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Three remarks on the use of Čebyšev polynomials for solving equations of the second kind


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THREE REMARKS ON THE USE OF ČEŠYŠEV POLYNOMIALS
FOR SOLVING EQUATIONS OF THE SECOND KIND (*)

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Abstract — Three methods are considered: the Čebyśev-Euler method, the Čebyśev semi-iterative method and a refinement of the projection method for the approximation of the quasi inverse for self-adjoint operators A such that unity does not belong to the spectrum of A.

In this communication we consider three methods for the approximate solution of equations of the second kind

$$x - Ax = y$$

in a (complex) Hilbert space with a selfadjoint bounded linear operator A. We only assume that unity does not belong to the spectrum of A.

We give a new proof and an error estimate for the Čebyśev-Euler method, we discuss the Čebyśev-semi-iterative method (cf. Varga [7]) and we consider a refinement of the projection method of type $Q_\sigma$, introduced in [6].

1. INTRODUCTION

Let A be a bounded linear selfadjoint operator in a complex Hilbert space $H$. Let $(E_\lambda)$ be its spectral decomposition, $\sigma$ an interval containing the spectrum $\sigma(A)$ of A. Then

$$A = \int_\sigma \lambda \, dE$$

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and for each continuous function $p : \sigma \to \mathbb{R}$

$$p(A) = \int_{\sigma} p(\lambda) \, dE_\lambda$$

and

$$\| p(A) \| \leq \sup_{\lambda \in \sigma} | p(\lambda) | .$$

Especially, if $1 \notin \sigma$,

$$\| (1 - A)^{-1} - p(A) \| \leq \sup_{\lambda \in \sigma} \left| \frac{1}{1 - \lambda} - p(\lambda) \right| .$$

If $p$ is an arbitrary polynomial of degree $n$, then $\| (1 - A)^{-1} - p(A) \|$ is minimal, if $p$ is the proximum of $r$ (with $r(\lambda) = (1 - \lambda)^{-1}$) in the space of all polynomials of degree $n$ on the spectrum of $A$ with respect of the sup-norm.

We call a method for the approximate solution of $x - Ax = y$ polynomial, if the approximate solution $\hat{x}$ is of the form $\hat{x} = p(A) y$, where $p(A)$ is an operator polynomial.

2. THE ČEBYŠEV-EULER METHOD

If $\sigma = [a, b], b < 1$ then the Čebyšev-Euler method consists of determining the proximum $p_n$ to $r$ in $[a, b]$ by polynomials degree $n$. Then

$$x_n = p_n(A) y$$

is the Čebyšev-Euler approximation of the solution $x$ of $x - Ax = y$.

This approximation is easy to calculate: it is known Čebyšev [3], Bernstein [2], Meinardus [4], that in the interval $[-1, 1]$ the proximum of

$$s_n(\lambda) = \frac{1}{\lambda - \alpha} \quad \alpha > 1$$

is given by the polynomials $q_n$ of degree $n$ which fulfill

$$\frac{1}{\lambda - \alpha} - q_n(\lambda) = \gamma_n \cos (n\varphi + \delta)$$

$$\gamma_n = \frac{(\alpha - \sqrt{\alpha^2 - 1})^n}{\alpha^2 - 1}, \quad \lambda = \cos \varphi, \quad \frac{\alpha \lambda - 1}{\lambda - 1} = \cos \delta .$$

Using the Čebyšev polynomials $t_n$ and $v_n$ of first resp. second kind, (2) is equivalent to

$$(\alpha - \lambda) q_n(\lambda) = 1 - \gamma_n(\alpha \lambda - 1) t_n(\lambda) + \gamma_n \sqrt{\alpha^2 - 1} (1 - \lambda^2) v_{n-1}(\lambda) .$$
The recursion formulas for \( r_n \) and \( v_n \) lead to the recursion formula for \( q_n \):

\[
q_{n+1}(\lambda) = 2 \gamma \lambda q_n(\lambda) - \gamma^2 q_{n-1}(\lambda) - 2 \gamma
\]

\[
\gamma = \alpha - \sqrt{\alpha^2 - 1}, \quad q_0(\lambda) = \frac{\alpha}{1 - \alpha^2}, \quad q_1(\lambda) = \frac{\lambda + \sqrt{\alpha^2 - 1}}{\alpha^2 - 1}.
\]

