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# AN ELASTO-PLASTIC CONTACT PROBLEM (*) (1)

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Communiqué par P. G. Ciarlet

Abstract. — *We study the problem of finding the stresses and the displacements in an elasto-plastic body $\mathscr{E}$ in frictionless contact with a rigid body which is pressed against $\mathscr{E}$. We prove existence of a solution and then we consider finite element methods for finding approximate solutions of the problem.*

## INTRODUCTION

Duvaut [1] has studied the problem of finding the stresses in an elasto-plastic body $\mathscr{E}$ in frictionsless contact with a rigid body $\mathscr{B}$ which is pressed against $\mathscr{E}$. In this note we extend the study of Duvaut by looking also for the displacements of $\mathscr{E}$ and $\mathscr{B}$. We shall consider a stationary case corresponding to Henky's law. For simplicity we shall assume that $\mathscr{E}$ is isotropic.

In Section 1 we prove existence of a solution to the contact problem assuming that $\mathscr{E}$ is elastic-perfectly plastic. In this case the displacements of $\mathscr{E}$ may be discontinuous (and even non-unique) and we have to use a formulation requiring little regularity of the displacements. One way of obtaining more regular displacements is to assume a suitable hardening of the elasto-plastic material. Such a case is studied in Section 2. Then in Sections 3 and 4 we consider finite element methods for finding approximate solutions of the contact problem.

## 1. ELASTIC-PERFECTLY PLASTIC MATERIAL

Suppose that initially the elasto-plastic body $\mathscr{E}$ occupies the bounded region $\Omega \subset \mathbf{R}^3$ with boundary $\Gamma$ and that $\Gamma$ contains an open set $\Gamma_1$ in the plane $\{ x = (x_1, x_2, x_3) \in \mathbf{R}^3 : x_3 = 0 \}$. Moreover, suppose that initially the rigid body $\mathscr{B}$ occupies the region

$$B = \{ x \in \mathbf{R}^3 : -x_3 \geqq \varphi (x_1, x_2), (x_1, x_2) \in \bar{\Gamma}_0 \},$$

where $\Gamma_0$ is an open set compactly contained in $\Gamma_1$ with smooth boundary and $\varphi : \Gamma_1 \to \mathbf{R}$ is smooth, nonnegative and $\varphi(\bar{x}) = 0$ for some $\bar{x} \in \Gamma_0$ (see *Fig.*).

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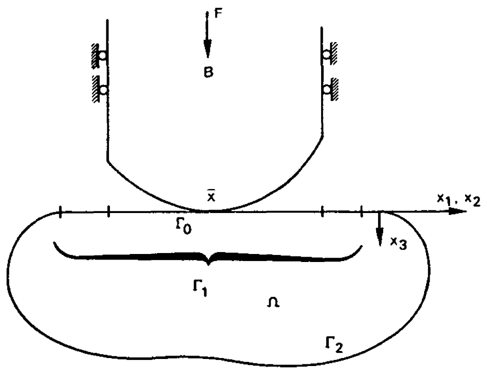


Let the boundary of $\mathscr{E}$ be fixed on the portion $\Gamma_2 = \Gamma \setminus \Gamma_1$ and free on $\Gamma_1$. Let $\mathscr{B}$ be acted upon by the vertical force $F\,(F > 0)$ and suppose that $\mathscr{B}$ is free to move vertically, whereas rotation and horisontal displacement are prevented. We want to find the vertical displacement $U$ of $\mathscr{B}$, the stress $\sigma = \{ \sigma_{ij} \}$, $i, j = 1, 2, 3$, in $\mathscr{E}$, and the displacement $u = \{ u_i \}$, $i = 1, 2, 3$, of $\mathscr{E}$, where $u_i$ is the displacement in the $x_i$-direction. The reference configuration is the one in *figure*. We shall assume that the displacements are small; in particular this means that the relation $u_3 \geqq U - \varphi$ can be used to describe the compatibility of the displacements of $\mathscr{B}$ and $\mathscr{E}$.

