Stefano Finzi Vita

$L_\infty$-error estimates for variational inequalities with Hölder continuous obstacle

RAIRO. Analyse numérique, tome 16, no 1 (1982), p. 27-37

© AFCET, 1982, tous droits réservés.

L’accès aux archives de la revue « RAIRO. Analyse numérique » implique l’accord avec les conditions générales d’utilisation (http://www.numdam.org/legal.php). Toute utilisation commerciale ou impression systématique est constitutive d’une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/
$L^\infty$-ERROR ESTIMATES FOR VARIATIONAL INEQUALITIES WITH HÖLDER CONTINUOUS OBSTACLE (*)

by Stefano FINZI VITA

Communicé par E MAGENES

Abstract — An error estimate is derived, using a linear finite element method, for the $L^\infty$-approximation of the solution of variational inequalities with Hölder continuous obstacle. If the obstacle is in $C^{0,\alpha}(\Omega)$ ($0 < \alpha \leq 1$), then the $L^\infty$-error for the linear element solution is in the order of $h^{n-\epsilon}$ ($\forall \epsilon > 0$).

Resumé. — On démontre que l’erreur d’approximation dans la norme $L^\infty$ de la solution d’une inéquation variationnelle, avec obstacle $\alpha$-holdérien ($0 < \alpha \leq 1$), par la méthode des éléments finis linéaires, est de l’ordre $h^{n-\epsilon}$, pour tout $\epsilon > 0$.

1. INTRODUCTION

The interest for the study of variational inequalities (V.I.) with « irregular » obstacles has recently increased. Regularity properties of solutions have been proved for V.I. with Hölder continuous ([4], [7], [8], [12]), continuous [12], or one-sided Hölder continuous [13] obstacles.

The importance of such results lies in particular in their application to the theory of quasi-variational inequalities (Q.V.I.), namely V.I. with the obstacle depending on the solution itself. Such an implicit obstacle, in fact, is in general "fairly irregular" (see [3] for some examples connected to stochastic control theory).

From a numerical point of view, some recent results are known concerning the approximation of solutions of Q.V.I. connected to some stochastic impulse control problems (see [11], [15]), by means of finite element methods.

The aim of this paper is to show an error estimate in the $L^\infty$ norm, for the approximation, by means of linear finite elements, of the solution of variational inequalities with Hölder continuous obstacles.

(*) Reçu le 15 décembre 1980

(*) Istituto Matematico « G. Castelnuovo », Università di Roma, Piazzale A. Moro 5, 00100 Roma, Italie

R A I R O Analyse numérique/Numerical Analysis, 0399-0516/1982/27/$ 5 00

© Bordas-Dunod
inequalities with Hölder continuous obstacle. If the obstacle is in $C^{0,\alpha}(\Omega)$, $0 < \alpha \leq 1$ (so that, according to the mentioned regularity results, the solution itself is in $C^{0,\alpha}(\Omega)$), then, under reasonable hypotheses on the triangulation, the $L^\infty$-error of such an approximation is in the order of $h^{\alpha - \epsilon}$ (for each $\epsilon > 0$), that is the expected order of convergence.

In § 2 we introduce some notations and we recall the regularity of solutions. In § 3 the discretization is studied, and we state our principal result (theorem 3.2) together with some remarks and corollaries. In § 4 we indicate some useful results which are needed, in § 5, to prove theorem 3.2.

2. FORMULATION OF THE PROBLEM

Let $\Omega$ be a convex bounded domain of $\mathbb{R}^N$, with sufficiently smooth boundary $\Gamma$ (we suppose for example $\Gamma \in C^2$).

With classical notations, $C^{0,\alpha}(\Omega)$, $0 < \alpha < 1$ [$\alpha = 1$], is the space of all the Hölder [Lipschitz] continuous functions of exponent $\alpha$ over $\Omega$, with the semi-norm

$$[v]_\alpha = \sup_{x,y \in \Omega} \frac{|v(x) - v(y)|}{|x - y|^\alpha}.$$ 

For $p \geq 1$, we let $L^p(\Omega)$ denote the classical Banach space consisting of measurable functions on $\Omega$ that are $p$-integrable, with the norm

$$\| v \|_p = \left( \int_\Omega |v|^p \, dx \right)^{1/p} \quad \text{if} \quad 1 \leq p < + \infty ,$$

$$\| v \|_\infty = \text{ess. sup} \frac{|v|}{\Omega} \quad \text{if} \quad p = \infty .$$

