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ON SPECTRAL APPROXIMATION PART 2. ERROR ESTIMATES FOR THE GALERKIN METHOD (*)

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Abstract. — One considers an isolated eigenvalue λ of finite multiplicity of an operator A which is approximated by a Galerkin method. Using Osborn's technics, one derives several error estimates for λ .

1. SITUATION AND RESULTS

In part 1 of this paper [3], we have been concerned with the problem of convergence in spectral approximation; since the theory we have developed has received concrete applications for non compact operators only in connection with the Galerkin method, we shall now restrict ourself to this case.

Let X be a complex Banach space of norm $\| \cdot \|$ and $\{ X_h \}$ be a sequence of finite dimensional subspaces of X . One gives two continuous sesquilinear forms a and b on X and one supposes a coercive. Then, by Lax-Milgram, one can define the continuous operators $A : X \rightarrow X$ and $A_h : X_h \rightarrow X_h$ by

$$a(Au, v) = b(u, v), \quad \forall u, v \in X, \quad a(A_h u, v) = b(u, v), \quad \forall u, v \in X_h.$$

All along this paper we shall suppose that the two following conditions are satisfied (see [3]):

$$P1 : \lim_{h \rightarrow 0} \| (A - A_h)|_{X_h} \| = 0; \quad P2 : \forall x \in X, \quad \lim_{h \rightarrow 0} \inf_{x_h \in X_h} \| x - x_h \| = 0.$$

Let $\lambda \in \mathbb{C}$ be an isolated eigenvalue of A of finite algebraic multiplicity m ; since a is coercive $\lambda \neq 0$ and there exists a closed disc Δ of center λ and boundary Γ such that $0 \notin \Delta$ and $\Delta \cap \sigma(A) = \{ \lambda \}$ where $\sigma(A)$ denotes the spectrum of A . Let $\mu_{1h}, \dots, \mu_{m(h),h}$ be the eigenvalues of A_h , repeated following their algebraic multiplicities and contained in Δ . In [3], section 2,

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we have proved:

- a) $m(h) = m$ for h small enough;
- b) $\lim_{h \rightarrow 0} \mu_{ih} = \lambda, i = 1, 2, \dots, m.$

The purpose of this part 2 of our paper is to give estimates of λ by the μ_{ih} 's. In fact, we shall adapt to the situation described above Osborn's method [5]; note that, independently of the fact that A_h is a Galerkin approximation, we have simplified the presentation of Osborn's main argument and strengthened his results. See also Grigorieff [4].

At this point, we recall some standard notations. For an operator D , $R_z(D) = (z - D)^{-1}$ is the resolvent operator. Let Y and Z be closed subspaces of X ; then for $x \in X$,

$$\delta(x, Z) = \inf_{z \in Z} \|x - z\|, \quad \delta(Y, Z) = \sup_{\substack{y \in Y \\ \|y\|=1}} \delta(y, Z)$$

and

$$\hat{\delta}(Y, Z) = \max(\delta(Y, Z), \delta(Z, Y)).$$

Let us also open a short parenthesis on duality. Let X^* be the adjoint space of X , i. e. the set of antilinear continuous forms on X . By Lax-Milgram, the operator $C : X^* \rightarrow X$ defined by the relation $a(v, C\varphi) = \bar{\varphi}(v), \forall v \in X, \varphi \in X^*$, is an isomorphism between X^* and X which allows to identify these two spaces. With this identification if $D : X \rightarrow X$ is a bounded linear operator, its adjoint $D^* : X \rightarrow X$ will be characterized by the relation $a(Du, v) = a(u, D^*v), \forall u, v \in X$; one verifies also immediately the relation $\|D^*\| \leq \|C\| \cdot \|C^{-1}\| \cdot \|D\|$.