We shall use the following notation: For $m$ a positive integer and $1 \leqq p \leqq \infty$, let $\| \cdot \|_{m,p}$ denote the norm in the usual Sobolev space $[W_p^m(\Omega)]^n$ with $n$ a positive integer. If $m = 0$ we omit this index and write $\| \cdot \|_p$ instead of $\| \cdot \|_{0,p}$. Let $(.,.)$ and $\| \cdot \|$ denote the scalar product and norm in $[L^2(\Omega)]^n$. Furter we define

$$H = \{ \tau = \{\tau_{ij}\} \in [L^2(\Omega)]^9 : \quad \tau_{ij} = \tau_{ji}, i, j = 1, 2, 3, \},$$

$$\mathscr{W} = [W]^3, \qquad W = \{ w \in W_2^1(\Omega) : \quad w = 0 \text{ on } \Gamma_2 \},$$

$$K = \{ (u, U) \in \mathscr{W} \times \mathbf{R} : \quad u_3 + \varphi - U \geqq 0 \text{ on } \bar{\Gamma}_0 \},$$

and the deformation $\varepsilon(u) = \{ \varepsilon_{ij}(u) \}$ associated with $u \in \mathscr{W}$ by

$$\varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}), \qquad i, j = 1, 2, 3,$$

where $w_{,j} = \partial w / \partial x_j$. We recall Green's formula:

$$(\tau, \varepsilon(v)) = \int_\Gamma \tau_{ij} n_j u_i \, ds - (\operatorname{div} \tau, v), \tag{1.1}$$

where $n = (n_1, n_2, n_3)$ is the outward unit normal to $\Gamma$ and

$$\operatorname{div}\tau = ((\operatorname{div}\tau)_i), \qquad i = 1, 2, 3,$$

$$(\operatorname{div}\tau)_i = \tau_{ij,\,j}.$$

Here and below we use the summation convention: repeated indices indicate summation from 1 to 3.

Let $D \subset \mathbf{R}^9$ be a given closed convex set with $0 \in D$ and define the set of plastically admissible stresses

$$P = \{\tau \in H : \quad \tau^d(x) \in D \text{ a.e. in } \Omega\},$$

where

$$\tau^d = \tau - \frac{1}{3}\operatorname{tr}(\tau)\,\delta,$$

$$\operatorname{tr}(\tau) = \tau_{kk},$$

$$\delta = \{\delta_{ij}\}, \qquad \delta_{ij} = \begin{cases} 1 & \text{if } \quad i = j, \\ 0 & \text{if } \quad i \neq j. \end{cases}$$

The tensor $\tau^d$ is the so called *stress deviatoric* associated to $\tau$. If $\sigma(x) \in$ (interior of $D$), then $\mathscr{E}$ is in an elastic state at the point $x \in \Omega$ and then we have the linear strain-stress relation

$$\varepsilon(u) = A\,\sigma,$$

$$(A\,\sigma)_{ij} = \lambda\operatorname{tr}(\sigma)\,\delta_{ij} + \nu\,\sigma_{ij}^d,$$

where $\lambda$ and $\nu$ are positive constants. For notational simplicity we shall assume below that $\nu = 1$. We define the bilinear form

$$\cancel{a(\sigma, \tau) = \int_\Omega (\lambda\operatorname{tr}(\sigma)\operatorname{tr}(\tau) + \sigma_{ij}^d\tau_{ij}^d)\,dx},$$

and we note that

$$a(\tau, \tau) \geqq \alpha\,\|\tau\|^2, \qquad \tau \in H, \tag{1.2}$$

where $\alpha = \min(1, 9\lambda)$.

A natural formulation of the contact problem is now the following: Find $(\sigma, (u, V)) \in P \times K$ such that

$$a(\sigma, \tau - \sigma) - (\varepsilon(u), \tau - \sigma) \geqq 0, \qquad \forall \tau \in P, \tag{1.3 a}$$

$$(\sigma, \varepsilon(v - u)) \geqq F(V - U), \qquad \forall (v, V) \in K, \tag{1.3 b}$$

or, equivalently, find a saddle point $(\sigma, (u, V)) \in P \times K$ for the functional $L : P \times K \to \mathbf{R}$ defined by