Then for $p \geq 1$, $m \in \mathbb{N}$, $W^{m,p}(\Omega)$ is the classical Sobolev space defined by

$$W^{m,p}(\Omega) = \{ v : D^\gamma v \in L^p(\Omega), \text{ for all } |\gamma| \leq m \} ;$$

in $W^{m,p}(\Omega)$ we introduce the norm

$$\| v \|_{m,p} = \sum_{|\gamma| \leq m} \| D^\gamma v \|_p ,$$

and we set $H^m(\Omega) = W^{m,2}(\Omega)$; then $H^0_0(\Omega)$ is the closure, in the norm of $W^{1,2}(\Omega)$, of $C^0(\Omega)$, the space of all continuous functions with compact support in $\Omega$, having all first derivatives continuous in $\Omega$.

In the following $c$ will be the notation for positive constants involved in calculation, and the terms on which $c$ depends will be clarified each time.

R.A.I.R.O. Analyse numérique/Numerical Analysis
Let $A$ be the second order linear elliptic operator defined by
\[
A = - \sum_{i,j=1}^{N} \frac{\partial}{\partial x_j} \left( a_{ij}(x) \frac{\partial}{\partial x_i} \right) + \sum_{i=1}^{N} b_i(x) \frac{\partial}{\partial x_i} + c_0(x),
\]
with the following assumptions:

i) $a_{ij} \in C^1(\overline{\Omega})$, $b_i$, $c_0 \in L^\infty(\Omega)$, $i, j = 1, 2, \ldots, N$;

ii) There is a constant $\nu > 0$ such that (uniform ellipticity):
\[
\sum_{i,j=1}^{N} a_{ij}(x) \xi_i \xi_j \geq \nu |\xi|^2, \text{ a.e. in } \Omega, \forall \xi \in \mathbb{R}^N - \{0\};
\]

iii) $c_0(x) \geq \bar{c} > 0$, $\forall x \in \Omega$, with $\bar{c}$ sufficiently large (such that $A$ is a coercive operator on the space $H^1_0(\Omega)$).

Let $a(\cdot, \cdot) : H^1_0(\Omega) \times H^1_0(\Omega) \to \mathbb{R}$ be the continuous and coercive bilinear form on $H^1_0(\Omega)$ associated with the operator $A$, namely, $\forall u, v \in H^1_0(\Omega)$,
\[
a(u, v) = \sum_{i,j=1}^{N} \int_{\Omega} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \, dx + \sum_{i=1}^{N} \int_{\Omega} b_i(x) \frac{\partial u}{\partial x_i} v \, dx + \int_{\Omega} c_0(uv) \, dx.
\]

Let us now consider an "obstacle problem" for the operator $A$, i.e. the following V.I. with homogeneous boundary conditions:
\[
a(u, v - u) \geq (f, v - u), \quad \forall v \in \mathcal{K}, \quad u \in \mathcal{K} \tag{2.1}
\]
where $\mathcal{K} = \{ v \in H^1_0(\Omega) : v \geq \psi \text{ in } \Omega \}$ is a closed convex subset of $H^1_0(\Omega)$, and
\[
\begin{align*}
f &\in L^\alpha(\Omega), \\
\psi &\in C^{0,\alpha}(\overline{\Omega}), \quad 0 < \alpha \leq 1,
\end{align*} \tag{2.2} \tag{2.3}
\]
are two given functions. We assume $\psi |_{\Gamma} \leq 0$, in order to avoid $\mathcal{K}$ being empty. Then the following regularity result is known:

**Theorem 2.1**: Under the assumptions (2.2) and (2.3), the unique solution $u$ of problem (2.1) is in $C^{0,\alpha}(\overline{\Omega})$.

The proof in the interior of $\Omega$ can be deduced for example from Caffarelli-Kinderlehrer [7], where it is shown that the solution of problem (2.1) has the same modulus of continuity of the obstacle. For a general proof we refer to Frehse [12], where the nonlinear case has been considered. For the case $\alpha = 1$, see also Chipot [8]. Lastly we mention the result of Biroli [4] : $u \in C^{0,\alpha'}(\overline{\Omega})$, $\alpha' < \alpha$, if more general boundary conditions are involved.
3. DISCRETIZATION AND PRINCIPAL RESULT

Let $\Omega_h$ denote a polyhedral domain inscribed in $\Omega$, such that the diameter of every "face" of $\Gamma_h = \partial \Omega_h$ has length less than $h$. Let us consider that over $\Omega_h$ a "triangulation" $\mathcal{T}_h$ is defined (in the usual way, see [9]), regular, in the sense that, setting $\forall T \in \mathcal{T}_h$:

$$h_T = \text{diam} \ (T),$$
$$\rho_T = \sup \{ \text{diam} (B) : B \subset T \text{ is a ball in } \mathbb{R}^N \},$$

then:

i) there is a constant $\sigma$ such that, $\forall T \in \mathcal{T}_h$, $h_T \leq \sigma$;

ii) $h \geq \max_{T \in \mathcal{T}_h} h_T$.

