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J. A. NITSCHE

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ON KORN'S SECOND INEQUALITY (*)

by J. A. NITSCHE ([1])

Resume — *On donne trois démonstrations élémentaires de la seconde inéquation de Korn, selon l'une des hypothèses suivantes la frontière du domaine est* (1) *continûment dérivable,* (2) *un polygone en dimension* 2, (3) *Lipschitzienne bornée*

Abstract — *Three elementary proofs are given for Korn's second inequality according to the assumptions The boundary of the domain is* (1) *continuously differentiable,* (2) *a polygon in two dimensions,* (3) *Lipschitz bounded*

0. NOTATIONS AND INTRODUCTION

In this paper we use the notations standard in the theory of elliptic equations. In addition by a dot on top of a letter denoting a function space the (not closed) subspace of functions vanishing outside some sphere is meant, e.g. $\dot{H}_1 = \dot{H}_1(\Omega)$ consists of all $u \in H_1$ such that there is a sphere with finite radius which contains the support of u. Of course this is only relevant for unbounded domains. Differentiation is indicated by $u_{|i} := \partial u/\partial x_i$. We will also consider vector-valued functions in which case we write $\underline{u} = (u_1, ..., u_N)$ with N being the dimension. Then $\underline{u} \in \underline{H}_k$ means $u_i \in H_k$ for $i = 1, 2, ..., N$ and

$$(\underline{u}, \underline{v})_k = (u_i, v_i)_k \tag{1}$$

(the summation convention is used throughout the paper).

We will also use the abbreviations

$$(\nabla \underline{u}, \nabla \underline{v}) = (u_{i|k}, v_{i|k}) \tag{2}$$

and

$$\| \nabla \underline{u} \| = (\nabla \underline{u}, \nabla \underline{u})^{1/2} . \tag{3}$$

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([1]) Institut für Angewandte Mathematik, Albert-Ludwigs-Universität, Freiburg R F A

In linear elasticity associated with the displacement vector $\underline{u}$ are the strain- and stress-tensors defined by

$$\begin{aligned} \varepsilon(\underline{u}) &: \varepsilon_{ik}(\underline{u}) = \frac{1}{2}(u_{i\,|\,k} + u_{k\,|\,i}) \\ \sigma(\underline{u}) &: \sigma_{ik}(\underline{u}) = 2\,\mu\varepsilon_{ik}(\underline{u}) + \lambda(u_{j\,|\,j})\,\delta_{ik} \end{aligned} \tag{4}$$

with λ, μ being the Lamé constants and δ_{ik} the Kronecker symbol. The elastic energy corresponding to the displacement $\underline{u}$ is given by

$$\begin{aligned} a(\underline{u}, \underline{u}) &= (\varepsilon_{ik}(\underline{u}), \sigma_{ik}(\underline{u})) \\ &= \iint_\Omega \{ 2\,\mu\varepsilon_{ik}(\underline{u})\,\varepsilon_{ik}(\underline{u}) + \lambda(u_{i\,|\,i})^2 \}\, dx\,. \end{aligned} \tag{5}$$

Within the frame work of Hilbert-space-theory the coerciveness of the bilinear form $a(.,.)$ in the space $\underline{H}_1$ is essential. This means : Is there a constant c_1 independent of $\underline{u}$ such that

$$\| \underline{u} \|_1^2 \leqslant c_1 \{ a(\underline{u}, \underline{u}) + \| \underline{u} \|^2 \} \tag{6}$$

holds for all $\underline{u} \in \underline{H}_1$. For more details we refer to Fichera [1972].

Because of the conditions $\lambda \geqslant 0$, $\mu > 0$ it is sufficient to have the inequality

$$\| \nabla\underline{u} \|^2 \leqslant c_2 \{ \| \varepsilon(\underline{u}) \|^2 + \| \underline{u} \|^2 \} := c_2 \{ (\varepsilon_{ik}(\underline{u}), \varepsilon_{ik}(\underline{u})) + (u_i, u_i) \}\,. \tag{7}$$

Korn's first inequality states the validity of (7) for $u \in \overset{\circ}{\underline{H}}_1$ even without the second term on the right hand side. The proof is quite simple leading to the explicite bound $c_2 = 2$, see Friedrichs [1947] or Velte [1976], p. 67.

