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ERROR ESTIMATES FOR THE ASSUMED STRESSES HYBRID METHODS IN THE APPROXIMATION OF 4TH ORDER ELLIPTIC EQUATIONS (*)

by Alfio QUARTERONI [1]

Communiqué par E. MAGENES

Abstract. — *We analyse the* assumed stresses *hybrid approximation of the biharmonic problem. We provide optimal error bounds for the displacements in the L^2-norm and in the energy norm.*

Résumé. — *On analyse une approximation du problème biharmonique par éléments finis hybrides du type* « assumed stresses ». *On donne une estimation optimale de l'erreur pour les déplacements, dans la norme L^2 et dans la norme de l'énergie.*

INTRODUCTION

In this paper we consider the *assumed stresses* (or *dual*) hybrid methods due to Pian and Tong (*see* [10]) for the approximation of the model problem:

$$\text{(P)} \qquad w \in H_0^2(\Omega), \qquad \Delta^2 w = p \quad \text{in } \Omega,$$

where Ω is a bounded convex polygonal domain of $\mathbb{R}^2$ and $p \in L^2(\Omega)$. For any regular decomposition $\mathscr{T}_h$ of Ω the problem (P) is translated into a saddle point problem in which two unknowns are independently approximated:

(i) a function ψ biharmonic at the interior of any $K \in \mathscr{T}_h$, having the same traces of w on the internal boundaries $\{\partial K\}$;

(ii) the field $\sigma = (\sigma_{ij})$ $(i, j = 1, 2)$ of the second derivatives of w inside each K.

The numerical analysis of such a method has been done in [3]; a very large family of discretizations is constructed and sufficient conditions are given in order to have convergence. Moreover, if ψ_h and σ_h are the approximate solutions, optimal error bounds for ψ-ψ_h and $\sigma - \sigma_h$ in the norms of $H_0^2(\Omega)$ and $L^2(\Omega)$ are found.

Here we study the order of convergence of $\psi - \psi_h$ in the L^2-norm. Let $\nu = \nu(\Omega) \in]0, 1]$ be a real number such that: if $\Phi \in H_0^2(\Omega)$ and $\Delta^2 \Phi \in L^2(\Omega)$ then $\Phi \in H_0^2(\Omega) \cap H^{3+\nu}(\Omega)$. Setting $E = \psi - \psi_h$ we prove that $\| E \|_{L^2(\Omega)} = O(h^{1+\nu}) . \| E \|_{H_0^2(\Omega)}$.

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We remark that the computation of the known terms is one of the main difficulties arising in the practical solution of the dual hybrid problem (*cf.* [3], [8], [9]). Here we also propose a different theoretical approach in order to avoid the above mentioned difficulty. Such an approach is based on the idea, firstly presented in [1], to consider the dual hybrid formulation as a pure displacement formulation of the model problem (P). Let $\tilde{\psi}_h$ be the approximate solution obtained in this way; setting $\tilde{E}=\psi-\tilde{\psi}_h$ we prove:

$$\|\tilde{E}\|_{H_0^2(\Omega)}\sim\|E\|_{H_0^2(\Omega)} \quad \text{and} \quad \|\tilde{E}\|_{L^2(\Omega)}\sim\|E\|_{L^2(\Omega)}.$$

These results are applied in some nonlinear problems in elasticity (*cf.* [11]).

The scheme of the paper is the following: in section 1 we summarize the main results related to the assumed stresses hybrid methods and obtained in [3]. Section 2 is devoted to the study of the norm of E in $L^2(\Omega)$; in section 3 we provide the estimates for E in the L^2-norm and in the energy norm.

1. THE HYBRID PROBLEM: PRELIMINARY RESULTS

Let Ω be a convex polygonal domain of $\mathbb{R}^2$ and $\partial\Omega$ be its boundary; moreover let p be a function in $L^2(\Omega)$ and consider the problem

$$w\in H_0^2(\Omega), \qquad \Delta^2 w=p \quad \text{in } \Omega. \tag{1.1}$$

Let us define

$$W=H_0^2(\Omega), \tag{1.2}$$

$$H=\{v=(v_{ij})\,|\,v_{ij}\in L^2(\Omega),\ v_{12}=v_{21}\,(i,j=1,2)\}. \tag{1.3}$$

According to their arguments the norm notations shall apply to different spaces: namely, for any positive real number s $\|\cdot\|_s$ will denote the norm of $H^s(\Omega)$ and $H\cap(H^s(\Omega))^4$. Moreover $\|\cdot\|_0$ is the norm of $L^2(\Omega)$ (for the definitions and the properties of the Sobolev and Hilbert spaces here used *see*, e.g., [7]). Let

$$\|\varphi\|=|\varphi|_{2,\Omega}=\left(\int_\Omega \varphi_{/ij}\,\varphi_{/ij}\ dx\right)^{1/2} \tag{1.4}$$

be the norm of W; here and in the following we adopt the convention of summation on repeated indices; moreover: $\cdot_{/i}=\partial/\partial x_i\,(i=1,2)$ and $dx=dx_1\,dx_2$. Let

$$((u,v))=\int_\Omega u_{/ij}\,v_{/ij}\,dx \tag{1.5}$$

be the inner product of H and

$$|v| = \sqrt{((v, v))} \tag{1.6}$$

be the norm of H. Let T be the second derivative operator

$$T: \quad W \to H, \qquad v = T\varphi \quad \text{iff} \quad v_{ij} = \varphi_{/ij} \quad (i, j = 1, 2) \tag{1.7}$$

and T^* be the formal adjoint of T:

$$\langle T^* v, \Phi \rangle = ((v, T\varphi)), \qquad \forall \varphi \in W, \quad \forall v \in H; \tag{1.8}$$

