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## AN ELASTO-PLASTIC CONTACT PROBLEM (\*) (1)

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Communiqué par P. G. Ciarlet

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*Abstract.* — We study the problem of finding the stresses and the displacements in an elasto-plastic body  $\mathcal{E}$  in frictionless contact with a rigid body which is pressed against  $\mathcal{E}$ . We prove existence of a solution and then we consider finite element methods for finding approximate solutions of the problem.

### INTRODUCTION

Duvaut [1] has studied the problem of finding the stresses in an elasto-plastic body  $\mathcal{E}$  in frictionless contact with a rigid body  $\mathcal{B}$  which is pressed against  $\mathcal{E}$ . In this note we extend the study of Duvaut by looking also for the displacements of  $\mathcal{E}$  and  $\mathcal{B}$ . We shall consider a stationary case corresponding to Henky's law. For simplicity we shall assume that  $\mathcal{E}$  is isotropic.

In Section 1 we prove existence of a solution to the contact problem assuming that  $\mathcal{E}$  is elastic-perfectly plastic. In this case the displacements of  $\mathcal{E}$  may be discontinuous (and even non-unique) and we have to use a formulation requiring little regularity of the displacements. One way of obtaining more regular displacements is to assume a suitable hardening of the elasto-plastic material. Such a case is studied in Section 2. Then in Sections 3 and 4 we consider finite element methods for finding approximate solutions of the contact problem.

### 1. ELASTIC-PERFECTLY PLASTIC MATERIAL

Suppose that initially the elasto-plastic body  $\mathcal{E}$  occupies the bounded region  $\Omega \subset \mathbf{R}^3$  with boundary  $\Gamma$  and that  $\Gamma$  contains an open set  $\Gamma_1$  in the plane  $\{x = (x_1, x_2, x_3) \in \mathbf{R}^3 : x_3 = 0\}$ . Moreover, suppose that initially the rigid body  $\mathcal{B}$  occupies the region

$$B = \{x \in \mathbf{R}^3 : -x_3 \geq \varphi(x_1, x_2), (x_1, x_2) \in \bar{\Gamma}_0\},$$

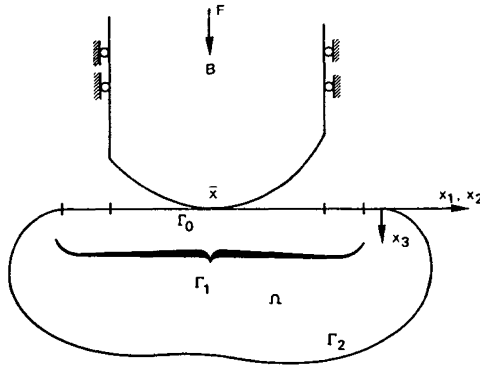
where  $\Gamma_0$  is an open set compactly contained in  $\Gamma_1$  with smooth boundary and  $\varphi : \Gamma_1 \rightarrow \mathbf{R}$  is smooth, nonnegative and  $\varphi(\bar{x}) = 0$  for some  $\bar{x} \in \Gamma_0$  (see *Fig.*).

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(1) Texte d'une conférence présentée aux Journées « Éléments Finis », Rennes, 4-6 mai 1977

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Let the boundary of  $\mathcal{E}$  be fixed on the portion  $\Gamma_2 = \Gamma \setminus \Gamma_1$  and free on  $\Gamma_1$ . Let  $\mathcal{B}$  be acted upon by the vertical force  $F (F > 0)$  and suppose that  $\mathcal{B}$  is free to move vertically, whereas rotation and horizontal displacement are prevented. We want to find the vertical displacement  $U$  of  $\mathcal{B}$ , the stress  $\sigma = \{ \sigma_{ij} \}$ ,  $i, j = 1, 2, 3$ , in  $\mathcal{E}$ , and the displacement  $u = \{ u_i \}$ ,  $i = 1, 2, 3$ , of  $\mathcal{E}$ , where  $u_i$  is the displacement in the  $x_i$ -direction. The reference configuration is the one in *figure*. We shall assume that the displacements are small; in particular this means that the relation  $u_3 \cong U - \varphi$  can be used to describe the compatibility of the displacements of  $\mathcal{B}$  and  $\mathcal{E}$ .

We shall use the following notation: For  $m$  a positive integer and  $1 \leq p \leq \infty$ , let  $\| \cdot \|_{m,p}$  denote the norm in the usual Sobolev space  $[W_p^m(\Omega)]^n$  with  $n$  a positive integer. If  $m = 0$  we omit this index and write  $\| \cdot \|_p$  instead of  $\| \cdot \|_{0,p}$ . Let  $(\cdot, \cdot)$  and  $\| \cdot \|$  denote the scalar product and norm in  $[L^2(\Omega)]^n$ . Further we define

$$H = \{ \tau = \{ \tau_{ij} \} \in [L^2(\Omega)]^9 : \tau_{ij} = \tau_{ji}, i, j = 1, 2, 3, \},$$

$$\mathcal{W} = [W]^3, \quad W = \{ w \in W_2^1(\Omega) : w = 0 \text{ on } \Gamma_2 \},$$

$$K = \{ (u, U) \in \mathcal{W} \times \mathbf{R} : u_3 + \varphi - U \geq 0 \text{ on } \bar{\Gamma}_0 \},$$

and the deformation  $\varepsilon(u) = \{ \varepsilon_{ij}(u) \}$  associated with  $u \in \mathcal{W}$  by

$$\varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad i, j = 1, 2, 3,$$

where  $w_{,j} = \partial w / \partial x_j$ . We recall Green's formula:

$$(\tau, \varepsilon(v)) = \int_{\Gamma} \tau_{ij} n_j u_i ds - (\text{div } \tau, v), \tag{1.1}$$

where  $n = (n_1, n_2, n_3)$  is the outward unit normal to  $\Gamma$  and

$$\begin{aligned} \operatorname{div} \tau &= ((\operatorname{div} \tau)_i), \quad i = 1, 2, 3, \\ (\operatorname{div} \tau)_i &= \tau_{ij,j}. \end{aligned}$$

Here and below we use the summation convention: repeated indices indicate summation from 1 to 3.