In the literature the proof in the general case, i.e. Korn's second inequality, is referred to " anything but trival ", see Fichera [1972], p. 382 or Ciarlet [1977], p. 24. The original proof of Korn [1909] is doubtful. In the meantime proofs are given by Duvaut-Lions [1972], p. 110, Fichera [1972], Friedrichs [1947], and Paine-Weinberger [1961]. Besides the different assumptions concerning the boundary, for instance only the proof of Fichera covers domains with corners, all these proofs are quite sophisticated. In order even to understand them a profound knowledge of the theory of partial differential equations is necessary.

The aim of this paper are three elementary proofs under different assumptions. In section 1 we treat the case of a domain with C^1-boundary. The special case of a polygonal domain in two dimensions is handled in section 2. This could be generalized to domains with piecewise smooth boundaries in arbitrary many dimensions but omitted here. The general case of bounded Lipschitz-domains is discussed in section 3. The main tool is the construction of extension operators $E : \underline{H}_1(\Omega) \to \overset{\circ}{\underline{H}}_1(\mathbb{R}^N)$ which are strain-preserving, i.e. such that an inequality of the type

$$\| \varepsilon(E\underline{u}) \|_{\mathbb{R}^N} \leqslant c_3 \{ \| \varepsilon(\underline{u}) \|_\Omega + \| \underline{u} \|_\Omega \} \tag{8}$$

holds true. The extensions are appropriate modifications of those typical in the elliptic theory, see Fichera [1965], p. 52 and Stein [1970], p. 180.

1. THE CASE OF SMOOTH DOMAINS

In this section we prove the validity of Korn's inequality for domains with C^1-boundaries in the following way : Firstly we construct a strain-energy preserving extension operator from the upper half-space to the whole space. In this way a simple proof is given for the half-space. Secondly we generalize this result to special C^1-domains and thirdly we turn over to the general case.

LEMMA 1 : *To any $\underline{u}$ defined in $\mathbb{R}^N_+$ there is a reflection $\underline{\tilde{u}}$ defined in $\mathbb{R}^N_-$ such that the extension operator E defined by*

$$E\underline{u} = \begin{cases} \underline{u} \text{ in } \mathbb{R}^N_+ \\ \underline{\tilde{u}} \text{ in } \mathbb{R}^N_- \end{cases}, \tag{9}$$

fulfills.

PROPOSITION 1.1 : *E maps $\underline{C}^0(\overline{\mathbb{R}}^N_+)$ into $\underline{C}^0(\mathbb{R}^N)$.*

PROPOSITION 1.2 : *E maps $\dot{\underline{H}}_1(\mathbb{R}^N_+)$ into $\overset{\circ}{\underline{H}}_1(\mathbb{R}^N)$ such that in addition*

$$\| \varepsilon(E\underline{u}) \|_{\mathbb{R}^N} \leqslant c_4 \| \varepsilon(\underline{u}) \|_{\mathbb{R}^N_+} \tag{10}$$

holds true.

Remark 1 : The constant c_4 is independent of the radius of the sphere outside of which $\underline{u}$ vanishes.

Remark 2 : Because of Korn's first inequality we get

$$\begin{aligned} \| \nabla \underline{u} \|_{\mathbb{R}^N_+} &\leqslant \| \nabla(E\underline{u}) \|_{\mathbb{R}^N} \\ &\leqslant 2 \| \varepsilon(E\underline{u}) \|_{\mathbb{R}^N} \\ &\leqslant 2\, c_4 \| \varepsilon(\underline{u}) \|_{\mathbb{R}^N_+} . \end{aligned} \tag{11}$$

This implies Korn's second inequality in the space $\dot{\underline{H}}_1(\mathbb{R}^N_+)$.

