $L^\infty$, *such that,* $\forall n$,

$$\psi^n \in C^1(\overline{\Omega}), \quad \psi^n|_\Gamma = 0, \tag{5.1}$$

$$\|\psi^n - \psi\|_\infty \leqslant cn^{-\alpha}, \tag{5.2}$$

$$\|\psi^n\|_{C^1(\overline{\Omega})} \leqslant cn^{1-\alpha}, \tag{5.3}$$

*where c depends on* $\psi, \alpha, \Omega$, *but not on n.*

*Proof* : See [4] ; (5.1) can be shown using convolutions of $\psi$ with suitable mollifiers and cut-off functions. ∎

Let us call $u^n$ the solution of the V.I. (2.1) with obstacle $\psi^n$, and $u_h^n$ the solution of the corresponding discrete problem (where now the obstacle is $\psi_I^n$).

*Step* 2 : *Elliptic regularization.*

LEMMA 5.2 : *For every fixed n, there is a sequence* $\{\psi^{n,m}\}$ *converging, for* $m \to +\infty$, *to* $\psi^n$ *in* $L^\infty$, *such that* $\forall m$, $\psi^{n,m}$ *is the solution of*

$$\begin{bmatrix} m^{-1} A\psi^{n,m} + \psi^{n,m} = \psi^n \\ \psi^{n,m}|_\Gamma = 0 \end{bmatrix}$$

*and*

$$\psi^{n,m} \in W^{2,p}(\Omega), \quad \forall p < +\infty ;$$

$$\| \psi^{n,m} - \psi^n \|_\infty \leqslant c m^{-1/2} \| \psi^n \|_{1,p}, \quad \forall p < +\infty, \tag{5.4}$$

$$\| A\psi^{n,m} \|_\infty \leqslant c m^{1/2} \| \psi^n \|_{1,p}, \quad \forall p < +\infty, \tag{5.5}$$

*where c does not depend on m and n.*

(For the proof see [4] again.)

As we did in Step 1, let us call $u^{n,m}$ the solution of the V.I. (2.1) with obstacle $\psi^{n,m}$, and $u_h^{n,m}$ the solution of the corresponding discrete problem. Of course $u^{n,m} \in H_0^1(\Omega) \cap W^{2,p}(\Omega)$, $\forall p < +\infty$ ; it follows

$$\| u^{n,m} \|_{2,p} \leqslant c \| Au^{n,m} \|_p \leqslant c \| Au^{n,m} \|_\infty .$$

Furthermore the following inequality of Lewy-Stampacchia's type holds (see e.g. [16]) :

$$f \leqslant Au^{n,m} \leqslant (A\psi^{n,m}) \vee f ;$$

this yields, recalling (5.5),

$$\| u^{n,m} \|_{2,p} \leqslant c \| A\psi^{n,m} \|_\infty \leqslant c m^{1/2} \| \psi^n \|_{1,p}, \quad \forall p < +\infty .$$

Likewise,

$$\| \psi^{n,m} \|_{2,p} \leqslant c m^{1/2} \| \psi^n \|_{1,p}, \quad \forall p < +\infty .$$

Applying theorem 4.3, then

$$\| u^{n,m} - u_h^{n,m} \|_\infty \leqslant c m^{1/2} h^{2-\varepsilon(p)} \| \psi^n \|_{1,p}, \quad \forall p < +\infty, \tag{5.6}$$

where for shortness we have set : $h^{2-\varepsilon(p)} = h^{2-N/p} | \log h |$.

*Step* 3 : *Inversion of Step* 2.

LEMMA 5.3 : *The following estimate holds* :

$$\| u^n - u_h^n \|_\infty \leqslant c h^{1-\varepsilon(p)} \| \psi^n \|_{1,p}, \quad \forall n \in \mathbb{N}, \quad \forall p < +\infty . \tag{5.7}$$

*Proof* : For every choice of index $m$, we have

$$\| u^n - u_h^n \|_\infty \leqslant \| u^n - u^{n,m} \|_\infty + \| u^{n,m} - u_h^{n,m} \|_\infty + \| u_h^{n,m} - u_h^n \|_\infty ,$$

and, by lemma 4.1 and (5.4), $\forall p$,

$$\| u^n - u^{n,m} \|_\infty \leqslant c m^{-1/2} \| \psi^n \|_{1,p} .$$

Likewise, using lemma 4.2,

$$\| u_h^{n,m} - u_h^n \|_\infty \leqslant \| \psi_I^{n,m} - \psi_I^n \|_\infty \leqslant cm^{-1/2} \| \psi^n \|_{1,p};$$

then, from (5.6), we obtain

$$\| u^n - u_h^n \|_\infty \leqslant c(m^{-1/2} + m^{1/2} h^{2-\varepsilon(p)}) \| \psi^n \|_{1,p}, \quad \forall p < +\infty .$$

If we now choose a suitable $m$, i.e. such that $1/h^2 \leqslant m \leqslant (1/h^2) + 1$, then the proof is complete. ■

*Step* 4 : *Inversion of Step* 1.

To complete the proof of theorem 3.2, let us use the same trick of Step 3, obtaining

$$\| u - u_h \|_\infty \leqslant \| u - u^n \|_\infty + \| u^n - u_h^n \|_\infty + \| u_h^n - u_h \|_\infty ;$$

according to (5.3), from (5.7) we get

$$\| u^n - u_h^n \|_\infty \leqslant cn^{1-\alpha} h^{1-\varepsilon(p)} ;$$

then, using lemmas 4.1 and 4.2, and (5.2),

$$\| u - u_h \|_\infty \leqslant c(n^{-\alpha} + n^{1-\alpha} h^{1-\varepsilon(p)}) ;$$

if we now take $n$ such that $1/h \leqslant n \leqslant (1/h) + 1$, we finally have

$$\| u - u_h \|_\infty \leqslant ch^{\alpha-\varepsilon(p)}, \quad \forall p < +\infty ,$$

that is the thesis (3.5). ■

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