We need, for the following, to introduce some further operators. $\Pi_h : X \rightarrow X$ is the projector with range X_h defined by the relation $a(\Pi_h u - u, v) = 0, \forall v \in X_h$. One has $A_h = \Pi_h A|_{X_h}$ and we set $B_h = \Pi_h A \Pi_h : X \rightarrow X$; except for zero, B_h has the same spectrum as A_h and the same corresponding invariant subspaces. $E = (2 \Pi i)^{-1} \int_{\Gamma} R_z(A) dz$ is the spectral projector of A relative to λ and, for h small enough, $F_h = (2 \Pi i)^{-1} \int_{\Gamma} R_z(B_h) dz$ is the spectral projector of B_h relative to $\mu_{1h}, \dots, \mu_{mh}$. Now consider the adjoints of these operators as defined above. A^* has the isolated eigenvalue $\bar{\lambda}$ of algebraic multiplicity m ; Π_h^* will be the projector with range X_h satisfying the relation $a(v, \Pi_h^* u - u) = 0, \forall v \in X_h$; E^* and F_h^* will be the spectral projectors of A^* and $B_h^* = \Pi_h^* A^* \Pi_h^*$ associated respectively to $\bar{\lambda}$ and to the set $\bar{\mu}_{1h}, \dots, \bar{\mu}_{mh}$;

they will satisfy the relations

$$E^* = (2 \Pi i)^{-1} \int_{\bar{\Gamma}} R_z(A^*) dz \quad \text{and} \quad F_h^* = (2 \Pi i)^{-1} \int_{\bar{\Gamma}} R_z(B_h^*) dz$$

where $\bar{\Gamma}$ is the conjugate circle of Γ (positively oriented).

In applications $E(X)$ and $E^*(X)$, the m -dimensional invariant subspaces of A and A^* corresponding to λ and $\bar{\lambda}$, will be often composed of smooth functions so that it is reasonable to introduce the quantities

$$\gamma_h = \delta(E(X), X_h), \quad \gamma_h^* = \delta(E^*(X), X_h).$$

We can now state the results.

THEOREM 1 : *There exists a constant c , independent of h such that*

$$\hat{\delta}(F_h(X), E(X)) \leq c \gamma_h; \quad \hat{\delta}(F_h^*(X), E^*(X)) \leq c \gamma_h^*.$$

In section 2, we shall show that $F_h|_{E(X)}$ defines for h small enough, a bijection between $E(X)$ and $F_h(X)$; let Λ_h be this bijection; $\hat{A} = A|_{E(X)}$ and $\hat{B}_h = \Lambda_h^{-1} B_h \Lambda_h$ will be considered as operators in $E(X)$; \hat{A} has the eigenvalue λ of algebraic multiplicity m and \hat{B}_h has the eigenvalues $\mu_{1h}, \dots, \mu_{mh}$.

THEOREM 2 : *There exists a constant c , independent of h such that*

$$\|\hat{A} - \hat{B}_h\|_{E(X)} \leq c \gamma_h \gamma_h^*.$$

By the choice of a basis in $E(X)$, theorem 2 reduces our original task to a pure matricial problem. Let f be a holomorphic function defined in a neighborhood of λ ; writing $f(\hat{A})$ and $f(\hat{B}_h)$ by the mean of Dunford integrals, one verifies immediately that

$$\|f(\hat{A}) - f(\hat{B}_h)\|_{E(X)} \leq c \| \hat{A} - \hat{B}_h \|_{E(X)}$$

where c depends on f but not on h ; using the classical properties of traces and determinants, one obtains theorem 3 a, b ; theorem 3 c, d is a direct application of results quoted in [7], pp. 80-81; here α is the ascent of the eigenvalue λ of \hat{A} , β is the number of Jordan blocs of the canonical form of \hat{A} .

THEOREM 3: *There exists a constant c , independent of h such that for h small enough:*

- a) $|f(\lambda) - \frac{1}{m} \sum_{i=1}^m f(\mu_{ih})| \leq c \gamma_h \gamma_h^*$,
- b) $|f^m(\lambda) - \prod_{i=1}^m f(\mu_{ih})| \leq c \gamma_h \gamma_h^*$,
- c) $\max_{i=1 \dots m} |\lambda - \mu_{ih}| \leq c (\gamma_h \gamma_h^*)^{1/\alpha}$,
- d) $\min_{i=1 \dots m} |\lambda - \mu_{ih}| \leq c (\gamma_h \gamma_h^*)^{\beta/m}$.

REMARKS: 1) In his original work, in a difference context, Osborn [5] has obtained theorem 3 a for $f(z) = z$ and theorem 3 c; in another context also Chatelin [1] proves, theorem 3 a for $f(z) = z$.

2) For $f(z) = 1/z$, theorem 3 a gives an estimate of $1/\lambda$ by the arithmetic mean of the $1/\mu_{ih}$'s; the result has been already obtained by [2]; we are indebted to Chatelin who showed us that it can also be deduced by Osborn's method.