$\langle\ ,\ \rangle$ denotes the duality between W' and W. Then $\Delta^2 = T^* T$ and from the definitions (1.4) and (1.6) we get

$$\forall \varphi \in W, \quad \|\varphi\| = |T\varphi|. \tag{1.9}$$

Let h be a parameter which tends to zero and $\{\mathscr{T}_h\}$ be a family of regular decompositions of Ω in convex polygons $\{K\}$. Define

$$W^*(\mathscr{T}_h) = \{\varphi \in W \mid \Delta^2 \varphi = 0 \text{ in each } K \in \mathscr{T}_h\}, \tag{1.10}$$

$$H^*(\mathscr{T}_h) = \{v \in H \mid T^* v = 0 \text{ in each } K \in \mathscr{T}_h\}. \tag{1.11}$$

Since $W^*(\mathscr{T}_h)$ and $H^*(\mathscr{T}_h)$ are closed subspaces respectively of W and H they will be equipped by the same norms of W and H. Let f be any function satisfying

$$f \in H, \quad T^* f = p \quad \text{in each } K \in \mathscr{T}_h. \tag{1.12}$$

We consider the problem

$$\left.\begin{array}{ll} & \{\sigma, \psi\} \in H^*(\mathscr{T}_h) \times W^*(\mathscr{T}_h), \\ 1. & \forall v \in H^*(\mathscr{T}_h), \quad ((\sigma + f, v)) - ((v, T\psi)) = 0, \\ 2. & \forall \varphi \in W^*(\mathscr{T}_h), \quad ((\sigma + f, T\varphi)) = (p, \varphi), \end{array}\right\} \tag{1.13}$$

$(\ ,\ )$ is the inner product of $L^2(\Omega)$. The problems (1.1) and (1.13) are equivalent in the sense that both have a unique solution and between w and $\{\sigma, \psi\}$ the following relations hold

$$\left.\begin{array}{c} \sigma + f = Tw, \\ \psi = w \quad \text{on } \Sigma, \\ \psi_{/i} = w_{/i} \quad \text{on } \Sigma, \qquad \Sigma = \{\cup \partial K : K \in \mathscr{T}_h\}. \end{array}\right\} \tag{1.14}$$

Let now H_h and W_h be two finite dimensional subspaces of $H^*(\mathscr{T}_h)$ and $W^*(\mathscr{T}_h)$ respectively, and consider the following approximation of problem (1.13):

$$\left.\begin{array}{ll} & \{\sigma_h, \psi_h\} \in H_h \times W_h, \\ 1. & \forall v_h \in H_h \quad ((\sigma_h + f, v_h)) - ((v_h, T\psi_h)) = 0, \\ 2. & \forall \varphi_h \in W_h \quad ((\sigma_h + f, T\varphi_h)) = (p, \varphi_h). \end{array}\right\} \quad (1.15)$$

From the abstract theory of [2] we have the result:

LEMMA 1 : *Suppose that H_h and W_h satisfy the* compatibility condition:

$$\text{(CC)} \quad \begin{cases} \text{there exists } \gamma > 0 \text{ independent of } h \text{ such that:} \\ \forall \varphi_h \in W_h, \quad \mathrm{Sup}\{|v_h|^{-1}((v_h, T\varphi_h)); v_h \in H_h - \{0\}\} \geqq \gamma \|\varphi_h\|. \end{cases}$$

Then (1.15) *has a unique solution which satisfies*

$$|\sigma - \sigma_h| + \|\psi - \psi_h\| \leqq C\{\operatorname*{Inf}_{v_h \in H_h} |\sigma - v_h| + \operatorname*{Inf}_{\varphi_h \in W_h} \|\psi - \varphi_h\|\}. \ \square \quad (1.16)$$

Throughout this paper C denotes a generic positive constant independent of the decomposition $\mathscr{T}_h$.