$$L(\tau, (v, V)) = \frac{1}{2}\|\tau\|_a^2 - (\tau; \varepsilon(v)) + F\,V,$$

where $\|\,.\,\|_a^2 = a(.,.)$. However, the regularity of the displacement $u$ needed in this formulation is in general not possible to achieve in the perfectly-plastic case. Therefore, we shall instead consider a formulation requiring less regularity of $u$ (*cf.* [3]). To be more precise, we shall seek $u$ in the space $Y_{3/2}$, where for $1 \leqq p \leqq \infty$,

$$Y_p = [L_p(\Omega)]^3.$$

To motivate this formulation we first note that be Green's formula (1.1), it follows that (1.3 *b*) is formally equivalent to the following relations:

$$\operatorname{div}\sigma = 0 \quad \text{in } \Omega, \tag{1.4}$$

$$-\int_{\Gamma_0} \sigma_{33}\, ds = F, \tag{1.5}$$

$$\sigma_{13} = \sigma_{23} = 0,\ \sigma_{33} \leqq 0 \quad \text{on } \Gamma_1, \tag{1.6}$$

$$\sigma_{33} = 0 \quad \text{on } \Gamma_1 \setminus \Gamma_0, \tag{1.7}$$

$$\sigma_{33}(x) = 0 \qquad \text{if} \quad (u_3 + \varphi - U)(x) > 0, \quad x \in \Gamma_0, \tag{1.8}$$

which is the intuitive way of formulating the statical relationship in the contact problem.

We shall seek the stress $\sigma$ in the space $\mathscr{P}_3 = P \cap \mathscr{H}_3$, where for $2 \leqq q < \infty$,

$$\mathscr{H}_q = \{\tau \in H : \ \operatorname{div}\tau \in Y_q \text{ and } \tau \text{ satisfies (1.6) and (1.7)}\}.$$

Here (1.6) and (1.7) are to be understood in the following sense:

$$\int_\Gamma \tau_{13}\, w ds = \int_\Gamma \tau_{23}\, w ds = 0, \qquad w \in \mathscr{W}, \tag{1.9}$$

$$\int_\Gamma \tau_{33}\, w ds \leqq 0, \qquad w \in \mathscr{W}, \quad w \geqq 0, \tag{1.10}$$

$$\int_\Gamma \tau_{33}\, w ds = 0 \qquad \text{for} \quad w \in \mathscr{W} \quad \text{such that } w = 0 \text{ on } \Gamma_0. \tag{1.11}$$

Note that if $\tau \in H$ and $\operatorname{div}\tau \in Y_2$, then (*see* [2]) $\tau_{ij}\, n_j \in H^{-\frac{1}{2}}(\Gamma)$ so that (1.9)-(1.11) are meaningful. Now, taking $\tau \in \mathscr{P}_3$ in (1.3 *a*) using Green's formula and (1.6), we find with $\psi \in \mathscr{W}$ satisfying $\psi = 1$ on $\Gamma_0$, that

$$\begin{aligned}
0 &\leqq a(\sigma, \tau - \sigma) - (\varepsilon(u), \tau - \sigma) \\
&= a(\sigma, \tau - \sigma) + (u, \operatorname{div}\tau - \operatorname{div}\sigma) + \int_\Gamma u_3 (\tau_{33} - \sigma_{33})\, ds \\
&= a(\sigma, \tau - \sigma) + (u, \operatorname{div}\tau - \operatorname{div}\sigma) + \int_\Gamma (u_3 + \psi(\varphi - U))(\tau_{33} - \sigma_{33})\, ds \\
&\quad + \int_\Gamma \psi(U - \varphi)(\tau_{33} - \sigma_{33})\, ds,
\end{aligned}$$

so that

$$a(\sigma, \tau-\sigma)+(u, \operatorname{div}\tau-\operatorname{div}\sigma)+\int_{\Gamma}\psi(U-\varphi)(\tau_{33}-\sigma_{33})\,ds \geqq 0, \tag{1.12}$$