A piecewise linear subspace $V_h$ can be defined on $\overline{\Omega}$ in the following way

$$V_h = \{ v \in C^0(\overline{\Omega}) : v |_T \text{ is a linear function}, \forall T \in \mathcal{T}_h; v \equiv 0 \text{ in } \overline{\Omega} - \Omega_h \}.$$ 

Let us denote by $\{ P_i \}_{i=1}^{r(h)}$ the internal nodes of $\mathcal{T}_h$. Then the functions $\{ \phi_{ij} \}_{i=1}^{r(h)}$ of $V_h$ such that

$$\phi_i(P_j) = \delta_{ij}, \quad i, j = 1, 2, ..., r(h),$$

form a basis of $V_h$; in particular for every $v \in C^0(\overline{\Omega}) \cap H^1_0(\Omega)$ the function

$$v_i(x) = \sum_{i=1}^{r(h)} v(P_i) \phi_i(x) \quad (3.1)$$

represents the interpolate of $v$ over $\mathcal{T}_h$.

Furthermore, from the definition of $\mathcal{T}_h$,

$$P_i \in \partial T \Rightarrow T \subset B(P_i, h), \quad i = 1, 2, ..., r(h), \quad \forall T \in \mathcal{T}_h,$$

where $B(P_i, h)$ is the ball of $\mathbb{R}^N$ with its center in $P_i$ and radius $h$; then

$$\text{supp} \ \phi_i \subset B(P_i, h), \quad i = 1, 2, ..., r(h). \quad (3.2)$$

Now let us consider the discrete problem associated with (2.1):

$$a(u_h, v_h - u_h) \geq (f, v_h - u_h), \quad \forall v_h \in \mathcal{K}_h$$
$$u_h \in \mathcal{K}_h \quad (3.3)$$

where $\mathcal{K}_h = \{ v_h \in V_h : v_h \geq \psi_h \}$, and $\psi_h$ is the piecewise linear function on $\Omega$.

R.A.I.R.O. Analyse numérique/Numencal Analysis
equal to $\psi$ at the nodes of $\mathcal{C}_h$ (and defined on every connected component of $\Omega - \Omega_h$ by a constant extension in directions normal to $\Gamma_h$, see [6]).

**Remark 3.1:** Such a choice of $\mathbb{K}_h$ means that the constraint $u_h \geq \psi$ is only imposed over the internal nodes of $\mathcal{C}_h$. It could in fact be defined in an equivalent way:

$$\mathbb{K}_h = \{ v_h \in V_h : v_h(P_I) \geq \psi(P_I), i = 1, 2, ..., r(h) \}$$

Let $M_h = (m_{ij})$ be the matrix of problem (3.3), i.e. the real $r(h) \times r(h)$ matrix whose generic term is

$$m_{ij} = a(\phi_i, \phi_j), \quad i, j = 1, 2, ..., r(h).$$

The following assumption is needed:

$$m_{ij} \leq 0 \quad \text{if} \quad i \neq j, \quad i, j = 1, 2, ..., r(h); \quad (3.4)$$

then, by the hypotheses on the coefficients of $A$, $M_h$ is an $M$-matrix, and the discrete problem (3.3) satisfies a discrete maximum principle, in the sense of [10] (where conditions of essentially geometric type on the triangulation $\mathcal{C}_h$ are given, under which (3.4) holds).

The solution $u_h$ of (3.3) represents the approximation of the solution $u$ of (2.1) in the linear finite element discretization. Under the previous assumptions we are able to obtain an error estimate, in $L^\infty$ norm, for such an approximation.

Namely, our principal result is:

**Theorem 3.2:** If (2.2), (2.3), (3.4) hold, then $\forall p > 1$

$$\| u - u_h \|_{\infty} \leq c h^{2-N/p} \log h,$$

where $c$ depends on $\Omega$, $\psi$, $p$, and $\alpha$, not on $h$.