A linear transformation of the interval \([-1, 1]\) onto \([a, b]\) gives the polynomials \( p_n \) of best approximation of \( r \) by

\[
p_n(\lambda) = -\frac{b - a}{2} q_n\left(\frac{2 \lambda - b - a}{b - a}\right)
\]

which leads to the recursion formula

\[
p_{n+1}(\lambda) = -\frac{2}{b - a} \left[ \frac{2 \gamma}{b - a} (2 \lambda - b - a) p_n(\lambda) - \gamma^2 p_{n-1}(\lambda) - 2 \gamma \right]
\]

\[
p_0(\lambda) = \frac{b - a}{2} \frac{\alpha}{\alpha^2 - 1}
\]

\[
p_1(\lambda) = \frac{1}{\alpha^2 - 1} \left( -\lambda + \frac{b + a}{2} - \frac{b - a}{\alpha^2 - 1} \right)
\]

\[
\alpha = \frac{2 - b - a}{b - a}
\]

\[
\gamma = \alpha - \sqrt{\alpha^2 - 1}.
\]

The error estimate is

\[
\max_{\lambda \in [a, b]} \left| \frac{1}{1 - \lambda} - p_n(\lambda) \right| = \max_{\lambda \in [-1, 1]} \frac{b - a}{2} \left| \frac{1}{\lambda} - \alpha - q_n(\lambda) \right|
\]

\[
= \frac{b - a}{2} \frac{\gamma^n}{\alpha^2 - 1} \max | \cos (n \varphi + \delta) | \leq \frac{b - a}{2(\alpha^2 - 1)} \cdot \gamma^n.
\]

If we replace \( \lambda \) by \( A \), we obtain the following result :

If \( \sigma(A) \subset [a, b], b < 1 \), then the best polynomial approximation method of \( x - Ax = y \) is given by the following semi-iterative method.

\[
x_{n+1} = -\frac{2}{b - a} \left[ \frac{2 \gamma}{b - a} (2 Ax_n - (b + a) x_n) - \gamma^2 x_{n-1} - 2 \gamma y \right]
\]

\[
x_0 = \frac{b - a}{2} \cdot \frac{\alpha}{\alpha^2 - 1} y, \quad x_1 = \frac{1}{\alpha^2 - 1} \left[ -Ay + \left( \frac{b + a}{2} - \frac{b - a}{2} \sqrt{\alpha^2 - 1} \right) y \right]
\]

\[
\alpha = \frac{2 - b - a}{b - a}, \quad \gamma = \alpha - \sqrt{\alpha^2 - 1}
\]

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with the error estimate
\[ \| x - x_n \| \leq \frac{b - a}{2(a^2 - 1)} \gamma^n. \]

This method also can be obtained by using methods of summability theory (cf. Niethammer [5]).

3. THE ČEBYŠEV SEMI-ITERATIVE METHOD

Let \( A \) be a linear selfadjoint bounded operator with \( 1 \notin \sigma(A) \) and \( (x_n) \) the Picard iteration sequence
\[ x_{n+1} = Ax_n + y, \quad x_0 = y. \]

This method calculates a linear combination
\[ \tilde{x}_n = \sum_{j=0}^{n} \gamma_j x_j \]
such that \( x - \tilde{x}_n \) has a small norm.

Since for
\[ \tilde{w}_n = \tilde{x}_n - x = \sum_{j=0}^{n} \gamma_j (x_j - x) + \sum_{j=0}^{n} (\gamma_j - 1) x \]

it is
\[ \tilde{w}_n = \sum_{j=0}^{n} \gamma_j A^i (x_0 - x) = p(A) (x_0 - x) \]

with the condition \( p(1) = \sum_{j=0}^{n} \gamma_j = 1 \), so
\[ \tilde{w}_n = p(A) (y - (I - A)^{-1} y) = \int_{\sigma} p(\lambda) \frac{\lambda}{1 - \lambda} dE\lambda y \]