In order to illustrate this theorem, we consider the example developed in section 4 of part 1 of this paper [3]; one can prove by Rappaz' method of elimination used in [6] the existence of an infinite number of isolated eigenvalues of finite multiplicities; by supposing the coefficients α, β, \dots sufficiently smooth, one verifies that the corresponding eigensubspaces are subsets of $H^2 \times (H^1)^2$; consequently $\gamma_h = O(h)$, $\gamma_h^* = O(h)$ and the estimates of theorem 3 a, b are of order h^2 .

We conclude this section by stating a very elementary result for the self-adjoint case. We suppose that the forms a and b are hermitian. Because of its coercivity, a is a scalar product for which X is a Hilbert space with norm $\|x\|_a^2 = a(x, x)$; then A, B_h and Π_h become hermitian. Let v be an eigenvalue of A , which is not supposed isolated or of finite multiplicity, and G be the corresponding eigensubspace. For the distance $\delta(v, \sigma(B_h))$ from v to the spectrum $\sigma(B_h)$ of B_h , one gets the estimate

$$\begin{aligned} \delta(v, \sigma(B_h)) &= \inf_{\substack{y \in X \\ \|y\|=1}} \|(B_h - v)y\|_a \leq \inf_{\substack{y \in X_h \\ \|y\|=1}} \|(B_h - v)y\|_a \\ &\leq \inf_{\substack{y \in X_h \\ \|y\|=1}} \|(A - v)y\|_a = \inf_{\substack{y \in X_h \\ \|y\|=1}} \|(A - v)(y - x)\|_a, \quad \forall x \in G, \end{aligned}$$

i. e., since the norms $\|\cdot\|$ and $\|\cdot\|_a$ are equivalent:

$$\delta(v, \sigma(B_h)) \leq c \inf_{\substack{x \in G \\ \|x\|=1}} \delta(x, X_h), \quad c \text{ independent of } h \text{ and } v. \quad (1)$$

REMARKS: 1) We have obtained the estimate (1) without supposing P1 or P2.

2) Examples show that it is not possible to replace the right member of (1), by $c \left\{ \inf_{\substack{x \in G \\ \|x\|=1}} \delta(x, X_h) \right\}^{1+\varepsilon}$, $\varepsilon > 0$.

2. PROOFS

In this section we prove theorem 1 and 2. We use the definitions and notations of section 1 and we suppose hypotheses P1 and P2. c will denote a "generic" constant.

We first recall a well-known result. Since a is continuous and coercive, the projectors Π_h are bounded uniformly with respect to h and there exists a constant c such that $\|x - \Pi_h x\| \leq c \delta(x, X_h)$, $\forall x \in X$; the Π_h^* 's possess the same properties.

Lemma 1 of section 2 of [3] shows that P1 implies the inequality $\sup_{\substack{x \in X_h \\ \|x\|=1}} \|R_z(A_h)x\| \leq c$, $\forall z \in \Gamma$ for h small enough, c independent of h .

We extend this result to B_h and B_h^* .

LEMMA 1: *There exists $h_0 > 0$ and c such that*

$$\|R_z(B_h)\| \leq c, \quad h < h_0, \quad z \in \Gamma$$

and

$$\|R_z(B_h^*)\| \leq c, \quad h < h_0, \quad z \in \bar{\Gamma}.$$

Proof: Since $R_z(B_h^*) = (R_z(B_h))^*$ we need to prove only the first statement; since B_h is compact it suffices to verify that $\|(z - B_h)x\| \geq c \|x\|$, $\forall x \in X$, $z \in \Gamma$. Taking in account the fact that $0 \notin \Gamma$ one has

$$\begin{aligned} \|x\| &\leq \|\Pi_h x\| + \|x - \Pi_h x\| \\ &\leq c \|(z - B_h)\Pi_h x\| + |z|^{-1} \|(z - B_h)(x - \Pi_h x)\| \leq c \|(z - B_h)x\|. \quad \blacksquare \end{aligned}$$

We note that we shall not use any more P1 explicitly. Consequently, in the proofs of lemma 3 and theorem 1, the statements for the adjoints operators are obtained in the same way as for the direct operators.