Proof of lemma 1 : We will use the splitting $x = (\xi, \xi')$ with $\xi = (x_1, \dots, x_{N-1})$ and $\xi' = x_N$. Further let α, β denote indices within the range $(1, \dots, N - 1)$. Now we put

$$\begin{aligned} \tilde{u}_\alpha &= p u_\alpha^\lambda + q u_\alpha^\mu , \\ \tilde{u}_N &= r u_N^\lambda + s u_N^\mu . \end{aligned} \tag{12}$$

Here p, q, r, s and λ, μ — the last in no context with the Lamé constants — are parameters to be fixed later and the upper indices λ, μ denote for any function v

$$\begin{aligned} v^\lambda &= v(\xi, -\lambda\xi') , \\ v^\mu &= v(\xi, -\mu\xi') . \end{aligned} \tag{13}$$

The conditions

$$\lambda > 0 , \qquad \mu > 0 \tag{14}$$

guarantee that $\tilde{u}_i$ is well defined for $x \in \mathbb{R}_-^N$. Proposition 1.1 holds in case of

$$\begin{aligned} p + q &= 1 , \\ r + s &= 1 . \end{aligned} \tag{15}$$

Then, of course, the first assertion of proposition 1.2 is also valid.
Obviously we have

$$\varepsilon_{\alpha\beta}(\tilde{\underline{u}}) = p\varepsilon_{\alpha\beta}(\underline{u})^\lambda + q\varepsilon_{\alpha\beta}(\underline{u})^\mu , \tag{16}$$

$$\varepsilon_{NN}(\tilde{\underline{u}}) = -\lambda r\varepsilon_{NN}(\underline{u})^\lambda - \mu s\varepsilon_{NN}(\underline{u})^\mu . \tag{17}$$

Thus only the pairs (α, N) remain to be considered. We get

$$2\,\varepsilon_{\alpha N}(\tilde{\underline{u}}) = -\lambda p u_{\alpha|N}^\lambda - \mu q u_{\alpha|N}^\mu + r u_{N|\alpha}^\lambda + s u_{N|\alpha}^\mu . \tag{18}$$

The choice

$$r = -\lambda p , \qquad s = -\mu q \tag{19}$$

leads to

$$\varepsilon_{\alpha N}(\tilde{\underline{u}}) = r\varepsilon_{\alpha N}(\underline{u})^\lambda + s\varepsilon_{\alpha N}(\underline{u})^\mu . \tag{20}$$

For $\lambda \neq \mu$ the parameters p, q, r, s are uniquely defined. A possible choice is $\lambda = 2$, $\mu = 1$, $p = -2$, $q = 3$, $r = 4$, $s = -3$. Having fixed the parameters we come from (16), (17) and (20) to (10) with some constant c_4.

Next we extend lemma 1 to C^1-domains of the following type. Let $f(\xi)$ be a given C^1-function in $N-1$ variables with the scaling

$$f(0) = 0\,, \qquad f_{|\alpha}(0) = 0\,. \tag{21}$$

Then we consider the domain

$$\Omega_+ = \{ x \mid x = (\xi, \xi') \wedge \xi' > f(\xi) \} \tag{22}$$

and in addition

$$\Omega_+^R = \Omega_+ \cap K_R(0) \tag{23}$$

with

$$K_R(0) = \{ x \mid |x| < R \}\,. \tag{24}$$

We will use notations like $\underline{u} \in \underline{H}_1^R(\Omega_+)$ if $\underline{u}$ is in $\underline{H}_1(\Omega_+)$ and vanishes outside Ω_+^R.