and \( \| \tilde{w}_n \| \) is minimal, if \( p \) is a polynomial of degree \( n \) with \( p(1) = 1 \) and
\[ \max_{\lambda \in \sigma(A)} \left| p(\lambda) \frac{\lambda}{1 - \lambda} \right| \leq \max_{\lambda \in \sigma(A)} \left| q(\lambda) \frac{\lambda}{1 - \lambda} \right|, \]

where \( q \) is an arbitrary polynomial of degree \( n \) with \( q(1) = 1 \). If both 1 and 0 do not belong to the spectrum of \( A \), then \( p \) is up to a constant the same polynomial as the polynomial \( q \) with \( q(1) = 1 \) and \( \max_{\lambda \in \sigma(A)} | q(\lambda) | \) is minimal.
In each case, this minimal polynomial does not lead to an easy semi-iterative method, so the usual minimization condition is to determine the polynomial $p$ of degree $n$ with $p(1) = 1$ and minimal norm.

It is well known that the transformed Čebyšev polynomials have this property that their norm on an interval is minimal. So one has to consider three cases

1° $\sigma(A) \subset [a, b], b < 1$
2° $\sigma(A) \subset [a, b], a > 1$
3° $\sigma(A) \subset [a_1, b_1] \cup [a_2, b_2]$.

In the first and second case

$$p_n(\lambda) = \frac{t_n\left(\frac{2 \lambda - b - a}{b - a}\right)}{t_n\left(\frac{2 - b - a}{b - a}\right)}$$

is the minimal polynomial with

$$\| p_n \| = \max_{\lambda \in [a, b]} | p_n(\lambda) | = \left\| \frac{1}{t_n\left(\frac{2 - b - a}{b - a}\right)} \right\|.$$

Using the recursing formula for the Čebyšev polynomials we obtain for

$$\rho_n = t_n\left(\frac{2 - b - a}{b - a}\right)^{-1}$$

$$\rho_{n+1}^{-1} = 2 \frac{2 - b - a}{b - a} \rho_n^{-1} - \rho_{n-1}^{-1}, \quad \rho_0 = 1, \quad \rho_1^{-1} = \frac{2 - b - a}{b - a}$$

and

$$p_{n+1}(\lambda) = \frac{2 \rho_{n+1}}{\rho_n} \frac{2 \lambda - b - a}{b - a} p_n(\lambda) - \frac{\rho_{n+1}}{\rho_{n-1}} \rho_{n-1}(\lambda)$$

$$p_0(\lambda) = 1, \quad p_1(\lambda) = \frac{2 \lambda - b - a}{2 - b - a}.$$

This gives after a short calculation, using the recursion formulas and

$$\tilde{w}_{n+1} = \tilde{x}_{n+1} - x = p_n(A) w_0$$
the semi-iterative method

\[ \tilde{x}_{n+1} = \frac{4 \rho_{n+1}}{\rho_n (b - a)} \left( A \tilde{x}_n + y - \frac{b + a}{2} \tilde{x}_n \right) - \frac{\rho_{n+1}}{\rho_{n-1}} \tilde{x}_{n-1} \]

\[ \tilde{x}_0 = y, \quad \tilde{x}_1 = \frac{2}{2 - b - a} Ay + y \]

and the error estimate

\[ \| x - \tilde{x}_n \| \leq \| p_n(A) (x - y) \| \leq |\rho_n| \| x - y \|. \]

In the third case we assume that there is known a number \( \eta \) such that

\[ \sigma(A) \subset [-\rho, 1 - \eta] \cup [1 + \eta, \rho]. \]

Since the polynomials

\[ q_{2n}(\lambda) = t_n \left( \frac{2 \lambda^2 - 1 - \alpha^2}{1 - \alpha^2} \right) \]

are the polynomials of minimal norm on the intervals \([-1, -\alpha] \cup [\alpha, 1]\) of degree \(2n\) with \(q_{2n}(1) = 1\) (cf. Achieser [1], p. 287) a linear transformation of \([-1, -\alpha] \cup [\alpha, 1]\) onto \([-\rho, 1 - \eta] \cup [1 + \eta, \rho + 2]\) (resp. \([2 - \rho, 1 - \eta] \cup [1 + \eta, \eta]\) if more convenient) and the substitution of \( \lambda \) by \( A \) leads to the semi-iterative method