Let r, s, m be some positive integers satisfying: $r \geqq 3, s \geqq 1, m \geqq 1$ and define two families of subspaces of $W^*(\mathscr{T}_h)$ and $H^*(\mathscr{T}_h)$:

$$W_h(r, s) = \left\{ \varphi_h \in W^*(\mathscr{T}_h) \,|\, \varphi_h|_{\partial K} \in P^r(\partial K), \left.\frac{\partial \varphi_h}{\partial n}\right|_{\partial K} \in P^s(\partial K), \forall K \in \mathscr{T}_h \right\}, \quad (1.17)$$

$$H_h(m) = \left\{ v_h \in H^*(\mathscr{T}_h) \,|\, v_h|_K \in (P^m(K))^4, \forall K \in \mathscr{T}_h \right\}. \quad (1.18)$$

For any K n is the outward normal direction to ∂K; $P^t(\partial K)$ is the space of polynomials of degree t on each side of ∂K, not necessarily continuous at the vertices; $P^t(K)$ is the space of polynomials of degree t inside K. Since condition "$\varphi_h \in W^*(\mathscr{T}_h)$" implies, in some sense, that φ_h and its first derivatives are continuous, on each side of the decomposition at least the values of φ_h and $\varphi_{h/i}$ $(i = 1, 2)$ at the vertices are to be imposed as degrees of freedom. Therefore it is reasonable to suppose $r \geqq 3$, $s \geqq 1$. Moreover in order to satisfy (CC) m must be chosen depending on the values of r, s. (In [3] these dependence conditions are specified). One can prove:

LEMMA 2: *Let $\{v, \varphi\} \in H^*(\mathscr{T}_h) \times W^*(\mathscr{T}_h)$ and suppose that for any $K \in \mathscr{T}_h$ $v_{ij}|_K \in H^{m+1}(K)$ $(i, j = 1, 2)$ and $\varphi|_K \in H^{q+2}(K)$ with $q = \min(r-1, s)$. Then*

$$\operatorname*{Inf}_{v_h \in H_h(m)} |v - v_h| \leqq Ch^{m+1} |v|_{m+1,h}, \quad (1.19)$$

$$\operatorname*{Inf}_{\varphi_h \in W_h(r,s)} \| \varphi - \varphi_h \| \leqq C h^q |\varphi|_{q+2,h}, \tag{1.20}$$

where

$$|v|_{m+1,h} = \left(\sum_{i,j=1}^{2} \sum_{K \in \mathcal{T}_h} |v_{ij}|^2_{m+1,K} \right)^{1/2},$$

$$|\varphi|_{q+2,h} = \Big(\sum_{K \in \mathcal{T}_h} |\varphi|^2_{q+2,K} \Big)^{1/2}. \quad \square$$

2. L^2-ERROR ESTIMATE FOR THE DISPLACEMENTS

In this section we are going to derive some estimates for the error $\|\psi - \psi_h\|_0$ where ψ and ψ_h are respectively solutions of (1.13) and (1.15). We remark that if $\psi|_K \in H^{q+2}(K)$ and $\sigma_{ij}|_K \in H^{m+1}(K)$ $(i, j = 1, 2)$ in any $K \in \mathcal{T}_h$ then from (1.16), (1.19) and (1.20) we get

$$\|\psi - \psi_h\| \leqq C h^\theta F(\psi, \sigma), \tag{2.1}$$

$$\theta = \min(q, m+1), \qquad F(\psi, \sigma) = |\psi|_{\theta+2,h} + |\sigma|_{\theta,h}. \tag{2.2}$$

Then trivially $\|\psi - \psi_h\|_0 = O(h^\theta)$; the following theorem improves this result:

THEOREM 1: *If Ω is a convex polygonal domain of R^2, under some suitable regularity assumptions for ψ and σ there exists a positive constant $\nu \in]0, 1]$ depending on Ω such that*

$$\|\psi - \psi_h\|_0 \leqq C h^{\min(s, 1+\nu)} \|\psi - \psi_h\| \tag{2.3}$$

Proof: Let us set $E = \psi - \psi_h$ and let Φ satisfy:

$$\Phi \in W, \qquad \Delta^2 \Phi = E \quad \text{in } \Omega. \tag{2.4}$$

There exists a real $\nu = \nu(\Omega) \in]0, 1]$ depending on the maximum angle of $\partial\Omega$ such that $\Phi \in W \cap H^{3+\nu}(\Omega)$ (*cf.* [5, 6]); moreover (*cf.* [7]):

$$\|\Phi\|_{3+\nu} \leqq C \|E\|_0. \tag{2.5}$$

Let us set $\chi = \Phi$ and $\tau = T\chi$; then $T^*\tau = E$ therefore $\{\chi, \tau\}$ satisfy

$$\left.\begin{array}{ll} & \{\chi, \tau\} \in W \times H, \\ 1. & \forall v \in H, \quad ((\tau, v)) - ((T\chi, v)) = 0, \\ 2. & \forall \varphi \in W, \quad ((\tau, T\varphi)) = (E, \varphi). \end{array}\right\} \tag{2.6}$$

One can easily check that (2.6) admits a unique solution. Let us consider an approximation of the above problem:

$$\left.\begin{array}{l}\{\chi_h, \tau_h\} \in W_h(r, s) \times H_h(m), \\ \forall v_h \in H_h(m), \quad ((\tau_h, v_h)) - ((T\chi_h, v_h)) = 0, \\ \forall \varphi_h \in W_h(r, s), \quad ((\tau_h, T\varphi_h)) = (E, \varphi_h).\end{array}\right\} \tag{2.7}$$