since by (1.7)-(1.8),

$$\int_{\Gamma}(u_3+\psi(\varphi-U))\,\sigma_{33}\,ds = 0,$$

and by (1.10) and (1.11) assuming that $(u, U) \in K$,

$$\int_{\Gamma}(u_3+\psi(\varphi-U))\,\tau_{33}\,ds \leqq 0.$$

We are thus led to the following formulation of the contact problem: Find $(\sigma, u, U) \in \mathscr{P} \times Y \times \mathbf{R}$ such that

$$a(\sigma, \tau-\sigma)+(u, \operatorname{div}\tau-\operatorname{div}\sigma)$$

$$+\int_{\Gamma}\psi(U-\varphi)(\tau_{33}-\sigma_{33})\,ds \geqq 0, \qquad \tau \in \mathscr{P}, \tag{1.13 a}$$

$$(v, \operatorname{div}\sigma) = 0, \qquad v \in Y, \tag{1.13 b}$$

$$-\int_{\Gamma}\psi\,\sigma_{33}\,ds = F, \tag{1.13 c}$$

where $\mathscr{P} = \mathscr{P}_3$ and $Y = Y_{3/2}$.

*Remark:* Note that the condition $u_3+\varphi-U \geqq 0$ on $\Gamma_0$ does not appear explicitety in this formulation, which is natural since the trace of $u \in Y$ on $\Gamma$ may not be defined.

To prove existence of a solution of (1.13) we shall need the following "safe load hypothesis":

$$\text{There exists } \delta > 0 \quad \text{and} \quad \chi \in \mathscr{P} \cap E$$

$$\cancel{\text{such that } \operatorname{dist}(\chi(x), \partial D) > \delta \text{ for } x \in \Omega, \qquad (1.14)}$$

where $\partial D$ denotes the boundary of $D$ and

$$E = \left\{ \tau \in H: \ \operatorname{div}\tau = 0 \text{ in } \Omega, \ -\int_{\Gamma}\psi\,\tau_{33} = F \right\}.$$

Note that with $\delta = 0$, this is a necessary condition for existence of a solution.

THEOREM 1: *If* (1.14) *holds then there exists* $(\sigma, u, U) \in \mathscr{P} \times Y \times \mathbf{R}$, *satisfying* (1.13). *Moreover* $\sigma$ *is uniquely determined.*

*Proof:* The proof will be divided into three parts: First we prove existence of a solution of a regularized problem depending on a parameter $\mu > 0$. Then we establish some *a priori* estimates for the solution of this problem and finally we obtain a solution of the original problem by passing to the limit as $\mu$ tends to zero. The uniqueness of $\sigma$ is easy to prove.

**(a) The regularized problem**

For $\mu > 0$ we consider the following problem: Find a saddle point $(\sigma_\mu, (u_\mu, U_\mu)) \in H \times K$ for the regularized Lagrangian $L_\mu : H \times K \to \mathbf{R}$ defined by

$$L_\mu(\tau, (v, V)) = \frac{1}{2} \| \tau \|_a^2 + J_\mu(\tau) - (\tau, \varepsilon(v)) + F V,$$

where

$$J_\mu(\tau) = \frac{1}{2\mu} \| \tau - \pi\tau \|^2,$$

and $\pi$ is the orthogonal projection in $\mathbf{R}^9$ onto $D$. In other words, we seek $(\sigma_\mu, (u_\mu, U_\mu)) \in H \times K$ satisfying

$$a(\sigma_\mu, \tau) + (J'_\mu(\sigma_\mu), \tau) - (\tau, \varepsilon(u_\mu)) = 0, \qquad \tau \in H, \tag{1.15a}$$

$$(\sigma_\mu, \varepsilon(v - u_\mu)) \geqq F(V - U_\mu), \qquad (v, V) \in K, \tag{1.15b}$$

where $J'(\tau) = (1/\mu)(\tau - \pi\tau)$. Existence of a saddle point $(\sigma_\mu, (u_\mu, U_\mu))$ will follow easily if we can show that the problem

$$\sup_{(v, V) \in K} g_\mu(v, V), \tag{1.16}$$

where

$$g_\mu(v, V) = \inf_{\tau \in H} L_\mu(\tau, (v, V)), \tag{1.17}$$

has a solution. The infimum in (1.17) is attained for $\tau = \bar\tau$ satisfying