Estimate (3.5) is quasi-optimal. In fact the interpolation error in $L^\infty$ for Hölder continuous functions in $C^{0,\alpha}(\Omega)$ is a $O(h^\alpha)$. Here this result is shown under the hypotheses:

$$u \bigg|_{\Gamma} = 0; \quad (3.6)$$

$$\text{dist} (\Gamma, \Gamma_h) \leq c h^2. \quad (3.7)$$

Condition (3.6) can be easily eliminated. It should also be noted that, under the assumptions made on $\Omega$ (convex, with $\Gamma \in C^2$), it is always possible to construct $\Omega_h$ such that (3.7) holds. (We remark that, in the non-convex case, assuming condition (3.7) as an hypothesis, we still obtain an estimate such as (3.5).)
Lemma 3.3: If \( u \in C^{0,\alpha}(\Omega) \), \( 0 < \alpha \leq 1 \), and conditions (3.6), (3.7) are satisfied, then

\[ \| u - u_f \|_\infty \leq c h^2, \]

where \( c \) depends only on \( u, \alpha \) and \( \Omega \).

Proof. — From the definition (3.1) \( (\text{since } \sum_{i=1}^{r(h)} \phi_i(x) \leq 1, \forall x \in \Omega) \):

\[ | u(x) - u_f(x) | \leq \left( 1 - \sum_{i=1}^{r(h)} \phi_i(x) \right) | u(x) | + \sum_{i=1}^{r(h)} \phi_i(x) | u(x) - u(P_i) |; \quad (3.8) \]

the first term in the right hand side of (3.8) is either equal to zero (when \( x \) belongs to the convex envelope of the internal nodes, \( \sum_{i=1}^{r(h)} \phi_i(x) = 1 \)), or, in the other case, it is less than \( c h^{2\alpha} \) (from (3.7)). For the second term we have

\[ \sum_{i=1}^{r(h)} \phi_i(x) | u(x) - u(P_i) | \leq [u]_\alpha \sum_{i=1}^{r(h)} \phi_i(x) | x - P_i |^\alpha \]

\[ \leq [u]_\alpha h^\alpha, \]

since, from (3.2), \( \phi_i(x) \neq 0 \) implies \( | x - P_i | < h \).

As a corollary of theorem 3.2 we have an approximation result for the set \( D = \{ x \in \Omega : u(x) > \psi(x) \} \), where the solution does not touch the obstacle. The boundary of \( D \) is the so-called free boundary, and it is in many cases the real unknown of problems such as (2.1). Usually the convergence of \( u_h \) to \( u \) is not enough to ensure the convergence to \( D \) (in set theoretical sense) of sets \( D_h = \{ x \in \Omega : u_h(x) > \psi(x) \} \). However, theorem 3.2 implies:

Corollary 3.4: Under the same assumptions of theorem 3.2, the sequence \( \{ D_h, \varepsilon \} \), where

\[ D_{h,\varepsilon} = \{ x \in \Omega : u_h(x) > \psi(x) + h^{\alpha - \varepsilon} \}, \]

"converges from the interior" to \( D, \forall \varepsilon > 0, \) in the sense that:

a) \( \lim_{h \to 0^+} D_{h,\varepsilon} = D \) \( \text{ (in set theoretical sense) ;} \)

b) \( D_{h,\varepsilon} \subset D, \) if \( h \) is sufficiently small.

(See [2] for the proof.)
4. PRELIMINARY RESULTS

Let us state some useful results in order to prove theorem 3.2.

— A priori estimates

The following relation between solutions and obstacles of two different V.I. is well known (see [5]):

**Lemma 4.1**: Let \( u \) [resp. \( w \)] \( \in H_0^1(\Omega) \) be the unique solution of a V.I. such as (2.1), with obstacle \( \psi \) [resp. \( \varphi \)] \( \in L^\infty(\Omega) \); then

\[
\| u - w \|_\infty \leq \| \psi - \varphi \|_\infty.
\]

The discrete analogue of lemma 4.1 is also valid (see [11]):

**Lemma 4.2**: Let \( u_h \) [resp. \( w_h \)] \( \in V_h \) denote the approximation of \( u \) [resp. \( w \)] given by problem (3.3); if \( M_h \) satisfies (3.4), then

\[
\| u_h - w_h \|_\infty \leq \| \psi_h - \varphi_h \|_\infty.
\]