Let  $D \subset \mathbf{R}^9$  be a given closed convex set with  $0 \in D$  and define the set of plastically admissible stresses

$$P = \{ \tau \in H : \tau^d(x) \in D \text{ a.e. in } \Omega \},$$

where

$$\begin{aligned} \tau^d &= \tau - \frac{1}{3} \operatorname{tr}(\tau) \delta, \\ \operatorname{tr}(\tau) &= \tau_{kk}, \\ \delta &= \{ \delta_{ij} \}, \quad \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \end{aligned}$$

The tensor  $\tau^d$  is the so called *stress deviatoric* associated to  $\tau$ . If  $\sigma(x) \in$  (interior of  $D$ ), then  $\mathcal{E}$  is in an elastic state at the point  $x \in \Omega$  and then we have the linear strain-stress relation

$$\begin{aligned} \varepsilon(u) &= A \sigma, \\ (A \sigma)_{ij} &= \lambda \operatorname{tr}(\sigma) \delta_{ij} + \nu \sigma_{ij}^d, \end{aligned}$$

where  $\lambda$  and  $\nu$  are positive constants. For notational simplicity we shall assume below that  $\nu = 1$ . We define the bilinear form

$$a(\sigma, \tau) = \int_{\Omega} (\lambda \operatorname{tr}(\sigma) \operatorname{tr}(\tau) + \sigma_{ij}^d \tau_{ij}^d) dx,$$

and we note that

$$a(\tau, \tau) \geq \alpha \|\tau\|^2, \quad \tau \in H, \tag{1.2}$$

where  $\alpha = \min(1, 9\lambda)$ .

A natural formulation of the contact problem is now the following: Find  $(\sigma, (u, V)) \in P \times K$  such that

$$a(\sigma, \tau - \sigma) - (\varepsilon(u), \tau - \sigma) \geq 0, \quad \forall \tau \in P, \tag{1.3 a}$$

$$(\sigma, \varepsilon(v - u)) \geq F(V - U), \quad \forall (v, V) \in K, \tag{1.3 b}$$

or, equivalently, find a saddle point  $(\sigma, (u, V)) \in P \times K$  for the functional  $L : P \times K \rightarrow \mathbf{R}$  defined by

$$L(\tau, (v, V)) = \frac{1}{2} \|\tau\|_a^2 - (\tau, \varepsilon(v)) + FV,$$

where  $\| \cdot \|_a^2 = a(\cdot, \cdot)$ . However, the regularity of the displacement  $u$  needed in this formulation is in general not possible to achieve in the perfectly-plastic case. Therefore, we shall instead consider a formulation requiring less regularity of  $u$  (cf. [3]). To be more precise, we shall seek  $u$  in the space  $Y_{3/2}$ , where for  $1 \leq p \leq \infty$ ,

$$Y_p = [L_p(\Omega)]^3.$$

To motivate this formulation we first note that be Green's formula (1.1), it follows that (1.3 b) is formally equivalent to the following relations:

$$\operatorname{div} \sigma = 0 \quad \text{in } \Omega, \quad (1.4)$$

$$-\int_{\Gamma_0} \sigma_{33} ds = F, \quad (1.5)$$

$$\sigma_{13} = \sigma_{23} = 0, \quad \sigma_{33} \leq 0 \quad \text{on } \Gamma_1, \quad (1.6)$$

$$\sigma_{33} = 0 \quad \text{on } \Gamma_1 \setminus \Gamma_0, \quad (1.7)$$

$$\sigma_{33}(x) = 0 \quad \text{if } (u_3 + \varphi - U)(x) > 0, \quad x \in \Gamma_0, \quad (1.8)$$

which is the intuitive way of formulating the statical relationship in the contact problem.

We shall seek the stress  $\sigma$  in the space  $\mathcal{P}_3 = P \cap \mathcal{H}_3$ , where for  $2 \leq q < \infty$ ,

$$\mathcal{H}_q = \{ \tau \in H : \operatorname{div} \tau \in Y_q \text{ and } \tau \text{ satisfies (1.6) and (1.7)} \}.$$

Here (1.6) and (1.7) are to be understood in the following sense:

$$\int_{\Gamma} \tau_{13} w ds = \int_{\Gamma} \tau_{23} w ds = 0, \quad w \in \mathcal{W}, \quad (1.9)$$

$$\int_{\Gamma} \tau_{33} w ds \leq 0, \quad w \in \mathcal{W}, \quad w \geq 0, \quad (1.10)$$

$$\int_{\Gamma} \tau_{33} w ds = 0 \quad \text{for } w \in \mathcal{W} \text{ such that } w = 0 \text{ on } \Gamma_0. \quad (1.11)$$