$$\varepsilon(v) - A\bar\tau = \frac{1}{\mu}(\bar\tau - \pi\bar\tau),$$

i. e.,

$$\varepsilon(v)^d - \bar\tau^d = \frac{1}{\mu}(\bar\tau^d - \pi\bar\tau^d), \tag{1.18}$$

$$\operatorname{tr}(\varepsilon(v)) - \lambda \operatorname{tr}(\bar\tau) = 0,$$

since by the definition of $D$, tr $(\tau - \pi\tau) = 0$ for $\tau \in H$. But (1.18) implies that $\pi\bar\tau^d = \pi\varepsilon(v)^d$ and thus

$$\bar\tau = \frac{1}{\lambda}\operatorname{tr}(\varepsilon(v)) + \frac{\mu}{1+\mu}\varepsilon(v)^d + \frac{1}{1+\mu}\pi\varepsilon(v)^d,$$

which gives after a simple computation

$$g_\mu(v, V) = \frac{1}{2(1+\mu)} \| \varepsilon(v)^d - \pi\varepsilon(v)^d \|^2 - \frac{1}{2} \| \varepsilon(v)^d \|^2$$

$$- \frac{1}{2\lambda} \| \operatorname{tr}(\varepsilon(v)) \|^2 + F V.$$

Since $g_\mu : \mathscr{W} \times \mathbf{R} \to \mathbf{R}$ is concave (being the infimum of a set of linear functions) and continuous and $K$ is closed and convex in $\mathscr{W} \times \mathbf{R}$, to prove existence of a solution of the problem (1.16) it remains only to prove that $g_\mu$ is coercive, i. e.,

$$g_\mu(v, V) \to -\infty \qquad \text{as} \quad \| (v, V) \|_{\mathscr{W} \times K} \to \infty, (v, V) \in \mathscr{W} \times K.$$

But this follows easily from Korn's inequality (*see* [2]),

$$\| v \|_{\mathscr{W}} \leqq C \| \varepsilon(v) \|, \qquad v \in \mathscr{W},$$

the trace inequality,

$$\| v_3 \|_{L^1(\Gamma_0)} \leqq C \| v \|_{\mathscr{W}}, \qquad v \in \mathscr{W},$$

and the fact that

$$V \leqq v_3 + \varphi \quad \text{on } \Gamma_0,$$

if $(v, V) \in K$. Thus the problem (1.16) has a solution $(u_\mu, U_\mu) \in K$.

The extremality relation can be written

$$(\sigma_\mu, \varepsilon(v - u_\mu)) \geqq F(V - U_\mu), \qquad (v, V) \in K,$$

with

$$\sigma_\mu = \frac{1}{\lambda} \operatorname{tr}(\varepsilon(u_\mu)) + \frac{\mu}{1+\mu} \varepsilon(u_\mu)^d + \frac{1}{1+\mu} \pi\varepsilon(u_\mu)^d.$$

Thus (1.15 *b*) holds and it is easy to check that also (1.15 *a*) is satisfied and therefore $(\sigma_\mu, (u_\mu, U_\mu)) \in H \times K$ is a saddle point for $L_\mu$

**(b)** ***A priori* estimates**

By varying $(v, V) \in \mathrm{X}$ in (1.15 *b*) one concludes that $\sigma_\mu$ satisfies the relations (1.9)-(1.11) and

$$\operatorname{div} \sigma_\mu = 0 \quad \text{in } \Omega, \tag{1.19}$$

$$- \int_{\Gamma_1} \psi \, \sigma_{\mu, 33} \, ds = 0, \tag{1.20}$$

$$\int_{\Gamma_1} (u_{\mu, 3} + \psi(\varphi - U_\mu)) \sigma_{\mu, 33} \, ds = 0. \tag{1.21}$$

Thus, replacing $\tau$ in (1.15 *a*) by $\tau-\sigma_\mu$ where $\tau \in \mathscr{P}$ and applying Green's formula, paralleling the proof of (1.12) we find that

$$a(\sigma_\mu, \tau-\sigma_\mu)+(J'_\mu(\sigma_\mu), \tau-\sigma_\mu) \\ +(u_\mu, \operatorname{div}\tau-\operatorname{div}\sigma_\mu)+\int_\Gamma \psi(U_\mu-\varphi)(\tau_{33}-\sigma_{33,\mu})\,ds \geqq 0. \quad (1.22)$$