LEMMA 2 : *There exists an extension operator E which fulfills.*

PROPOSITION 2.1 : *E maps $\underline{C}^0(\overline{\Omega}_+)$ into $\underline{C}^0(\mathbb{R}^N)$.*

PROPOSITION 2.2 : *E maps $\underline{H}_1^R(\Omega_+)$ into $\overset{\circ}{\underline{H}}_1(\mathbb{R}^N)$ such that in addition*

$$\| \varepsilon(E\underline{u}) \|_{\mathbb{R}^N} \leqslant c_5 \| \varepsilon(\underline{u}) \|_{\Omega_+} + c_6\, \kappa \| \nabla \underline{u} \|_{\Omega_+} \tag{25}$$

holds true with two numerical constants c_5, c_6 and κ defined by

$$\kappa = \sup \{ | f_{|\alpha}(\xi) | \mid \alpha = 1, \ldots, N-1 \wedge |\xi| < R \}\,. \tag{26}$$

Remark 3 : Similar to remark 2 we get now

$$\| \nabla u \|_{\Omega_+} \leqslant 2\, c_5 \| \varepsilon(\underline{u}) \|_{\Omega_+} + 2\, c_6\, \kappa \| \nabla \underline{u} \|_{\Omega_+}\,. \tag{27}$$

Thus for $\kappa < 1/(2\, c_6)$ we get Korn's second inequality in the space $\underline{H}_1^R(\Omega_+)$.

Proof of lemma 2 : We use the definition of $\tilde{u}$ (12) but now with the meaning

$$\begin{aligned} v^\lambda &= v(\xi, f(\xi) + \lambda(f(\xi) - \xi'))\,, \\ v^\mu &= v(\xi, f(\xi) + \mu(f(\xi) - \xi'))\,. \end{aligned} \tag{28}$$

In the relations (16), (17), (20) correction terms enter now, we have e.g.

$$\begin{aligned} \varepsilon_{\alpha\beta}(\underline{\tilde{u}}) = p\varepsilon_{\alpha\beta}(\underline{u})^\lambda + q\varepsilon_{\alpha\beta}(\underline{u})^\mu + \frac{1}{2} \{ p(1+\lambda) f_{|\beta}\, u^\lambda_{\alpha|N} + p(1+\lambda) f_{|\alpha}\, u^\lambda_{\beta|N} + \\ + q(1+\mu) f_{|\beta}\, u^\mu_{\alpha|N} + q(1+\mu) f_{|\alpha}\, u^\mu_{\beta|N} \}\,. \end{aligned} \tag{29}$$

Since these additional terms are bounded in norm by $c\kappa \, \| \nabla \underline{u} \|_{\Omega_+}$ the lemma is proven.

In the last step now let Ω be a bounded domain with boundary $\partial\Omega$ continuously differentiable. We may cover $\partial\Omega$ by a set of spheres K_ν with centers on $\partial\Omega$ such that $\Omega_\nu = \Omega \cap K_\nu$ coincides after a translation and rotation with a domain of the above type and such that the corresponding κ_ν are less than $1/(2\, c_6)$. Further we choose a domain $\Omega_0 \subset \subset \Omega$ such that Ω_0 and the Ω_ν are a covering of Ω. Let φ_ν be a partition of unity with respect to the Ω_ν, i.e. $\varphi_\nu \in C^\infty$, supp $(\varphi_\nu) \subseteq \Omega_\nu$ and $\Sigma\varphi_\nu = 1$. We introduce the splitting

$$\underline{u} = \Sigma \underline{u}^\nu \quad \text{with} \quad \underline{u}^\nu = \varphi_\nu \, \underline{u} \,. \tag{30}$$

Applying Korn's first inequality to $\underline{u}^0$ and lemma 2 to $\underline{u}^\nu$ for $\nu \geqslant 1$ we come to

$$\| \nabla \underline{u}^\nu \| \leqslant c_7 \, \| \varepsilon(\underline{u}^\nu) \| \tag{31}$$

with a numerical constant c_7. Because of

$$\varepsilon_{ik}(\underline{u}^\nu) = \varphi_\nu \, \varepsilon_{ik}(\underline{u}) + \frac{1}{2}(u_i \, \varphi_{\nu\,|\,k} + u_k \, \varphi_{\nu\,|\,i}) \tag{32}$$

we get

$$\| \varepsilon(\underline{u}^\nu) \| \leqslant c_8 \, \{ \, \| \varepsilon(\underline{u}) \| + \| \underline{u} \| \, \} \tag{33}$$

with c_8 depending on the covering and henceforth on $\partial\Omega$ and N.