Note that if  $\tau \in H$  and  $\operatorname{div} \tau \in Y_2$ , then (see [2])  $\tau_{ij} n_j \in H^{-\frac{1}{2}}(\Gamma)$  so that (1.9)-(1.11) are meaningful. Now, taking  $\tau \in \mathcal{P}_3$  in (1.3 a) using Green's formula and (1.6), we find with  $\psi \in \mathcal{W}$  satisfying  $\psi = 1$  on  $\Gamma_0$ , that

$$\begin{aligned} 0 &\leq a(\sigma, \tau - \sigma) - (\varepsilon(u), \tau - \sigma) \\ &= a(\sigma, \tau - \sigma) + (u, \operatorname{div} \tau - \operatorname{div} \sigma) + \int_{\Gamma} u_3 (\tau_{33} - \sigma_{33}) ds \\ &= a(\sigma, \tau - \sigma) + (u, \operatorname{div} \tau - \operatorname{div} \sigma) + \int_{\Gamma} (u_3 + \psi(\varphi - U)) (\tau_{33} - \sigma_{33}) ds \\ &\quad + \int_{\Gamma} \psi(U - \varphi) (\tau_{33} - \sigma_{33}) ds, \end{aligned}$$

so that

$$a(\sigma, \tau - \sigma) + (u, \operatorname{div} \tau - \operatorname{div} \sigma) + \int_{\Gamma} \psi(U - \varphi)(\tau_{33} - \sigma_{33}) ds \geq 0, \quad (1.12)$$

since by (1.7)-(1.8),

$$\int_{\Gamma} (u_3 + \psi(\varphi - U)) \sigma_{33} ds = 0,$$

and by (1.10) and (1.11) assuming that  $(u, U) \in K$ ,

$$\int_{\Gamma} (u_3 + \psi(\varphi - U)) \tau_{33} ds \leq 0.$$

We are thus led to the following formulation of the contact problem: Find  $(\sigma, u, U) \in \mathcal{P} \times Y \times \mathbf{R}$  such that

$$a(\sigma, \tau - \sigma) + (u, \operatorname{div} \tau - \operatorname{div} \sigma) + \int_{\Gamma} \psi(U - \varphi)(\tau_{33} - \sigma_{33}) ds \geq 0, \quad \tau \in \mathcal{P}, \quad (1.13 a)$$

$$(v, \operatorname{div} \sigma) = 0, \quad v \in Y, \quad (1.13 b)$$

$$-\int_{\Gamma} \psi \sigma_{33} ds = F, \quad (1.13 c)$$

where  $\mathcal{P} = \mathcal{P}_3$  and  $Y = Y_{3/2}$ .

*Remark:* Note that the condition  $u_3 + \varphi - U \geq 0$  on  $\Gamma_0$  does not appear explicitly in this formulation, which is natural since the trace of  $u \in Y$  on  $\Gamma$  may not be defined.

To prove existence of a solution of (1.13) we shall need the following "safe load hypothesis":

$$\begin{aligned} &\text{There exists } \delta > 0 \quad \text{and} \quad \chi \in \mathcal{P} \cap E \\ &\text{such that } \operatorname{dist}(\chi(x), \partial D) > \delta \text{ for } x \in \Omega, \end{aligned} \quad (1.14)$$

where  $\partial D$  denotes the boundary of  $D$  and

$$E = \left\{ \tau \in H : \operatorname{div} \tau = 0 \text{ in } \Omega, -\int_{\Gamma} \psi \tau_{33} = F \right\}.$$

Note that with  $\delta = 0$ , this is a necessary condition for existence of a solution.

**THEOREM 1:** *If (1.14) holds then there exists  $(\sigma, u, U) \in \mathcal{P} \times Y \times \mathbf{R}$ , satisfying (1.13). Moreover  $\sigma$  is uniquely determined.*

*Proof:* The proof will be divided into three parts: First we prove existence of a solution of a regularized problem depending on a parameter  $\mu > 0$ . Then we establish some *a priori* estimates for the solution of this problem and finally we obtain a solution of the original problem by passing to the limit as  $\mu$  tends to zero. The uniqueness of  $\sigma$  is easy to prove.

**(a) The regularized problem**

For  $\mu > 0$  we consider the following problem: Find a saddle point  $(\sigma_\mu, (u_\mu, U_\mu)) \in H \times K$  for the regularized Lagrangian  $L_\mu : H \times K \rightarrow \mathbf{R}$  defined by

$$L_\mu(\tau, (v, V)) = \frac{1}{2} \|\tau\|_a^2 + J_\mu(\tau) - (\tau, \varepsilon(v)) + FV,$$

where

$$J_\mu(\tau) = \frac{1}{2\mu} \|\tau - \pi\tau\|^2,$$

and  $\pi$  is the orthogonal projection in  $\mathbf{R}^9$  onto  $D$ . In other words, we seek  $(\sigma_\mu, (u_\mu, U_\mu)) \in H \times K$  satisfying

$$a(\sigma_\mu, \tau) + (J'_\mu(\sigma_\mu), \tau) - (\tau, \varepsilon(u_\mu)) = 0, \quad \tau \in H, \quad (1.15a)$$

$$(\sigma_\mu, \varepsilon(v - u_\mu)) \geq F(V - U_\mu), \quad (v, V) \in K, \quad (1.15b)$$

where  $J'(\tau) = (1/\mu)(\tau - \pi\tau)$ . Existence of a saddle point  $(\sigma_\mu, (u_\mu, U_\mu))$  will follow easily if we can show that the problem

$$\sup_{(v, V) \in K} g_\mu(v, V), \quad (1.16)$$

where

$$g_\mu(v, V) = \inf_{\tau \in H} L_\mu(\tau, (v, V)), \quad (1.17)$$

has a solution. The infimum in (1.17) is attained for  $\tau = \bar{\tau}$  satisfying

$$\varepsilon(v) - A\bar{\tau} = \frac{1}{\mu}(\bar{\tau} - \pi\bar{\tau}),$$

i. e.,

$$\varepsilon(v)^d - \bar{\tau}^d = \frac{1}{\mu}(\bar{\tau}^d - \pi\bar{\tau}^d), \quad (1.18)$$