If we now take $\tau = \chi$, where $\chi$ is given by assumption (1.14) and use the fact $\sigma_\mu$ as well as $\chi$ satisfies (1.19) and (1.20), we get

$$\|\sigma_\mu\|_a^2+(J'_\mu(\sigma_\mu), \sigma_\mu-\chi) \\ \leqq \int_\Gamma \psi\varphi(\chi_{33}-\sigma_{33,\mu})\,ds+a(\sigma_\mu, \chi). \quad (1.23)$$

But, as is easily seen, (1.14) implies that

$$\|J'_\mu(\sigma_\mu)\|_1 \leqq \frac{1}{\delta}(J'_\mu(\sigma_\mu), \sigma_\mu-\chi).$$

Using also the estimate (*see* [2]):

$$\|\tau\|_{H^{-\frac{1}{2}}(\Gamma)} \leqq C(\|\tau\|+\|\operatorname{div}\tau\|), \quad (1.24)$$

and (1.19), we now conclude from (1.23) that

$$\|\sigma_\mu\| \leqq C(\|\psi\varphi\|_{H^{1/2}(\Gamma)}+\|\chi\|) \leqq C, \quad (1.25)$$

$$\|J'_\mu(\sigma_\mu)\|_1 \leqq C, \quad (1.26)$$

with $C$ independent of $\mu$. Using the equation (1.15 *a*), it then follows that

$$\|\varepsilon(u_\mu)\|_1 \leqq C, \quad (1.27)$$

so that

$$\|u_\mu\|_{3/2} \leqq C, \quad (1.28)$$

by using the estimate

$$\|v\|_{3/2} \leqq C\|\varepsilon(v)\|_1, \qquad v \in \mathscr{W}.$$

A proof of this result in the case $\Gamma_2 = \psi$ can be found in [3]. The proof can easily be modified to cover also the present case.

Further, to bound $U_\mu$ we note that by the easy to prove trace inequality

$$\int_{\Gamma_0} |w|\,ds \leqq C\|w,_3\|_1, \qquad w \in W,$$

and (1.27), we have

$$\int_{\Gamma_0} |u_{\mu 3}|\,ds \leqq C.$$

Since $U_\mu \leqq u_{\mu 3}+\varphi$ on $\Gamma_0$ we thus find that $U_\mu \leqq C$. Moreover, taking $\tau = \sigma_\mu$ in (1.15 *a*) and $(v, V) = 0$ in (1.15 *b*) and adding we see that

$$\|\sigma_\mu\|_a^2+(J'_\mu(\sigma_\mu), \sigma_\mu) \leqq F U_\mu .$$

But since $J'_\mu$ is monotone and $J'_\mu(0) = 0$, $(J'_\mu(\sigma_\mu), \sigma_\mu) \geqq 0$ and thus $U_\mu \geqq 0$ so that

$$|U_\mu| \leqq C. \tag{1.29}$$

**(c) Passage to the limit**

From (1.19), (1.24)-(1.27), (1.28) and (1.29) it follows that there exists $(\sigma, (u, U)) \in \mathscr{P} \times Y \times \mathbf{R}$ with $\operatorname{div} \sigma = 0$ and a sequence $\mu$ tending to zero such that

$$\left.\begin{aligned} \sigma_\mu &\to \sigma \quad \text{weakly in } H, \\ \sigma_{33,\mu} &\to \sigma_{33} \quad \text{weakly in } H^{-1/2}(\Gamma), \\ u_\mu &\to u \quad \text{weakly in } Y, \\ U_\mu &\to U. \end{aligned}\right\} \tag{1.30}$$

Passing to the limit in the relation

$$a(\sigma_\mu, \tau-\sigma_\mu)+(u_\mu, \operatorname{div}\tau-\operatorname{div}\sigma_\mu) + \int_\Gamma \psi(U_\mu-\varphi)(\tau_{33}-\sigma_{33,\mu})\, ds \geqq 0, \qquad \tau \in \mathscr{P},$$

which follows from (1.22) using the monotonicity of $J'_\mu$, we now obtain (1.13 *a*) without any difficulty recalling that $\operatorname{div}\sigma_\mu = \operatorname{div}\sigma = 0$. Finally, (1.13 *c*) follows from (1.30) by passing to the limit in (1.20). This completes the proof.