If (CC) is verified we have (*cf.* [2]):

$$|\tau-\tau_h| + \|\chi-\chi_h\| \leq C\{\inf_{v_h \in H_h(m)} |\tau - v_h| + \inf_{\varphi_h \in W_h(r, s)} \|\chi - \varphi_h\|\}. \tag{2.8}$$

From $(2.6)_1$ we get (setting $v=\sigma-\sigma_h$):

$$((\tau, \sigma-\sigma_h)) - ((\sigma-\sigma_h, T\chi)) = 0. \tag{2.9}$$

From $(1.13)_1$ and $(1.15)_1$ we obtain (setting $v_h=\tau_h$):

$$((\sigma-\sigma_h, \tau_h)) - ((\tau_h, TE)) = 0. \tag{2.10}$$

Finally $(1.13)_2$ and $(1.15)_2$ give (setting $\varphi_h=\chi_h$):

$$((\sigma-\sigma_h, T\chi_h)) = 0. \tag{2.11}$$

Now from (2.6) we get

$$\begin{aligned}\|E\|_0^2 &= ((\tau, TE)) = ((\tau-\tau_h, TE)) + ((\tau_h, TE)) \\ &= (\text{using } (2.10))\, ((\tau-\tau_h, TE)) + ((\sigma-\sigma_h, \tau)) + ((\sigma-\sigma_h, \tau_h-\tau)) \\ &= (\text{thanks to } (2.9))\, ((\tau-\tau_h, TE)) + ((\sigma-\sigma_h, T\chi)) + ((\sigma-\sigma_h, \tau_h-\tau)) \\ &= (\text{thanks to } (2.11))\, ((\tau-\tau_h, TE)) + ((\sigma-\sigma_h, T(\chi-\chi_h))) + ((\sigma-\sigma_h, \tau_h-\tau)).\end{aligned}$$

Thus

$$\|E\|_0^2 \leq C\{\|E\| . |\tau-\tau_h| + |\sigma-\sigma_h| (|\tau-\tau_h| + \|\chi-\chi_h\|). \tag{2.12}$$

Finally from (2.8) we get

$$\|E\|_0^2 \leq C(\inf_{v_h \in H_h(m)} |\tau-v_h| + \inf_{\varphi_h \in W_h(r, s)} \|\chi-\varphi_h\|)\, \|E\|. \tag{2.13}$$

We have the result:

LEMMA 3: *Let W and H be defined in* (1.2) *and* (1.3) *and $W_h(r, s)$, $H_h(m)$ be defined in* (1.17) *and* (1.18). *Then*

$$\left.\begin{array}{l}\inf_{\varphi_h \in W_h(r, s)} \|\varphi-\varphi_h\| \leq Ch^{\mu} \|\varphi\|_{\mu+2}, \\ \forall \varphi \in W \cap H^{\mu+2}(\Omega), \quad \forall \mu, \qquad 0 \leq \mu \leq \min(r-1, s, 2) = \min(s, 2),\end{array}\right\} \tag{2.14}$$

$$\left.\begin{array}{l}\operatorname*{Inf}_{v_h \in H_h(m)} |v - v_h| \leqq C h^{\mu} \| v \|_{\mu}, \\ \forall v \in H \cap (H^{\mu}(\Omega))^4, \quad \forall \mu, \qquad 0 \leqq \mu \leqq \min(m+1, 2) = 2. \end{array}\right\} \tag{2.15}$$

Proof: By an immediate application of the standard interpolation results (*cf.* e.g. [4]). □

Let us remark that from (2.5) and the definitions of χ, τ we get

$$\chi \in W \cap H^{3+\nu}(\Omega), \qquad \tau \in H \cap (H^{1+\nu}(\Omega))^4, \tag{2.16}$$

$$\| \tau \|_{1+\nu} + \| \chi \|_{3+\nu} \leqq C \| E \|_0 \tag{2.17}$$

An application of the above lemma leads to (recall that $1+\nu \leqq 2$):

$$\operatorname*{Inf}_{\varphi_h \in W_h(r, s)} \| \chi - \varphi_h \| \leqq C h^{\min(s, 1+\nu)} \| \chi \|_{3+\nu}, \tag{2.18}$$

$$\operatorname*{Inf}_{v_h \in H_h(m)} | \tau - v_h | \leqq C h^{1+\nu} \| \tau \|_{1+\nu}. \tag{2.19}$$

From (2.13), (2.17), (2.18) and (2.19) we finally get

$$\| E \|_0^2 \leqq C h^{\min(s, 1+\nu)} \| E \|_0 \| E \|$$

and (2.3) holds. □

3. DUAL HYBRID METHODS AS DISPLACEMENT METHODS FOR PROBLEM (1.1): ERROR ESTIMATES IN $L^2(\Omega)$ AND IN THE ENERGY NORM