\[ x_{n+1} = \frac{4 \tau_{n+1}}{\tau_n ((\rho + 1)^2 - \eta^2)} \left[ (A^2 x_n - 2 Ax_n + x_n - Ay) - \frac{1}{2} ((\rho + 1)^2 + \eta^2) x_n \right] - \frac{\tau_{n+1}}{\tau_{n-1}} x_{n-1} \]

\[ x_0 = y \]

\[ x_1 = -\frac{2}{(\rho + 1)^2 + \eta^2} (A^2 y - Ay) + y \]

\[ \tau_{n+1}^{-1} = t_n \left( -\frac{(\rho + 1)^2 + \eta^2}{(\rho + 1)^2 - \eta^2} \right) \]

\[ \tau_{n+1}^{-1} = -\frac{2 (\rho + 1)^2 + \eta^2}{(\rho + 1)^2 - \eta^2} \tau_{n-1}^{-1} - \tau_{n-1}^{-1} \]

\[ \tau_0 = 1 \]

\[ \tau_1^{-1} = -\frac{(\rho + 1)^2 + \eta^2}{(\rho + 1)^2 - \eta^2}. \]
The order of convergence of this method is
\[ \| x_n - x \| \leq \| p(A) (y - x) \| = 0(\tau_n^{-1}) . \]

4. AČEBYŠEV PROJECTION METHOD

Let \( A \) be again a bounded linear selfadjoint operator in a Hilbert space \( H \) with spectrum in \([a, b]\), \( b < 1 \).

Let \( p_n : [a, b] \rightarrow \mathbb{R} \) be the polynomials from section 2, which are the proxima of \((1 - \lambda)^{-1}\) of degree \( n \) in \([a, b]\). Then for linear independent elements \( z_1, \ldots, z_k \) of \( H \) we determine
\[
Z = \sum_{j=1}^{n} \gamma_j z_j
\]
from the system of linear equations
\[
\langle z - Az - y + (1 - A) p_n(A) y, z_j \rangle = 0
\]
for \( j = 1, 2, \ldots, k \). Then
\[
\hat{x}_n = p_n(A) y + z
\]
is an approximation for the solution \( x \) of \( x - Ax = y \).

If \( p_n = 0 \), then this method is the usual Ritz-Galerkin method, if
\[
p_n(\lambda) = \sum_{j=0}^{n} \lambda^j,
\]
then this method is the projection method of type \( Q_{n+1} \) introduced in [6].

As usual in the theory of the Ritz-Galerkin method, the optimal rate of convergence for compact \( A \) is obtained, if \( z_1, \ldots, z_k \) are eigenvectors of \( A \). In this case we get with
\[
x = p_n(A) y + ((1 - A)^{-1} - p_n(A)) y
\]
\[
\hat{x}_n = p_n(A) y + z
\]
and a simple Hilbert space calculation
\[
z = \sum_{j=1}^{k} \left( \frac{1}{1 - \lambda_j} - p_n(\lambda_j) \right) \langle y, z_j \rangle z_j
\]
\[
\| x - x_n \|^2 = \sum_{j=k+1}^{\infty} \left| \frac{1}{1 - \lambda_j} - p_n(\lambda_j) \right|^2 \| y, z_j \|^2
\]
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so

\[ \| x - \tilde{x}_n \| \leq \sup_{j \geq k+1} \left| \frac{1}{1 - \lambda_j} - p_n(\lambda_j) \right| \leq \frac{b - a}{2(\alpha^2 - 1)} \gamma^n \]

where

\[ \alpha = \frac{2 - b - a}{b - a}, \quad \gamma = \alpha - \sqrt{\alpha^2 - 1} \]

as in a section 2. Also as in section 2 is shown \( p_n(A) y \) can be calculated by a semi-iterative method.

5. CONCLUDING REMARKS

Niethammer [5] has shown that the order of convergence of the Čebyšev semi-iterative method tends to the order of convergence of the Čebyšev-Euler method. In [8] M. Wolf has demonstrated that the Čebyšev projection method in general gives quite better approximations than the usual Ritz-Galerkin method.

REFERENCES