$$\text{tr}(\varepsilon(v)) - \lambda \text{tr}(\bar{\tau}) = 0,$$

since by the definition of  $D$ ,  $\text{tr}(\tau - \pi\tau) = 0$  for  $\tau \in H$ . But (1.18) implies that  $\pi\bar{\tau}^d = \pi\varepsilon(v)^d$  and thus

$$\bar{\tau} = \frac{1}{\lambda} \text{tr}(\varepsilon(v)) + \frac{\mu}{1+\mu} \varepsilon(v)^d + \frac{1}{1+\mu} \pi\varepsilon(v)^d,$$

which gives after a simple computation

$$g_\mu(v, V) = \frac{1}{2(1+\mu)} \|\varepsilon(v)^d - \pi\varepsilon(v)^d\|^2 - \frac{1}{2} \|\varepsilon(v)^d\|^2 - \frac{1}{2\lambda} \|\text{tr}(\varepsilon(v))\|^2 + FV.$$

Since  $g_\mu : \mathcal{W} \times \mathbf{R} \rightarrow \mathbf{R}$  is concave (being the infimum of a set of linear functions) and continuous and  $K$  is closed and convex in  $\mathcal{W} \times \mathbf{R}$ , to prove existence of a solution of the problem (1.16) it remains only to prove that  $g_\mu$  is coercive, i. e.,

$$g_\mu(v, V) \rightarrow -\infty \quad \text{as } \|(v, V)\|_{\mathcal{W} \times K} \rightarrow \infty, (v, V) \in \mathcal{W} \times K.$$

But this follows easily from Korn's inequality (see [2]),

$$\|v\|_{\mathcal{W}} \leq C \|\varepsilon(v)\|, \quad v \in \mathcal{W},$$

the trace inequality,

$$\|v_3\|_{L^1(\Gamma_0)} \leq C \|v\|_{\mathcal{W}}, \quad v \in \mathcal{W},$$

and the fact that

$$V \leq v_3 + \varphi \quad \text{on } \Gamma_0,$$

if  $(v, V) \in K$ . Thus the problem (1.16) has a solution  $(u_\mu, U_\mu) \in K$ .

The extremality relation can be written

$$(\sigma_\mu, \varepsilon(v - u_\mu)) \geq F(V - U_\mu), \quad (v, V) \in K,$$

with

$$\sigma_\mu = \frac{1}{\lambda} \text{tr}(\varepsilon(u_\mu)) + \frac{\mu}{1+\mu} \varepsilon(u_\mu)^d + \frac{1}{1+\mu} \pi\varepsilon(u_\mu)^d.$$

Thus (1.15 b) holds and it is easy to check that also (1.15 a) is satisfied and therefore  $(\sigma_\mu, (u_\mu, U_\mu)) \in H \times K$  is a saddle point for  $L_\mu$ .

**(b) A priori estimates**

By varying  $(v, V) \in X$  in (1.15 b) one concludes that  $\sigma_\mu$  satisfies the relations (1.9)-(1.11) and

$$\text{div } \sigma_\mu = 0 \quad \text{in } \Omega, \tag{1.19}$$

$$-\int_{\Gamma_1} \psi \sigma_{\mu, 33} ds = 0, \tag{1.20}$$

$$\int_{\Gamma_1} (u_{\mu, 3} + \psi(\varphi - U_\mu)) \sigma_{\mu, 33} ds = 0. \tag{1.21}$$



Thus, replacing  $\tau$  in (1.15 a) by  $\tau - \sigma_\mu$  where  $\tau \in \mathcal{P}$  and applying Green's formula, paralleling the proof of (1.12) we find that

$$a(\sigma_\mu, \tau - \sigma_\mu) + (J'_\mu(\sigma_\mu), \tau - \sigma_\mu) + (u_\mu, \operatorname{div} \tau - \operatorname{div} \sigma_\mu) + \int_\Gamma \psi(U_\mu - \varphi)(\tau_{33} - \sigma_{33, \mu}) ds \geq 0. \quad (1.22)$$

If we now take  $\tau = \chi$ , where  $\chi$  is given by assumption (1.14) and use the fact  $\sigma_\mu$  as well as  $\chi$  satisfies (1.19) and (1.20), we get

$$\begin{aligned} & \|\sigma_\mu\|_a^2 + (J'_\mu(\sigma_\mu), \sigma_\mu - \chi) \\ & \leq \int_\Gamma \psi \varphi (\chi_{33} - \sigma_{33, \mu}) ds + a(\sigma_\mu, \chi). \end{aligned} \quad (1.23)$$

But, as is easily seen, (1.14) implies that

$$\|J'_\mu(\sigma_\mu)\|_1 \leq \frac{1}{\delta} (J'_\mu(\sigma_\mu), \sigma_\mu - \chi).$$

Using also the estimate (see [2]):

$$\|\tau\|_{H^{-\frac{1}{2}}(\Gamma)} \leq C(\|\tau\| + \|\operatorname{div} \tau\|), \quad (1.24)$$

and (1.19), we now conclude from (1.23) that

$$\|\sigma_\mu\| \leq C(\|\psi \varphi\|_{H^{1/2}(\Gamma)} + \|\chi\|) \leq C, \quad (1.25)$$

$$\|J'_\mu(\sigma_\mu)\|_1 \leq C, \quad (1.26)$$

with  $C$  independent of  $\mu$ . Using the equation (1.15 a), it then follows that

$$\|\varepsilon(u_\mu)\|_1 \leq C, \quad (1.27)$$

so that

$$\|u_\mu\|_{3/2} \leq C, \quad (1.28)$$

by using the estimate

$$\|v\|_{3/2} \leq C\|\varepsilon(v)\|_1, \quad v \in \mathcal{W}.$$

A proof of this result in the case  $\Gamma_2 = \psi$  can be found in [3]. The proof can easily be modified to cover also the present case.