We omit the proof of the following trivial:

LEMMA 2: *Let Y and Z be two subspaces of X with the same finite dimension; 1 et $P: Y \rightarrow Z$ be a linear operator such that $\|P y - y\| \leq 0.5 \|y\|$, $\forall y \in Y$.*

Then P is bijective,

$$\|P^{-1}z\| \leq 2\|z\|, \quad \forall z \in Z$$

and

$$\sup_{\substack{z \in Z \\ \|z\|=1}} \|P^{-1}z - z\| \leq 2 \sup_{\substack{y \in Y \\ \|y\|=1}} \|Py - y\|.$$

LEMMA 3:

$$\begin{aligned} \|(E - F_h)|_{E(X)}\| &\leq c \|(A - B_h)|_{E(X)}\| \leq c\gamma_h, \\ \|(E^* - F_h^*)|_{E^*(X)}\| &\leq c \|(A^* - B_h^*)|_{E^*(X)}\| \leq c\gamma_h^*. \end{aligned}$$

Proof: For h small enough, by lemma 1, one has

$$\begin{aligned} \|(E - F_h)|_{E(X)}\| &\leq (2\Pi)^{-1} \int_{\Gamma} \|R_z(B_h)\| \cdot \|(A - B_h)R_z(A)|_{E(X)}\| \cdot |dz| \\ &\leq c \|(A - B_h)|_{E(X)}\|; \\ \|(A - B_h)|_{E(X)}\| &\leq \|(I - \Pi_h)A|_{E(X)}\| + \|\Pi_h A(I - \Pi_h)|_{E(X)}\| \\ &\leq c \|(I - \Pi_h)|_{E(X)}\| \leq c\gamma_h. \quad \blacksquare \end{aligned}$$

Proof of theorem 1: Lemma 3 implies that $\delta(E(X), F_h(X)) \leq c\gamma_h$. Set, as in Section 1, $\Lambda_h = F_h|_{E(X)} : E(X) \rightarrow F_h(X)$; for h small enough $E(X)$ and $F_h(X)$ have the same dimension m ; on the other side P2 implies $\lim_{h \rightarrow 0} \gamma_h = 0$;

by lemma 2, Λ_h^{-1} exists for h small enough and is uniformly bounded with respect to h ; furthermore $\sup_{\substack{x \in F_h(X) \\ \|x\|=1}} \|\Lambda_h^{-1}x - x\| \leq c\gamma_h$, i. e.

$\delta(F_h(X), E(X)) \leq c\gamma_h$. \blacksquare

Proof of theorem 2: Let $S_h = \Lambda_h^{-1}F_h - I : X \rightarrow X$; S_h is uniformly bounded with respect to h (see proof of theorem 1); from the identity

$$(\hat{A} - \hat{B}_h)x = (A - B_h)x + S_h(A - B_h)x, \quad x \in E(X),$$

one obtains for $x \in E(X)$, $y \in E^*(X)$, since $F_h S_h = 0$,

$$a((\hat{A} - \hat{B}_h)x, y) = a((A - B_h)x, y) + a(S_h(A - B_h)x, (I - F_h^*)y); \quad (2)$$

$$\begin{aligned} \|\hat{A} - \hat{B}_h\|_{E(X)} &\leq c \sup_{\substack{x \in E(X), y \in X \\ \|x\|=\|y\|=1}} a((\hat{A} - \hat{B}_h)x, y) \\ &\leq c \sup_{\substack{x \in E(X), y \in E^*(X) \\ \|x\|=\|y\|=1}} a((\hat{A} - \hat{B}_h)x, y); \quad (3) \end{aligned}$$

for $x \in E(X)$, $y \in E^*(X)$, $\|x\| = \|y\| = 1$, one has (using lemma 3):

$$|a(S_h(A - B_h)x, (I - F_h^*)y)| \leq c \|(A - B_h)x\| \cdot \|(I - F_h^*)y\| \leq c \gamma_h \gamma_h^*; \quad (4)$$

$$\begin{aligned} a((A - B_h)x, y) &= a((I - \Pi_h)Ax, (I - \Pi_h^*)y) \\ &\quad + a((I - \Pi_h)x, (I - \Pi_h^*)A^*y) \\ &\quad + a((I - \Pi_h)x, A^*(\Pi_h^* - I)y); \end{aligned}$$

$$|a((A - B_h)x, y)| \leq c \gamma_h \gamma_h^*; \quad (5)$$

theorem 2 follows from (2), (3), (4) and (5). ■

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