— V.I. with \( W^{2,p} \)-obstacle

Let us consider a V.I. such as (2.1), with the assumption (2.2), but now let \( \psi \in W^{2,p}(\Omega) \). Then it is well known [14] that the solution \( u \) is in \( W^{2,p}(\Omega) \). Baiocchi [1] and Nitsche [17] have already studied the approximation for the solution of this problem. In particular we have:

**Theorem 4.3**: Let \( f \in L^p(\Omega) \), \( \psi \in W^{2,p}(\Omega), \forall p < + \infty \); if (3.4) holds, then

\[
\| u - u_h \|_\infty \leq c h^{2-N/p} \{ \| u \|_{2,p} + \| \psi \|_{2,p} \}, \quad \forall p < + \infty,
\]

(4.1)

\( c \) independent of \( h \).

Proof of theorem 4.3 can be easily derived from [1], by means of the interpolation theory (see [9]), and of error estimates in \( L^\infty \) for solutions of equations. Estimates such as (4.1) hold in fact for equations with solutions in \( W^{2,p}(\Omega) \): they can be stated using Nitsche’s techniques of weighted norms; when \( A = -\Delta \), see also [18], where a quasi-optimality result in \( L^\infty \) is given for the \( H_0^1 \)-projection into finite element spaces.
5. PROOF OF THEOREM 3.2

Without loss of generality, let us consider \( \psi_\| r = 0 \) (such that in problem (3.3) now \( \psi_h = \psi_f \)); it can be shown in fact that solution \( u \) of (2.1) is equal to solution \( \hat{u} \) of

\[
a(\hat{u}, z - \hat{u}) \geq (f, z - \hat{u}), \quad \forall z \in H^1_0(\Omega), \quad z \geq \hat{\psi}
\]
\[
\hat{u} \in H^1_0(\Omega), \quad \hat{\psi} \geq \hat{\psi}
\]

where \( \hat{\psi} = \psi \vee u_0 \), and \( u_0 \) is the solution of the related equation

\[
a(u_0, v) = (f, v), \quad v \in H^1_0(\Omega)
\]
\[
u_0 \in H^1_0(\Omega).
\]

We have \( u_0 \in W^{2,p}(\Omega), \forall p < +\infty \) : hence \( \hat{\psi} \in C^{0,\alpha}(\overline{\Omega}) \), with the same \( \alpha \) of \( \psi \).

The proof of theorem 3.2 is based on a regularization procedure, consisting in the “approximation” of the initial problem by means of “more regular” V.I. (namely with \( W^{2,p} \)-obstacle, \( \forall p < +\infty \)), for which we can apply theorem 4.3. We then conveniently “go back” to problem (2.1), through continuity results. This procedure can be divided into four steps.

Step 1: Regularization by convolution.

**Lemma 5.1:** There is a sequence \( \{ \psi^n \} \) converging to \( \psi \) in \( L^\infty \), such that, \( \forall n, \)

\[
\psi^n \in C^1(\overline{\Omega}), \quad \psi^n |_\Gamma = 0, \quad (5.1)
\]
\[
\| \psi^n - \psi \|_\infty \leq c n^{-\alpha}, \quad (5.2)
\]
\[
\| \psi^n \|_{C^1(\overline{\Omega})} \leq c n^{1-\alpha}, \quad (5.3)
\]

where \( c \) depends on \( \psi, \alpha, \Omega \), but not on \( n \).

**Proof:** See [4]; (5.1) can be shown using convolutions of \( \psi \) with suitable mollifiers and cut-off functions. ■

Let us call \( u^n \) the solution of the V.I. (2.1) with obstacle \( \psi^n \), and \( u^n_h \) the solution of the corresponding discrete problem (where now the obstacle is \( \psi^n_h \)).

Step 2: Elliptic regularization.

**Lemma 5.2:** For every fixed \( n \), there is a sequence \( \{ \psi^{n,m} \} \) converging, for \( m \to +\infty \), to \( \psi^n \) in \( L^\infty \), such that \( \forall m, \psi^{n,m} \) is the solution of

\[
\begin{bmatrix}
m^{-1} A\psi^{n,m} + \psi^{n,m} = \psi^n \\
\psi^{n,m} |_\Gamma = 0
\end{bmatrix}
\]

R.A.I.R.O. Analyse numérique/Numerical Analysis
and
\[
\psi^{n,m} \in W^{2,p}(\Omega), \quad \forall p < + \infty;
\]
\[
\| \psi^{n,m} - \psi^n \|_\infty \leq cm^{-1/2} \| \psi^n \|_{1,p}, \quad \forall p < + \infty, \quad (5.4)
\]
\[
\| Au^{n,m} \|_\infty \leq cm^{1/2} \| \psi^n \|_{1,p}, \quad \forall p < + \infty, \quad (5.5)
\]

where \( c \) does not depend on \( m \) and \( n \).