Further, to bound  $U_\mu$  we note that by the easy to prove trace inequality

$$\int_{\Gamma_0} |w| ds \leq C\|w, 3\|_1, \quad w \in W,$$

and (1.27), we have

$$\int_{\Gamma_0} |u_{\mu 3}| ds \leq C.$$

Since  $U_\mu \leq u_{\mu 3} + \varphi$  on  $\Gamma_0$  we thus find that  $U_\mu \leq C$ . Moreover, taking  $\tau = \sigma_\mu$  in (1.15 a) and  $(v, V) = 0$  in (1.15 b) and adding we see that

$$\|\sigma_\mu\|_a^2 + (J'_\mu(\sigma_\mu), \sigma_\mu) \leq FU_\mu.$$

But since  $J'_\mu$  is monotone and  $J'_\mu(0) = 0$ ,  $(J'_\mu(\sigma_\mu), \sigma_\mu) \geq 0$  and thus  $U_\mu \geq 0$  so that

$$|U_\mu| \leq C. \tag{1.29}$$

**(c) Passage to the limit**

From (1.19), (1.24)-(1.27), (1.28) and (1.29) it follows that there exists  $(\sigma, (u, U)) \in \mathcal{P} \times Y \times \mathbf{R}$  with  $\text{div } \sigma = 0$  and a sequence  $\mu$  tending to zero such that

$$\left. \begin{aligned} \sigma_\mu &\rightarrow \sigma \text{ weakly in } H, \\ \sigma_{33, \mu} &\rightarrow \sigma_{33} \text{ weakly in } H^{-1/2}(\Gamma), \\ u_\mu &\rightarrow u \text{ weakly in } Y, \\ U_\mu &\rightarrow U. \end{aligned} \right\} \tag{1.30}$$

Passing to the limit in the relation

$$\begin{aligned} a(\sigma_\mu, \tau - \sigma_\mu) + (u_\mu, \text{div } \tau - \text{div } \sigma_\mu) \\ + \int_\Gamma \psi(U_\mu - \varphi)(\tau_{33} - \sigma_{33, \mu}) ds \geq 0, \quad \tau \in \mathcal{P}, \end{aligned}$$

which follows from (1.22) using the monotonicity of  $J'_\mu$ , we now obtain (1.13 a) without any difficulty recalling that  $\text{div } \sigma_\mu = \text{div } \sigma = 0$ . Finally, (1.13 c) follows from (1.30) by passing to the limit in (1.20). This completes the proof.

**2. HARDENING MATERIAL**

To describe the hardening of the elasto-plastic material (cf. [4]) we shall use a hardening parameter  $\xi = \{\xi_i\}$ ,  $i = 1, \dots, m$ , where  $m$  is a positive integer. We shall use the notation

$$\begin{aligned} \hat{H} &= \{\hat{\sigma} = (\sigma, \xi) : \sigma \in H, \xi \in [L^2(\Omega)]^m\}, \\ [\hat{\sigma}, \hat{\tau}] &= a(\sigma, \tau) + \gamma(\xi, \eta), \quad \hat{\sigma} = (\sigma, \xi), \tau = (\tau, \eta) \in \hat{H}, \\ \|\tau\|_a &= [\hat{\tau}, \hat{\tau}]^{1/2}, \quad \hat{\tau} \in \hat{H}, \end{aligned}$$

where  $\gamma$  is a positive constant. Let now  $\hat{D}$  be a closed convex set in  $R^{9+m}$ , the set of admissible combinations of stress deviatorics and hardening, such that  $0 \in \hat{D}$  and define

$$\hat{P} = \{\hat{\tau} \in \hat{H} : (\tau^d, \eta)(x) \in \hat{D} \text{ a. e. in } \Omega\}.$$

The elasto-plastic contact problem can now be formulated in the following way: Find  $(\hat{\sigma}, (u, U)) \in \hat{P} \times X$  such that

$$[\hat{\sigma}, \hat{\tau} - \hat{\sigma}] - (\varepsilon(u), \tau - \sigma) \geq 0, \quad \forall \hat{\tau} \in \hat{P}, \quad (2.1 a)$$

$$(\sigma, \varepsilon(v - u)) \geq F(V - U), \quad \forall (v, V) \in K, \quad (2.1 b)$$

or, equivalently, find a saddle point  $(\hat{\sigma}, (u, U))$  for the functional  $\hat{L} : \hat{P} \times K \rightarrow \mathbf{R}$  defined by

$$\hat{L}(\hat{\tau}, (v, V)) = \frac{1}{2} \|\hat{\tau}\|_a^2 - (\tau, \varepsilon(v)) + FV.$$

To prove existence of a solution of (2.1) we shall use the same method as that used in Section 1 for the regularized problem (1.15). Thus, we consider the problem

$$\sup_{(v, V) \in K} g(v, V), \quad (2.2)$$

where

$$\begin{aligned} g(v, V) &\equiv \inf_{\hat{\tau} \in \hat{P}} \hat{L}(\hat{\tau}, (v, V)) \\ &= \frac{1}{2} \|\hat{\varepsilon}(v) - \hat{\pi}\hat{\varepsilon}(v)\|^2 - \frac{1}{2} \|\varepsilon(v)\|^2 + FV, \end{aligned}$$

with

$$\hat{\varepsilon}(v) = (\varepsilon(v), 0) \in \hat{H},$$

and  $\hat{\pi}$  being the projection in  $\hat{H}$  onto  $\hat{P}$ . Since  $g(v, V)$  is clearly concave and continuous on  $\mathcal{W} \times \mathbf{R}$ , existence of a solution of (2.2) will follow if  $g$  is coercive on  $K$ , i. e., if

$$g(v, V) \rightarrow -\infty \quad \text{if} \quad \|(v, V)\|_{\mathcal{W}} \rightarrow \infty, \quad (v, V) \in K. \quad (2.3)$$