A hybrid approximation for the model problem (1.1), equivalent to the one given in (1.15), can be obtained as a variant of the usual displacement approach. Let f be a function satisfying (1.12); moreover let $b\colon H \times W \to R$ be the continuous bilinear form defined by

$$b(v, \varphi) = \sum_{K \in \mathscr{T}_h} \int_{\partial K} (\varphi_{/r} v_{rs} n_s - \varphi v_{rs/s} n_r) \, d\gamma. \tag{3.1}$$

Let now φ be any function in W; from (1.18) and Green's formula we get

$$\langle \Delta^2 w, \varphi \rangle = \langle T^* T w, \varphi \rangle = ((Tw, T\varphi)),$$

$$(p, \varphi) = \sum_{K \in \mathscr{T}_h} \int_K T^* f . \varphi \, dx = ((f, T\varphi)) - b(f, \varphi).$$

From the above equalities and (1.1) we get

$$\left.\begin{array}{l} w \in W, \\ \forall \varphi \in W \quad ((Tw, T\varphi)) = ((f, T\varphi)) - b(f, \varphi). \end{array}\right\} \tag{3.2}$$

This problem is a weak displacement formulation of (1.1). Since the hybrid method is related to the approximation of w and its first derivatives at the interelement boundaries Σ we are looking for an approximation of the following function

$$\psi \in W^*(\mathcal{T}_h), \quad \psi = w, \quad \psi_{/i} = w_{/i} \quad (i=1, 2) \quad \text{on } \Sigma, \tag{3.3}$$

ψ is the second component of the solution of problem (1.13) [*cf.* (1.14)]. Again from a Green's formula we get

$$\forall \varphi \in W^*(\mathcal{T}_h), \quad ((Tw, T\varphi)) = \sum_{K \in \mathcal{T}_h} \int_{\partial K} (\varphi_{/rs} w_{/r} n_s - \varphi_{/rss} w n_r)\, dy$$

$$= \sum_{K \in \mathcal{T}_h} \int_{K} (\varphi_{/rs} \psi_{/r} n_s - \varphi_{/rss} \psi n_r)\, d\gamma = ((T\psi, T\varphi)).$$

Thus ψ solves the problem

$$\psi \in W^*(\mathcal{T}_h), \quad ((T\psi, T\varphi)) = ((f, T\varphi)) - b(f, \varphi), \qquad \forall \varphi \in W^*(\mathcal{T}_h). \tag{3.4}$$

Let us consider a conforming finite element approximation of the above problem:

$$\bar{\psi}_h \in W_h(r, s), \quad ((T\bar{\psi}_h, T\varphi_h)) = ((f, T\varphi_h)) - b(f, \varphi_h), \qquad \forall \varphi_h \in W_h(r, s). \tag{3.5}$$

From the definition of $W_h(r, s)$ we see that the terms $T\varphi_h$, with φ_h belonging to $W_h(r, s)$, are not piecewise polynomials, then the inner products appearing in (3.5) cannot be easily computed. On the other side, $b(f, \varphi_h)$ depends only on the values of f and φ_h over Σ, so these terms are computable. In order to avoid the above mentioned difficulty let us consider the space $H_h(m)$ previously used [*see* (1.18)]. We denote with Π the orthogonal projection operator from H upon $H_h(m)$; finally we consider instead of (3.5) the problem

$$\mathring{\psi}_h \in W_h(r, s), \quad ((\Pi T\mathring{\psi}_h, \Pi T\varphi_h)) = ((f, \Pi T\varphi_h)) - b(f, \varphi_h),$$
$$\forall \varphi_h \in W_h(r, s). \tag{3.6}$$

One can easily verify that in (3.6) all the terms are computable. From the Lax-Milgram lemma we get that (3.6) has a unique solution iff the following *Projection Hypothesis* holds

$$\text{(PH)} \qquad \begin{cases} \exists \alpha > 0 \text{ independent of } h \text{ such that:} \\ \forall \varphi_h \in W_h(r, s), \quad |T\varphi_h| \leq \alpha |\Pi T\varphi_h| \end{cases}$$

LEMMA 4: *The compatibility condition* (CC) *is equivalent to the projection hypothesis* (PH) (with $\alpha = 1/\gamma$).

Proof: (PH) $\Rightarrow$ (CC).

Let φ_h be any function of $W_h(r, s)$; since $\|\varphi_h\| = |T\varphi_h|$ setting $\bar{v}_h = \Pi T\varphi_h$ we get

$$((\bar{v}_h, T\varphi_h)) = ((\Pi T\varphi_h, T\varphi_h))$$
$$= |\Pi T\varphi_h|^2 \geqq \frac{1}{\alpha}|\Pi T\varphi_h| . |T\varphi_h| = \frac{1}{\alpha}|\bar{v}_h| . |T\varphi_h| = \frac{1}{\alpha}|\bar{v}_h| . \|\varphi_h\|$$

then (CC) holds.

(CC) $\Rightarrow$ (PH).