Because of (30), (31) and (33) we have proved the validity of (7) resp. Korn's second inequality in the case of a smooth domain.

2. THE CASE OF POLYGONAL DOMAINS

For simplicity we restrict ourselves to $N = 2$ dimensions in this section. Now the variable will be denoted by (x, y) and the displacement-vector by (u, v). The proof of Korn's second inequality will follow the lines of Section 1 but in the present case with the omission of the second step. The counterpart of lemma 1 is

LEMMA 3 : *Let* (u, v) *be defined in the angular domain*

$$\Omega_+ = \{ \, (x, y) \mid x > 0 \wedge y > \gamma x \, \} \,. \tag{34}$$

There is a reflection $(\tilde{u}, \tilde{v})$ *defined in*

$$\Omega_- = \{ \, (x, y) \mid x > 0 \wedge y < \gamma x \, \} \tag{35}$$

such that the extension operator $\hat{E}$ defined by

$$\hat{E}(u, v) = \begin{cases} (u, v) \text{ in } \Omega_+ \\ (\tilde{u}, \tilde{v}) \text{ in } \Omega_- \end{cases}, \tag{36}$$

fulfills.

PROPOSITION 3.1 : *$\hat{E}$ maps $C^0(\overline{\Omega}_+)$ into $C^0(\overline{\mathbb{R}}_r^2)$ with*

$$\mathbb{R}_r^2 = \{ (x, y) \mid x > 0 \} . \tag{37}$$

PROPOSITION 3.2 : *$\hat{E}$ maps $\dot{H}_1(\Omega_+)$ into $\dot{H}_1(\mathbb{R}_r^2)$ such that*

$$\| \varepsilon(\hat{E}(u, v)) \|_{\mathbb{R}_r^2} \leqslant c_9 \| \varepsilon(u, v) \|_{\Omega_+} . \tag{38}$$

Remark 4 : Let $(u, v) \in \underline{\dot{H}}_1(\Omega_+)$ be given. Then $\hat{E}(u, v)$ is defined in $\mathbb{R}_r^2$ and belongs to $\underline{\dot{H}}_1(\mathbb{R}_r^2)$. The extension E of lemma 1 applied to $\hat{E}(u, v)$ leads to $(\tilde{\tilde{u}}, \tilde{\tilde{v}}) = E \circ \hat{E}(u, v) \in \underline{\overset{\circ}{H}}_1(\mathbb{R}^2)$. We get

$$\begin{aligned} \| \nabla(u, v) \|_{\Omega_+} &\leqslant \| \nabla(\tilde{\tilde{u}}, \tilde{\tilde{v}}) \|_{\mathbb{R}^2} \\ &\leqslant 2 \| \varepsilon(\tilde{\tilde{u}}, \tilde{\tilde{v}}) \|_{\mathbb{R}^2} \\ &\leqslant 2 c_4 \| \varepsilon(\hat{E}(u, v)) \|_{\mathbb{R}^2} \\ &\leqslant 2 c_4 c_9 \| \varepsilon(u, v) \|_{\Omega_+} . \end{aligned} \tag{39}$$

This implies Korn's second inequality in the space $\underline{\dot{H}}_1(\Omega_+)$.

Proof of lemma 3 : Similar to the proof of lemma 1 we will use a reflection parallel to the y-axis. Corresponding to (12) we need two additional terms. For $(x, y) \in \Omega_-$ we put

$$\begin{aligned} \tilde{u} &= pu^\lambda + qu^\mu + \rho v^\lambda + \sigma v^\mu , \\ \tilde{v} &= rv^\lambda + sv^\mu . \end{aligned} \tag{40}$$

In the present case the upper index λ (and similar μ) indicates for a function w

$$w^\lambda = w(x, \gamma x + \lambda(\gamma x - y)) . \tag{41}$$

Proposition 3.1 and the first assertion of proposition 3.2 is met if the conditions

$$p + q = 1 , \qquad r + s = 1 , \qquad \rho + \sigma = 0 \tag{42}$$

hold true.