A solution  $(u, U) \in K$  of (2.2) is characterized by the variational inequality

$$(\hat{\pi}\hat{\varepsilon}(u), \hat{\varepsilon}(v - u)) \geq F(V - U), \quad (v, V) \in K. \quad (2.4)$$

Thus, having a solution  $(u, U)$  of (2.2) we obtain  $(\hat{\sigma}, (u, U)) \in \hat{P} \times K$  satisfying (2.1) by setting  $\hat{\sigma} = \hat{\pi}\hat{\varepsilon}(u)$ . We therefore have the following result:

**THEOREM 2:** *If (2.3) holds, then there exists  $(\hat{\sigma}, U) \in \hat{P} \times X$  satisfying (2.1). Moreover,  $\hat{\sigma}$  is uniquely determined.*

*Remark.* It is easy to verify that (2.3) holds in the following two cases important in applications (for definiteness we use here the von Mises yield criterion, cf. [4]).

(i) *Isotropic hardening:* In this case  $m = 1$  and

$$\hat{B} = \{(\tau, \eta) \in \mathbf{R}^9 \times \mathbf{R} : |\tau^d| \leq 1 + \gamma\eta\}.$$

(ii) *Kinematic hardening*: In this case  $m = 9$  and

$$B = \{(\tau, \eta) \in \mathbf{R}^9 \times \mathbf{R}^9 : |\tau^d - \eta| \leq 1\}.$$

It is easy to see that also  $(u, U)$  is uniquely determined in these two cases.

**3. FINITE ELEMENT METHODS: HARDENING MATERIAL**

We shall only briefly discuss the case of a hardening material assuming that the coercivity condition (2.3) holds. In this case we simply take a finite dimensional subspace  $\mathcal{W}_h$  of  $\mathcal{W}$ , define  $K_h = \mathcal{W}_h \cap K$  and seek a solution  $(u_h, U_h)$  of the problem

$$\sup_{(v, V) \in K_h} g(v, V),$$

or, equivalently, we seek  $(u_h, U_h) \in K_h$  satisfying

$$(\hat{\pi}\hat{\varepsilon}(u_h), \hat{\varepsilon}(v - u_h)) \geq F(V - U_h), \quad (v, V) \in K_h. \tag{3.1}$$

Since  $g$  is coercive it follows that such a  $(u_h, U_h)$  exists.

It is also easy to obtain an estimate for  $\|\hat{\pi}\hat{\varepsilon}(u) - \hat{\pi}\hat{\varepsilon}(u_h)\|$  in the following way: Using the fact that since  $\hat{P}$  is convex,

$$\|\hat{\pi}\hat{\varepsilon}(u) - \hat{\pi}\hat{\varepsilon}(u_h)\|^2 \leq (\hat{\pi}\hat{\varepsilon}(u) - \hat{\pi}\hat{\varepsilon}(u_h), \hat{\varepsilon}(u) - \hat{\varepsilon}(u_h)),$$

and adding (2.4) and (3.1) with  $(v, V) = (u_h, U_h)$  in (2.4), we obtain for all  $(v, V) \in K_h$ ,

$$\|\hat{\pi}\hat{\varepsilon}(u) - \hat{\pi}\hat{\varepsilon}(u_h)\|^2 \leq (\hat{\pi}\hat{\varepsilon}(u_h), v - u) + F(U - V).$$

Having an *a priori* estimate of the form

$$\|\hat{\pi}\hat{\varepsilon}(u_h)\| \leq C, \tag{3.2}$$

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this will then give an estimate for  $\|\hat{\pi}\hat{\varepsilon}(u) - \hat{\pi}\hat{\varepsilon}(u_h)\|$ . In the particular cases discussed in Section 2, (3.2) is easily seen to hold and in these cases we also have

$$\|\varepsilon(u) - \varepsilon(u_h)\| \leq C \|\hat{\pi}\hat{\varepsilon}(u) - \hat{\pi}\hat{\varepsilon}(u_h)\|.$$

**4. FINITE ELEMENT METHODS: PERFECTLY PLASTIC MATERIAL**

We shall now consider a finite element method based on the formulation (1.13). We shall restrict ourselves to a two-dimensional problem (plane stress or plane strain) and we shall then use the notation of Section 1 with the obvious change from three to two dimensions. The  $x_2$ -axis will now correspond to the  $x_3$ -axis in the three-dimensional case. For simplicity we shall assume that  $\Gamma_1$  and  $\Gamma_0$  are line segments (cf. Fig.). In the two-dimensional case Theorem 1 holds with  $\mathcal{P} = \mathcal{P}_2$  and  $Y = Y_2$ .