Let φ_h be any function of $W_h(r, s)$ and u_h belong to $H_h(m) - \{0\}$ and satisfy: $((u_h, T\varphi_h)) \geqq \gamma \|\varphi_h\| . |u_h|$. Then

$$|u_h|\,|\Pi T\varphi_h| \geqq ((u_h, \Pi T\varphi_h)) = ((u_h, T\varphi_h)) \geqq \gamma\,\|\varphi_h\| . |u_h|.$$

Hence $|\Pi T\varphi_h| \geqq \gamma |T\varphi_h|$ and (PH) holds. □

From the above lemma we get that problem (3.6) has a unique solution whenever (CC) is satisfied. We can prove now

LEMMA 5: *Assume that* (CC) *holds. Between the solutions:* $\mathring{\psi}_h$ *of* (3.6) *and* $\{\sigma_h, \psi_h\}$ *of* (1.15) *the following relations hold*

$$\psi_h = \mathring{\psi}_h, \tag{3.7}$$

$$\sigma_h = \Pi T\mathring{\psi}_h - \Pi f. \tag{3.8}$$

Proof: It is sufficient to show that the couple $\{\Pi T\mathring{\psi}_h - \Pi f, \mathring{\psi}_h\}$ solves the problem (1.15). From the definition of Π we get:

$$\forall v_h \in H_h(m), \quad ((\Pi T\mathring{\psi}_h - \Pi f + f, v_h)) - ((v_h, T\mathring{\psi}_h))$$
$$= ((\Pi T\mathring{\psi}_h - \Pi f + \Pi f, v_h)) - ((v_h, \Pi T\mathring{\psi}_h)) = 0$$

thus $(1.15)_1$ is satisfied. From (3.6) we have

$$\forall \varphi_h \in W_h(r, s) \quad ((\Pi T\mathring{\psi}_h - f, \Pi T\varphi_h)) + b(f, \varphi_h) = 0$$

then

$$((\Pi T\mathring{\psi}_h - \Pi f, \Pi T\varphi_h)) + b(f, \varphi_h) = 0, \qquad \forall \varphi_h \in W_h(r, s).$$

and

$$((\Pi T\mathring{\psi}_h - \Pi f, T\varphi_h)) + b(f, \varphi_h) = 0, \qquad \forall \varphi_h \in W_h(r, s) \tag{3.9}$$

Now from (3.9) we get

$$((\Pi T\mathring{\psi}_h-\Pi f+f,\ T\varphi_h))=((f,\ T\varphi_h))-b(f,\ \varphi_h)=(p,\ \varphi_h),\qquad \forall\,\varphi_h\in W_h(r,\ s)$$

therefore $\Pi T\mathring{\psi}_h-\Pi f$ satisfies $(1.15)_2$. The proof is now complete. □

From (3.7), (2.1) and (2.3) we get

$$\|\psi-\mathring{\psi}_h\|\leqq Ch^{\theta}F(\psi,\ \sigma), \tag{3.10}$$

$$\|\psi-\mathring{\psi}_h\|_0\leqq Ch^{\min(s,\ 1+\nu)}\|\psi-\mathring{\psi}_h\|. \tag{3.11}$$

Suppose now that $\tilde{f}$ be a tensor satisfying, instead of (1.12), the condition

$$\tilde{f}\in H,\quad T^*\tilde{f}=p\quad \text{in } \Omega. \tag{3.12}$$

We shall see that the knowledge of $\tilde{f}$ is not explicitely needed. From (3.12) we get

$$(p,\ \varphi)=\int_\Omega T^*\tilde{f}\,\varphi\,dx=\langle T^*\tilde{f},\ \varphi\rangle=((\tilde{f},\ T\varphi)),\qquad \forall\,\varphi\in W. \tag{3.13}$$

Hence the function ψ defined in (3.3) satisfies

$$\psi\in W^*(\mathscr{T}_h),\quad ((T\psi,\ T\varphi))=((\tilde{f},\ T\varphi)),\qquad \forall\,\varphi\in W^*(\mathscr{T}_h) \tag{3.14}$$

A *hybrid-displacement* approximation of (3.14) is given by

$$\tilde{\psi}_h\in W_h(r,\ s),\quad ((\Pi T\tilde{\psi}_h,\ \Pi T\varphi_h))=((\tilde{f},\ T\varphi_h)),\qquad \forall\,\varphi_h\in W_h(r,\ s). \tag{3.15}$$

We have not projected the right hand side terms $T\varphi_h$ upon $H_h(m)$ since from (3.13) we get: $((\tilde{f},\ T\varphi_h))=\int_\Omega p\,\varphi_h\,dx$ and these integrals can be computed with good precision whenever the values of the basis functions of $W_h(r,\ s)$ are known at the quadrature nodes. These evaluations may be performed only once on the master element, stocked on disks or cards, and successively used to solve different problems. In such a way we overcome the difficulties arising whenever an explicit solution of (1.12) is sought.

We are looking now for the estimates of the errors: $\|\psi-\tilde{\psi}_h\|$ and $\|\psi-\tilde{\psi}_h\|_0$. We have:

THEOREM 2: *If* (PH) *holds and some suitable regularity assumptions are satisfied from* σ *and* ψ *then*

$$\|\psi-\tilde{\psi}_h\|\leqq Ch^{\theta}F(\psi,\ \sigma), \tag{3.16}$$

$$\|\psi-\tilde{\psi}_h\|_0\leqq Ch^{\min(s,\ 1+\nu)}\|\psi-\tilde{\psi}_h\|, \tag{3.17}$$

where θ *and* $F(\psi,\ \sigma)$ *are defined in* (2.2), $\nu=\nu(\Omega)$ *is defined in theorem* 1.