Since the definition of v resp. u_N is unaltered formula (17) is valid. But now ε_{11} as well as ε_{12} are to be considered. We get

$$\varepsilon_{11}(\tilde{u}, \tilde{v}) = p(u_x + \gamma(1+\lambda)\, u_y)^\lambda + q(u_x + \gamma(1+\mu)\, u_y)^\mu + \\ + \rho(v_x + \gamma(1+\lambda)\, v_y)^\lambda + \sigma(v_x + \gamma(1+\mu)\, v_y)^\mu . \quad (43)$$

Our aim is to choose the parameters such that the right hand side is a linear combination of $\varepsilon_{ik}(u, v)^{\lambda,\mu}$. The conditions are

$$\rho = p\gamma(1+\lambda), \qquad \sigma = q\gamma(1+\mu). \quad (44)$$

Then we have

$$\varepsilon_{11}(\tilde{u}, \tilde{v}) = p\varepsilon_{11}^\lambda + q\varepsilon_{11}^\mu + 2\,\rho\varepsilon_{12}^\lambda + 2\,\sigma\varepsilon_{12}^\mu + \\ + \rho\gamma(1+\lambda)\,\varepsilon_{22}^\lambda + \sigma\gamma(1+\mu)\,\varepsilon_{22}^\mu . \quad (45)$$

In the same way we get

$$2\,\varepsilon_{12}(\tilde{u}, \tilde{v}) = -\lambda(pu_y + \rho v_y)^\lambda - \mu(qu_y + \sigma v_y)^\mu + \\ + r(v_x + \gamma(1+\lambda)\, v_y)^\lambda + s(v_x + \gamma(1+\mu)\, v_y)^\mu . \quad (46)$$

If the conditions (19) are fulfilled we get

$$2\,\varepsilon_{12}(\tilde{u}, \tilde{v}) = 2\, r\varepsilon_{12}^\lambda + 2\, s\varepsilon_{12}^\mu + (r\gamma(1+\lambda) - \lambda\rho)\,\varepsilon_{22}^\lambda + \\ + (s\gamma(1+\mu) - \mu\sigma)\,\varepsilon_{22}^\mu . \quad (47)$$

Concerning the conditions on p, q, r, s they are the same as in section 1. Let ρ, σ be defined by (44). Then (42.3) is valid automatically.

Since $\varepsilon_{ik}(\tilde{u}, \tilde{v})$ are linear combinations of $\varepsilon_{ik}(u, v)^{\lambda,\mu}$ we get

$$\| \varepsilon(\tilde{u}, \tilde{v}) \|_{\Omega_-} \leqslant c \, \| \varepsilon(u, v) \|_{\Omega_+} \quad (48)$$

what completes the proof of lemma 3.

The remaining step in order to prove Korn's inequality is similar to above. Let Ω be a polygonal domain. We use a covering of $\partial\Omega$ be circles K_ν with centres on $\partial\Omega$ such that $\Omega \cap K_\nu$ is either a half-circle or an angular domain intersected with a circle. Then by means of a partition of unity and the splitting (30) we apply Korn's first inequality, lemma 1 and lemma 3.

3. THE CASE OF LIPSCHITZ DOMAINS

In analogy to section 1 let $f(\xi)$ be now a Lipschitz-function in $N - 1$ variables and let the " special Lipschitz domain " Ω_+ be defined by

$$\Omega_+ = \{ x \mid x = (\xi, \xi') \wedge \xi' > f(\xi) \} . \quad (49)$$

The complement $\mathbb{R}^N\text{-}\Omega_+$ is denoted by Ω_-. The counterpart of lemma 1 is :

LEMMA 4 : *To any $\underline{u}$ defined in Ω_+ there is a reflection $\underline{\tilde{u}}$ defined in Ω_- such that the corresponding extension operator E fulfills.*

PROPOSITION 4.1 : *E maps $\underline{C}^0(\overline{\Omega}_+)$ into $\underline{C}^0(\mathbb{R}^N)$.*

PROPOSITION 4.2 : *E maps $\underline{\dot{H}}_1(\Omega_+)$ into $\underline{\overset{\circ}{H}}_1(\mathbb{R}^N)$ such that in addition*

$$\| \varepsilon(E\underline{u}) \|_{\mathbb{R}^N} \leqslant c_{10} \| \varepsilon(\underline{u}) \|_{\Omega_+} \tag{50}$$

holds true with c_{10} depending on the dimension N and the Lipschitz-constant M of the function f.