The finite element method will be the following: Given finite dimensional spaces  $\mathcal{P}_h \subset \mathcal{P}_2$  and  $Y_h \subset Y_2$ , find  $(\sigma_h, u_h, U_h) \in \mathcal{P}_h \times Y_h \times \mathbf{R}$  such that

$$a(\sigma_h, \tau - \sigma_h) + (u_h, \operatorname{div} \tau - \operatorname{div} \sigma_h) + \int_{\Gamma} \psi(U_h - \varphi)(\tau_{22} - \sigma_{22,h}) ds \geq 0, \quad \tau \in \mathcal{P}_h, \quad (4.1 a)$$

$$(v, \operatorname{div} \sigma_h) = 0, \quad v \in Y_h, \quad (4.1 b)$$

$$-\int_{\Gamma} \psi \sigma_{22,h} ds = 0. \quad (4.1 c)$$

We shall now consider a particular choice of the space  $\mathcal{P}_h$  and  $Y_h$ . For simplicity we shall assume that  $\Omega$  is polygonal. Let  $\{\mathcal{C}_h\}$ ,  $0 < h < 1$ , be a regular family of triangulations of  $\Omega$

$$\Omega = \bigcup_{K \in \mathcal{C}_h} K,$$

indexed by the parameter  $h$  denoting as usual the maximum of the diameters of the triangles  $K \in \mathcal{C}_h$ . We assume that nodes are placed at the endpoints of  $\Gamma_0$  and  $\Gamma_1$ . We shall construct a finite dimensional space  $\mathcal{H}_h \subset \mathcal{H}_2$  and then define  $\mathcal{P}_h = \mathcal{H}_h \cap P$ . The finite element method will be an equilibrium method, i. e., the spaces  $\mathcal{H}_h$  and  $Y_h$  will satisfy:

$$\text{If } \tau \in \mathcal{H}_h \text{ and } (\operatorname{div} \tau, v) = 0 \text{ for } v \in Y_h, \text{ then } \operatorname{div} \tau = 0 \text{ in } \Omega. \quad (4.2)$$

Methods of this type including the present one have been studied in [5] in the case of linear elasticity. To define  $\mathcal{H}_h$  each triangle  $K$  is divided into three subtriangles  $T_k$ ,  $k = 1, 2, 3$ , by connecting the center of gravity with the three nodes of  $K$ . For each  $K \in \mathcal{C}_h$  we introduce the finite dimensional space  $H_K$  defined by

$$H_K = \left\{ \tau = \{ \tau_{ij} \} : \tau_{ij} = \tau_{ji} \text{ is linear on } T_k, \right. \\ \left. k = 1, 2, 3, i, j = 1, 2, \text{ and } \operatorname{div} \tau \in [L^2(K)]^2 \right\}.$$

One can prove (see [5]) that an element  $\tau \in H_K$  is uniquely determined by the following 15 degrees of freedom:

$$\text{the value of } \tau \cdot n \text{ at two points of each side of } K, \quad (4.3)$$

$$\int_K \tau_{ij} dx, \quad i, j = 1, 2, \quad (4.4)$$

where  $\tau \cdot n = (\tau_{11} n_1 + \tau_{12} n_2, \tau_{21} n_1 + \tau_{22} n_2)$  and  $n = (n_1, n_2)$  is the outward unit normal to the boundary of  $K$ . The space  $\mathcal{H}_h$  can now be defined:

$$\mathcal{H}_h = \left\{ \tau : \tau|_K \in H_K, K \in \mathcal{C}_h, \text{ and } \operatorname{div} \tau \in Y_2 \right\}.$$

If  $\tau|_K \in H_K$ ,  $K \in \mathcal{C}_h$ , then  $\operatorname{div} \tau \in Y_2$  if and only if  $\tau \cdot n$  is continuous at the interelement boundaries, i. e., if for any side  $S$  common to the triangles  $K$  and  $K'$ ,

$$\tau|_K \cdot n = \tau|_{K'} \cdot n \quad \text{on } S,$$

where  $n$  is a normal to  $S$ . Therefore the degrees of freedom for an element  $\tau \in H_h$  can be chosen as follows: the value of  $\tau \cdot n$  at two points at each side of  $\mathcal{C}_h$  and the values given by (4.4) for  $K \in \mathcal{C}_h$ .

Finally, defining

$$Y_h = \{v \in Y_2 : v \text{ is linear on } K, K \in \mathcal{C}_h\},$$

the particular case of the finite element method (4.1) we want to consider has been fully described.

In addition to (4.2) the spaces  $\mathcal{H}_h$  and  $Y_h$  have the following property (see [5]) important for the analysis: There is an interpolation operator  $\pi_h : \mathcal{H}_2 \rightarrow \mathcal{H}_h$  such that

$$(\operatorname{div} \pi_h \tau, v) = (\operatorname{div} \tau, v), \quad v \in Y_h, \tag{4.5}$$

and

$$\|\tau - \pi_h \tau\| \leq Ch^2 \|\tau\|_{2,2}.$$

Given  $\tau \in \mathcal{H}_2$ , sufficiently regular e. g.  $\tau \in [W_1(\Omega)]^4$ , the interpolant  $\pi_h \tau$  is defined to be the unique element in  $H_h$  satisfying for any side  $S$  of  $\mathcal{C}_h$  with normal  $n$ ,

$$\int_K ((\tau - \pi_h \tau) \cdot n) \cdot v ds = 0 \quad \text{for } v \text{ linear,}$$

and for any  $K \in \mathcal{C}_h$ ,

$$\int_K (\tau_{ij} - (\pi_h \tau)_{ij}) ds = 0, \quad i, j = 1, 2.$$

We observe that if  $\operatorname{div} \tau = 0$  in  $\Omega$  then by (4.2) also  $\operatorname{div} \pi_h \tau = 0$  in  $\Omega$ .