Proof: We remark that $b(\tilde{f}, \varphi)=0$, $\forall \varphi \in W$, thus from (3.6) we get

$$\mathring{\psi}_h \in W_h(r, s), \quad ((\Pi T \mathring{\psi}_h, \Pi T \varphi_h))=((\tilde{f}, \Pi T \varphi_h)), \qquad \forall \varphi_h \in W_h(r, s). \quad (3.18)$$

From (3.15) and (3.18) we have

$$((\Pi T(\tilde{\psi}_h-\mathring{\psi}_h), \Pi T \varphi_h)) = ((\tilde{f}, (I-\Pi) T \varphi_h)), \qquad \forall \varphi_h \in W_h(r, s). \quad (3.19)$$

Let us set $e=\tilde{\psi}_h-\mathring{\psi}_h$; from the above equation we get

$$((\Pi Te, \Pi Te))=((\tilde{f}, (I-\Pi) Te)). \quad (3.20)$$

From (PH) we have

$$\|e\|^2 \leqq \alpha^2 ((\Pi Te, \Pi Te))=\alpha^2 \{((\tilde{f}, Te))-((\tilde{f}, \Pi Te))\} = (\text{from } (3.14), (3.18))$$
$$\alpha^2 \{((T\psi, Te))-((\Pi T \mathring{\psi}_h, \Pi Te))\} = \alpha^2 \{(((I-\Pi) T\psi, Te))$$
$$+((\Pi T(\psi-\mathring{\psi}_h), \Pi Te))\} \leqq C\{|(I-\Pi) T\psi|.\|e\|+\|\psi-\mathring{\psi}_h\|.\|e\|\}. \quad (3.21)$$

Since $T\psi \in H^*(\mathscr{T}_h)$ from lemma 2 we get

$$|(I-\Pi) T\psi|= \operatorname*{Inf}_{v_h \in H_h(m)} |T\psi-v_h| \leqq Ch^{m+1} |T\psi|_{m+1,h} \quad (3.22)$$

and also (since $\theta \leqq m+1$):

$$|(I-\Pi) T\psi| \leqq Ch^{\theta} |T\psi|_{\theta, h} \leqq Ch^{\theta} |\psi|_{\theta+2, h} \quad (3.23)$$

Then from (3.10), (3.21), (3.23) and (2.2) we get

$$\|e\| \leqq Ch^{\theta} F(\psi, \sigma). \quad (3.24)$$

Finally since $\|\psi-\tilde{\psi}_h\| \leqq \|\psi-\mathring{\psi}_h\|+\|e\|$ the result (3.16) follows from (3.10) and (3.24).

In order to prove (3.17) let us set $\tilde{E}=\psi-\tilde{\psi}_h$ and consider the problem

$$\zeta \in W, \quad \Delta^2 \zeta=\tilde{E} \quad \text{in } \Omega.$$

We get

$$\zeta \in W \cap H^{3+\nu}(\Omega), \quad \|\zeta\|_{3+\nu} \leqq C \|\tilde{E}\|_0. \quad (3.25)$$

Let us consider the problems

$$\forall \varphi \in W, \quad ((T\zeta, T\varphi))=(\tilde{E}, \varphi), \quad (3.26)$$

$$\zeta_h \in W_h(r, s), \quad ((\Pi T\zeta_h, \Pi T \varphi_h))=(\tilde{E}, \varphi_h), \qquad \forall \varphi_h \in W_h(r, s). \quad (3.27)$$

Let ζ_I be the orthogonal projection of ζ upon $W_h(r, s)$; from (2.14) we get

$$\|\zeta-\zeta_I\| \leqq Ch^{\min(s, 1+\nu)} \|\zeta\|_{3+\nu} \leqq Ch^{\min(s, 1+\nu)} \|\tilde{E}\|_0. \quad (3.28)$$

Moreover we have

LEMMA 6: *If the hypotheses of lemma 2 hold we get*

$$\|\zeta_h-\zeta_I\| \leqq Ch^{\min(s,\,1+\nu)}\|\tilde{E}\|_0. \tag{3.29}$$

Proof: We have

$$\|\zeta_h-\zeta_I\|^2 \leqq \alpha^2((\Pi\,T(\zeta_h-\zeta_I),\,\Pi\,T(\zeta_h-\zeta_I)))$$
$$=\alpha^2\{((\Pi\,T(\zeta_h-\zeta),\,\Pi\,T(\zeta_h-\zeta_I)))+((\Pi\,T(\zeta-\zeta_I),\,\Pi\,T(\zeta_h-\zeta_I))).$$