(For the proof see [4] again.)

As we did in Step 1, let us call \( u^{n,m} \) the solution of the V.I. (2.1) with obstacle \( \psi^{n,m} \), and \( u_h^{n,m} \) the solution of the corresponding discrete problem. Of course \( u^{n,m} \in H^1_0(\Omega) \cap W^{2,p}(\Omega), \forall p < + \infty \); it follows
\[
\| u^{n,m} \|_{2,p} \leq c \| Au^{n,m} \|_p \leq c \| Au^{n,m} \|_\infty.
\]

Furthermore the following inequality of Lewy-Stampacchia’s type holds (see e.g. [16]) :
\[
f \leq Au^{n,m} \leq (A\psi^{n,m}) \lor f;
\]
this yields, recalling (5.5),
\[
\| u^{n,m} \|_{2,p} \leq c \| A\psi^{n,m} \|_\infty \leq cm^{1/2} \| \psi^n \|_{1,p}, \quad \forall p < + \infty.
\]
Likewise,
\[
\| \psi^{n,m} \|_{2,p} \leq cm^{1/2} \| \psi^n \|_{1,p}, \quad \forall p < + \infty.
\]

Applying theorem 4.3, then
\[
\| u^{n,m} - u_h^{n,m} \|_\infty \leq cm^{1/2} h^{2-\varepsilon(p)} \| \psi^n \|_{1,p}, \quad \forall p < + \infty, \quad (5.6)
\]
where for shortness we have set : \( h^{2-\varepsilon(p)} = h^{2-N/p} \log h \).

Step 3 : Inversion of Step 2.

**Lemma 5.3 :** The following estimate holds :
\[
\| u^n - u_h^n \|_\infty \leq ch^{1-\varepsilon(p)} \| \psi^n \|_{1,p}, \quad \forall n \in \mathbb{N}, \quad \forall p < + \infty. \quad (5.7)
\]

**Proof** : For every choice of index \( m \), we have
\[
\| u^n - u_h^n \|_\infty \leq \| u^n - u^{n,m} \|_\infty + \| u^{n,m} - u_h^{n,m} \|_\infty + \| u_h^{n,m} - u_h^n \|_\infty,
\]
and, by lemma 4.1 and (5.4), \( \forall p, \)
\[
\| u^n - u^{n,m} \|_\infty \leq cm^{-1/2} \| \psi^n \|_{1,p}.
\]
Likewise, using lemma 4.2,
\[ \| u_h^n - u_h^m \|_{\infty} \leq \| \psi_t^{n,m} - \psi_t^n \|_{\infty} \leq cm^{-1/2} \| \psi^n \|_{1,p}; \]
then, from (5.6), we obtain
\[ \| u^n - u_h^n \|_{\infty} \leq c(m^{-1/2} + m^{1/2} h^{-\varepsilon(p)}) \| \psi^n \|_{1,p}, \quad \forall p < +\infty. \]
If we now choose a suitable \( m \), i.e. such that \( 1/h^2 \leq m \leq \left(1/h^2\right) + 1 \), then the proof is complete.

**Step 4: Inversion of Step 1.**

To complete the proof of theorem 3.2, let us use the same trick of Step 3, obtaining
\[ \| u - u_h \|_{\infty} \leq \| u - u^n \|_{\infty} + \| u^n - u_h^n \|_{\infty} + \| u_h^n - u_h \|_{\infty}; \]
according to (5.3), from (5.7) we get
\[ \| u^n - u_h^n \|_{\infty} \leq cn^{1-\alpha} h^{1-\varepsilon(p)}; \]
then, using lemmas 4.1 and 4.2, and (5.2),
\[ \| u - u_h \|_{\infty} \leq c(n^{-\alpha} + n^{1-\alpha} h^{-\varepsilon(p)}); \]
if we now take \( n \) such that \( 1/h \leq n \leq (1/h) + 1 \), we finally have
\[ \| u - u_h \|_{\infty} \leq ch^{\alpha-\varepsilon(p)}, \quad \forall p < +\infty, \]
that is the thesis (3.5). ■

**REFERENCES**