Remark 5 : In analogy to remarks 2-4 then we have Korn's second inequality in the space $\underline{\dot{H}}_1(\Omega_+)$.

Proof of lemma 4 : In order to construct the reflection we need the concept of the generalized distance. For $x \in \Omega_-$ let $d(x)$ denote the distance to Ω_+, i.e.

$$d(x) = \inf \{ \mid x - y \mid \mid y \in \Omega_+ \} . \tag{51}$$

By elementary geometrical considerations we get

$$0 < (1 + M^2)^{-1/2} (f(\xi) - \xi') \leqslant d(x) \leqslant f(\xi) - \xi' . \tag{52}$$

As long as f is only Lipschitz-bounded the distance has no higher regularity. It can be increased by an appropriate smoothing process. We will not go into details but refer to Stein [1970], p. 171 :

There is a C^∞-function $\delta(x)$ defined in Ω_- such that

$$0 < 2(f(\xi) - \xi') \leqslant \delta(x) \leqslant c_{11}(f(\xi) - \xi') . \tag{53}$$

Moreover, there are bounds for the derivatives of the type

$$| D^a \delta(x) | \leqslant c(| a |) \, \delta(x)^{1-|a|} . \tag{54}$$

Remark 6 : The choice of the factor 2 in (53) is advantageous here.

Remark 7 : Actually we need only the first and second derivatives of δ and (54) for $| a | \leqslant 2$.

The reflection we will work with is

$$\tilde{u}_i(x) := \int_1^2 \psi(\lambda) \{ u_i(x_\lambda) + \lambda \, \delta_{|i}(x) \, u_N(x_\lambda) \} \, d\lambda . \tag{55}$$

Here $\psi(\lambda)$ is a weight-function and

$$x_\lambda = (\xi, \xi')_\lambda = (\xi, \xi' + \lambda\, \delta(x)) . \tag{56}$$

Because of (53) for $x \in \Omega_-$ the points x_λ are in Ω_+ for $\lambda \geqslant 1$, i.e. $\underline{\tilde u}$ is defined.

There is some ambiguity in the choice of ψ. In our case the conditions $\psi \in C^0[1, 2]$ and

$$\int_1^2 \psi(\lambda)\, d\lambda = 1, \int_1^2 \lambda\psi(\lambda)\, d\lambda = 0 \tag{57}$$

are sufficient.

Obviously $\underline{\tilde u} \in C^0(\overline{\Omega}_-)$ is a consequence of $u \in C^0(\overline{\Omega}_+)$. Now let $x_\nu = (\xi, \xi'_\nu)$, $\xi'_\nu < f(\xi)$ converge to $\overline{x} = (\xi, f(\xi))$. Because of the choice of ψ we get with $x \in \Omega_-$ arbitrary

$$u_\iota(\overline{x}) = \int_1^2 \psi(\lambda)\, \{\, u_\iota(\overline{x}) + \lambda \delta_{|\iota}(x)\, u_N(\overline{x}) \,\}\, d\lambda . \tag{58}$$

By comparison with (55) we find for $\underline{u}$ continuous in $\overline{\Omega}_+$ at once $\underline{\tilde u}(x_\nu) \to \underline{u}(\overline{x})$, i.e. proposition 4.1 and the first assertion of proposition 4.2 are shown.