Let us set: $F=((\Pi\,T(\zeta-\zeta_I),\,\Pi\,T(\zeta_h-\zeta_I)))$. From (3.27) we have

$$\|\zeta_h-\zeta_I\|^2 \leqq \alpha^2\{(\tilde{E},\,\zeta_h-\zeta_I)-((\Pi\,T\zeta,\,\Pi\,T(\zeta_h-\zeta_I)))+F\}$$
$$=(\text{thanks to (3.26)})\ \alpha^2\{((T\zeta,\,T(\zeta_h-\zeta_I)))-((\Pi\,T\zeta,\,T(\zeta_h-\zeta_I)))+F\}$$
$$=\alpha^2\{(((I-\Pi)\,T\zeta,\,T(\zeta_h-\zeta_I)))+F\}$$
$$\leqq C\|\zeta_h-\zeta_I\|(|(I-\Pi)\,T\zeta|+\|\zeta-\zeta_I\|) \tag{3.30}$$

From (3.25), (2.14) and (2.15) we get

$$|(I-\Pi)\,T\zeta| \leqq Ch^{1+\nu}\|T\zeta\|_{1+\nu} \leqq Ch^{1+\nu}\|\tilde{E}\|_0, \tag{3.31}$$
$$\|\zeta-\zeta_I\| \leqq Ch^{\min(s,\,1+\nu)}\|\zeta\|_{3+\nu} \leqq Ch^{\min(s,\,1+\nu)}\|\tilde{E}\|_0. \tag{3.32}$$

Therefore in (3.30) we get

$$\|\zeta_h-\zeta_I\| \leqq Ch^{\min(s,\,1+\nu)}\|\tilde{E}\|_0$$

that is (3.29). □

From (3.28) and (3.29) it follows

$$\|\zeta-\zeta_h\| \leqq Ch^{\min(s,\,1+\nu)}\|\tilde{E}\|_0. \tag{3.33}$$

By (3.26) we get

$$\|\tilde{E}\|_0^2=((T\zeta,\,T\tilde{E}))$$
$$=((T\zeta-\Pi\,T\zeta_h,\,T\tilde{E}))+((\Pi\,T\zeta_h,\,T\psi))-((\Pi\,T\zeta_h,\,\Pi\,T\tilde{\psi}_h))$$
$$=(\text{thanks to (3.14) and (3.15)})\ ((T\zeta-\Pi\,T\zeta_h,\,T\tilde{E}))$$
$$+((\Pi\,T\zeta_h,\,T\psi))-((T\psi,\,T\zeta_h))=((T\zeta-\Pi\,T\zeta_h,\,T\tilde{E}))+((T\psi-\Pi\,T\tilde{\psi}_h,$$
$$(\Pi-I)\ \ T\zeta_h))=((T\zeta-\Pi\,T\zeta_h,\,T\tilde{E}))+((T\psi-\Pi\,T\tilde{\psi}_h,\,\Pi\,T\zeta_h-T\zeta))$$
$$+((T\psi-\Pi\,T\tilde{\psi}_h,\,T(\zeta-\zeta_h)))\leqq|T\zeta-\Pi\,T\zeta_h|\{\|\tilde{E}\|+|T\psi-\Pi\,T\tilde{\psi}_h|\}$$
$$+|T\psi-\Pi\,T\tilde{\psi}_h|\,\|\zeta-\zeta_h\|. \tag{3.34}$$

Moreover

$$|T\zeta-\Pi\,T\zeta_h| \leqq |(I-\Pi)\,T\zeta|+\|\zeta-\zeta_h\| \leqq (\text{from (3.31) and (3.33)},$$
$$Ch^{\min(s,\,1+\nu)}\|\tilde{E}\|_0,$$

$$|T\psi - \Pi T\tilde{\psi}_h| \leqq |(I-\Pi) T\psi| + \|\psi - \tilde{\psi}_h\|$$
$$\leqq (\text{from } (3.16) \text{ and } (3.23))\ Ch^{\theta} F(\psi, \sigma).$$

Then from (3.33) and the above estimates by (3.34) we get

$$\|\tilde{E}\|_0 \leqq Ch^{\min(s,\, 1+\nu)} \|\tilde{E}\|$$

and that (3.17) holds follows from the definition of $\tilde{E}$. □

Since σ depends on w and f the estimates (3.16), (3.17) depend on the regularity of w and f. On the other hand we do not need an explicit knowledge of f in order to apply the method discussed in this section. Therefore we can assume that f is regular; for example, setting $f = Tw$ we see that the estimates (3.16), (3.17) only depend on the regularity of w.

REMARK: If no better regularity assumption than $w \in H_0^2(\Omega)$ is made we still have $\psi \in W^*(\mathcal{T}_h)$ and $\sigma \in H^*(\mathcal{T}_h)$. Then from (2.3) and (3.17) we easily get

$$\|\psi - \mathring{\psi}_h\|_0 \leqq Ch^{\min(s,\, 1+\nu)} \|w\|, \tag{3.35}$$

$$\|\psi - \tilde{\psi}_h\|_0 \leqq Ch^{\min(s,\, 1+\nu)} \|w\|. \tag{3.36}$$

The above estimates will be applied in [11]. □

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