## 2. HARDENING MATERIAL

To describe the hardening of the elasto-plastic material (*cf.* [4]) we shall use a hardening parameter $\xi = \{\xi_i\}$, $i = 1, \ldots, m$, where $m$ is a positive integer. We shall use the notation

$$\hat{H} = \{\hat{\sigma} = (\sigma, \xi) : \quad \sigma \in H, \ \xi \in [L^2(\Omega)]^m\},$$

$$[\hat{\sigma}, \hat{\tau}] = a(\sigma, \tau)+\gamma(\xi, \eta), \qquad \hat{\sigma} = (\sigma, \xi), \ \tau = (\tau, \eta) \in \hat{H},$$

$$\|\tau\|_a = [\hat{\tau}, \hat{\tau}]^{1/2}, \qquad \hat{\tau} \in \hat{H},$$

where $\gamma$ is a positive constant. Let now $\hat{D}$ be a closed convex set in $R^{9+m}$, the set of admissible combinations of stress deviatorics and hardening, such that $0 \in \hat{D}$ and define

$$\hat{P} = \{\hat{\tau} \in \hat{H} : \quad (\tau^d, \eta)(x) \in \hat{D} \text{ a.e. in } \Omega\}.$$

The elasto-plastic contact problem can now be formulated in the following way: Find $(\hat{\sigma}, (u, U)) \in \hat{P} \times X$ such that

$$[\hat{\sigma}, \hat{\tau}-\hat{\sigma}]-(\varepsilon(u), \tau-\sigma) \geqq 0, \qquad \forall \hat{\tau} \in \hat{P}, \tag{2.1 a}$$

$$(\sigma, \varepsilon(v-u)) \geqq F(V-U), \qquad \forall (v, V) \in K, \tag{2.1 b}$$

or, equivalently, find a saddle point $(\hat{\sigma}, (u, U))$ for the functional $\hat{L} : \hat{P} \times K \rightarrow \mathbf{R}$ defined by

$$\hat{L}(\hat{\tau}, (v, V)) = \frac{1}{2} \| \hat{\tau} \|_a^2 - (\tau, \varepsilon(v)) + F V.$$

To prove existence of a solution of (2.1) we shall use the same method as that used in Section 1 for the regularized problem (1.15). Thus, we consider the problem

$$\sup_{(v, V) \in K} g(v, V), \tag{2.2}$$

where

$$\begin{aligned} g(v, V) &\equiv \inf_{\hat{\tau} \in \hat{P}} \hat{L}(\hat{\tau}, (v, V)) \\ &= \frac{1}{2} \| \hat{\varepsilon}(v) - \hat{\pi}\hat{\varepsilon}(v) \|^2 - \frac{1}{2} \| \varepsilon(v) \|^2 + F V, \end{aligned}$$

with

$$\hat{\varepsilon}(v) = (\varepsilon(v), 0) \in \hat{H},$$

and $\hat{\pi}$ being the projection in $\hat{H}$ onto $\hat{P}$. Since $g(v, V)$ is clearly concave and continuous on $\mathscr{W} \times \mathbf{R}$, existence of a solution of (2.2) will follow if $g$ is coercive on $K$, i. e., if

$$g(v, V) \rightarrow -\infty \qquad \text{if} \quad \| (v, V) \|_{\mathscr{W}} \rightarrow \infty, \qquad (v, V) \in K. \tag{2.3}$$

A solution $(u, U) \in K$ of (2.2) is characterized by the variational inequality

$$(\hat{\pi}\hat{\varepsilon}(u), \hat{\varepsilon}(v-u)) \geqq F(V-U), \qquad (v, V) \in K. \tag{2.4}$$

Thus, having a solution $(u, U)$ of (2.2) we obtain $(\hat{\sigma}, (u, U)) \in \hat{P} \times K$ satisfying (2.1) by setting $\hat{\sigma} = \hat{\pi}\hat{\varepsilon}(u)$. We therefore have the following result:

THEOREM 2: *If* (2.3) *holds, then there exists* $(\hat{\sigma}, U)) \in \hat{P} \times X$ *satisfying* (2.1). *Moreover,* $\hat{\sigma}$ *is uniquely determined.*

*Remark.* It is easy to verify that (2.3) holds in the following two cases important in applications (for definiteness we use here the von Mises yield criterion, *cf.* [4]).