Existence of a solution of the finite element problem can be proved under the following "discrete safe load hypothesis":

$$\left. \begin{aligned} \text{There exists } \delta > 0 \text{ and } \chi_h \in \mathcal{P}_h \cap E \text{ such that} \\ \operatorname{dist}(\chi_h(x), \partial D) \geq \delta, \quad x \in \Omega, \\ \|\chi_h\| \leq C, \end{aligned} \right\} \tag{4.6}$$

where  $C$  and  $\delta$  are independent of  $h$ . We note that if the  $\chi$  in the safe load hypothesis (1.14) for the continuous problem is sufficiently smooth (e. g. if  $\chi$  is continuous), then (4.6) will be true for  $h$  sufficiently small if we choose  $\chi_h = \pi_h \chi$ .

Let us now consider the convergence of the finite element method (4.1). We have the following result on weak convergence:

**THEOREM 3:** *If (4.6) holds, then for any  $p$ ,  $1 \leq p < 2$ , there exists  $(\sigma, u, U) \in \mathcal{P}_q \times Y_p \times R$  satisfying (1.13) with  $\mathcal{P} = \mathcal{P}_q$  and  $Y = Y_p$ , where  $(1/q) + (1/p) = 1$ , and a sequence  $\{h_i\}$  tending to zero, such that*

$$\sigma_h \rightarrow \sigma \text{ weakly in } H \text{ as } h \rightarrow 0,$$

$$u_{h_i} \rightarrow u \text{ weak star in } Y_p,$$

$$U_{h_i} \rightarrow U.$$

*Proof:* The theorem follows easily from the following *a priori* estimates:

$$\|\sigma_h\| \leq C, \tag{4.7}$$

$$\|u_h\| \leq C, \tag{4.8}$$

$$|U_h| \leq C. \tag{4.9}$$

The estimate (4.7) follows directly by taking  $\tau = \chi_h$  in (4.1 a), where  $\chi_h$  is given by (4.6).

Next, (4.9) follows by choosing  $\tau = \chi_h + \pi_h \bar{\chi}$  in (4.1 a), where  $\bar{\chi} \in C^\infty(\Omega) \cap H$  satisfies

$$\operatorname{div} \bar{\chi} = 0 \text{ in } \Omega,$$

$$\int_{\Gamma_1} \bar{\chi}_{22} ds \neq 0,$$

$$\|\bar{\chi}\|_\infty \leq \frac{\delta}{2}.$$

Such a  $\bar{\chi}$  can easily be constructed by solving a suitable linear elastic problem.

Finally, (4.8) will follow by choosing  $\tau = \chi_h + \pi_h \tilde{\chi}$  in (4.1 a), where  $\tilde{\chi} \in \mathcal{H}_2$  satisfies

$$\operatorname{div} \tilde{\chi} = g,$$

where  $g$  ranges over the ball  $\{g \in Y_q : \|g\|_q \leq \mu\}$ , with  $\mu > 0$  sufficiently small. For instance one can choose  $\tilde{\chi} = \varepsilon(\tilde{u})$ , where  $\tilde{u}$  satisfies

$$\begin{aligned} \operatorname{div}(\varepsilon(\tilde{u})) &= g \text{ in } \bar{\Omega}, \\ \varepsilon(\tilde{u}) \cdot n &= 0 \text{ on } \partial\bar{\Omega}, \end{aligned}$$

and  $\bar{\Omega}$  is a domain with smooth boundary such that  $\Omega \subset \bar{\Omega}$ ,  $\Gamma_1 \subset \partial\bar{\Omega}$  and  $g$  is suitably extended outside  $\Omega$ . To see that  $\tau = \chi_h + \pi_h \tilde{\chi} \in \mathcal{P}_h$  for  $\mu$  sufficiently

small, we note that by elliptic regularity (see [6]) one has

$$\|\tilde{\chi}\|_{1,q} \leq C \|g\|_q,$$

which by Sobolev's imbedding theorem implies that  $\tilde{\chi}$  is continuous and

$$\|\tilde{\chi}\|_{\infty} \leq C \|g\|_q.$$

Now, (4.1 a) and (4.5) together with the previously obtained estimates (4.7) and (4.9) show that

$$(u_h, g) = (u_h, \operatorname{div} \chi) = (u_h, \operatorname{div} \pi_h \tilde{\chi}) \leq C,$$

if  $\|g\|_q \leq \mu$ , which proves (4.8). This completes the proof.

Finally, we shall obtain an estimate for  $\sigma - \sigma_h$  in terms of the quantity

$$\alpha = \inf \{ \beta : \exists \tau_h \in \mathcal{H}_h \cap E \text{ such that } (1 - \beta) \tau_h \in P \text{ and } \|\sigma - \tau_h\| \leq \beta \}.$$

If  $\sigma$  is sufficiently regular then by choosing  $\tau_h = \pi_h \sigma$  we see that  $\alpha \rightarrow 0$  as  $h \rightarrow 0$ .

**THEOREM 4:** *If (4.6) holds then there exists a constant C independent of h such that for  $\alpha < 1$ ,*

$$\|\sigma - \sigma_h\| \leq C \sqrt{\alpha}.$$

*Proof:* Choosing  $\tau = (1 - 2\alpha) \tau_h$  in (4.1 a) where  $\tau_h \in \mathcal{H}_h \cap E$  satisfies  $\|\sigma - \tau_h\| \leq 2\alpha$ , and  $\tau = \sigma_h$  in (1.13 a) and adding, we easily find that

$$\frac{1}{C} \|\sigma - \sigma_h\|^2 \leq \|\sigma - \sigma_h\|_a^2 \leq a(\sigma_h, (1 - 2\alpha) \tau_h - \sigma).$$

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$$+ 2\alpha U_h F + \int_{\Gamma_1} \varphi(\sigma_{22} - \tau_{22,h}) ds + 2\alpha \int_{\Gamma_1} \varphi \tau_{22,h} ds.$$


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Thus, using the estimates (4.7), (4.9) and (1.24) we obtain the desired estimate.

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