By direct computation we get from (55)

$$\varepsilon_{\iota k}(\underline{\tilde u}(x)) = \int_1^2 \psi(\lambda)\, \{\, \varepsilon^\lambda_{\iota k} + \lambda\, \delta_{|\iota}\, \varepsilon^\lambda_{kN} + \lambda\, \delta_{|k}\, \varepsilon^\lambda_{\iota N} + \\ + \lambda^2\, \delta_{|\iota}\, \delta_{|k}\, \varepsilon^\lambda_{NN} + \lambda\, \delta_{|\iota k}\, u^\lambda_N \,\}\, d\lambda \tag{59}$$

with $\varepsilon^\lambda_{\iota k} = \varepsilon_{\iota k}(\underline{u}(x_\lambda))$. On the right hand side all terms except the last one are bounded by (numerical) multiples of integrals of the type

$$\tilde e(x) = \int_1^2 |\, e(x_\lambda) \,|\, d\lambda \tag{60}$$

with e being one of the strain components $\varepsilon_{\iota k}(\underline{u})$.

In the last term u_N instead of ε enters on the one hand and $\delta_{|\iota k}$ will not be bounded on the other hand. But still we can find a bound of the above type. Taylor's formula gives

$$u_N(x_\lambda) = u_N(x_1) + \delta \int_1^\lambda u_{N|N}(x_\mu)\, d\mu \tag{61}$$

leading because of (57_2) to

$$\delta_{|ik}(x) \int_1^2 \lambda\psi(\lambda)\, u_N(x_\lambda)\, d\lambda = \\ = \delta(x)\, \delta_{|ik}(x) \int_1^2 \varepsilon_{NN}(x_\mu)\, d\mu \int_\mu^2 \lambda\psi(\lambda)\, d\lambda\,. \quad (62)$$

The proof of lemma 4 is complete if the $L_2(\Omega_-)$-norm of $\tilde{e}$ is bounded up to a constant by the $L_2(\Omega_+)$-norm of e. The main part will be the integration with respect to the last variable. Let ξ be fixed. We will use a translation such that $f(\xi) = 0$. Then omitting the dependence on ξ we have for $x = (\xi, \xi') \in \Omega_-$, i.e. $\xi' < 0$

$$\tilde{e}(\xi') = \int_1^2 |\, e(\xi' + \lambda\, \delta(\xi'))\,|\, d\lambda = \int_{\xi'+\delta(\xi')}^{\xi'+2\delta(\xi')} \delta^{-1}(\xi')\, |\, e(s)\,|\, ds\,. \quad (63)$$

We find with the aid of (53)

$$\delta^{-1}(\xi') \leqslant \frac{1}{2} |\, \xi'\,|^{-1}, \xi' + \delta(\xi') \geqslant |\, \xi'\,|, \xi' + 2\,\delta(\xi') \leqslant 2\, c_{11}\, |\, \xi'\,|\,. \quad (64)$$

Therefore we get from (63) with $t = -\xi'$, $c = 2\, c_{11}$

$$2\, \tilde{e}(-t) \leqslant t^{-1} \int_t^{ct} |\, e(s)\,|\, ds\,. \quad (65)$$

Schwarz' inequality with $|\, e\,| = s^{1/4} \{ s^{-1/4}\, |\, e\,| \}$ gives

$$4\, \tilde{e}^2(-t) \leqslant \frac{2}{3} c^{3/2}\, t^{-1/2} \int_t^{ct} s^{-1/2}\, e^2(s)\, ds\,. \quad (66)$$

Integration from 0 to ∞ and the interchange of the order of integration lead to

$$\int_{-\infty}^0 \tilde{e}^2(\xi')\, d\xi' \leqslant c_{12} \int_0^\infty s^{-1/2}\, e^2(s)\, ds \int_{c^{-1}s}^s t^{-1/2}\, dt \leqslant 2\, c_{12} \int_0^\infty e^2(s)\, ds\,. \quad (67)$$

Without the mentioned translation the limits 0 in the integrals are to be replaced by $f(\xi)$. Now we integrate with respect to ξ over $\mathbb{R}^{N-1}$ getting

$$\|\, \tilde{e}\, \|_{L_2(\Omega_-)} \leqslant c_{13}\, \|\, e\, \|_{L_2(\Omega_+)}\,. \quad (68)$$

In this way lemma 4 is proved.

In order to get Korn's second inequality for general Lipschitz domains the arguments at the end of section 1 have to be applied. We will not repeat them here once more.

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