(i) *Isotropic hardening:* In this case $m = 1$ and

$$\hat{B} = \{ (\tau, \eta) \in \mathbf{R}^9 \times \mathbf{R} : \quad | \tau^d | \leqq 1 + \gamma\eta \}.$$

(ii) *Kinematic hardening:* In this case $m = 9$ and

$$B = \{(\tau, \eta) \in \mathbf{R}^9 \times \mathbf{R}^9 : |\tau^d - \eta| \leqq 1\}.$$

It is easy to see that also $(u, U)$ is uniquely determined in these two cases.

## 3. FINITE ELEMENT METHODS: HARDENING MATERIAL

We shall only briefly discuss the case of a hardening material assuming that the coercivity condition (2.3) holds. In this case we simply take a finite dimensional subspace $\mathscr{W}_h$ of $\mathscr{W}$, define $K_h = \mathscr{W}_h \cap K$ and seek a solution $(u_h, U_h)$ of the problem

$$\sup_{(v, V) \in K_h} g(v, V),$$

or, equivalently, we seek $(u_h, U_h) \in K_h$ satisfying

$$(\hat{\pi}\hat{\varepsilon}(u_h), \hat{\varepsilon}(v - u_h)) \geqq F(V - U_h), \qquad (v, V) \in K_h. \tag{3.1}$$

Since $g$ is coercive it follows that such *a* $(u_h, U_h)$ exists.

It is also easy to obtain an estimate for $\| \hat{\pi}\hat{\varepsilon}(u) - \hat{\pi}\hat{\varepsilon}(u_h) \|$ in the following way: Using the fact that since $\hat{P}$ is convex,

$$\| \hat{\pi}\hat{\varepsilon}(u) - \hat{\pi}\hat{\varepsilon}(u_h) \|^2 \leqq (\hat{\pi}\hat{\varepsilon}(u) - \hat{\pi}\hat{\varepsilon}(u_h), \hat{\varepsilon}(u) - \hat{\varepsilon}(u_h)),$$

and adding (2.4) and (3.1) with $(v, V) = (u_h, U_h)$ in (2.4), we obtain for all $(v, V) \in K_h$,

$$\| \hat{\pi}\hat{\varepsilon}(u) - \hat{\pi}\hat{\varepsilon}(u_h) \|^2 \leqq (\hat{\pi}\hat{\varepsilon}(u_h), v - u) + F(U - V).$$

Having an *a priori* estimate of the form

$$\| \hat{\pi}\hat{\varepsilon}(u_h) \| \leqq C, \tag{3.2}$$

this will then give an estimate for $\| \hat{\pi}\hat{\varepsilon}(u) - \hat{\pi}\hat{\varepsilon}(u_h) \|$. In the particular cases discussed in Section 2, (3.2) is easily seen to hold and in these cases we also have

$$\| \varepsilon(u) - \varepsilon(u_h) \| \leqq C \| \hat{\pi}\hat{\varepsilon}(u) - \hat{\pi}\hat{\varepsilon}(u_h) \|.$$

## 4. FINITE ELEMENT METHODS: PERFECTLY PLASTIC MATERIAL

We shall now consider a finite element method based on the formulation (1.13). We shall restrict ourselves to a two-dimensional problem (plane stress or plane strain) and we shall then use the notation of Section 1 with the obvious change from three to two dimensions. The $x_2$-axis will now correspond to the $x_3$-axis in the three-dimensional case. For simplicity we shall assume that $\Gamma_1$ and $\Gamma_0$ are line segments (cf. *Fig.*). In the two-dimensional case Theorem 1 holds with $\mathscr{P} = \mathscr{P}_2$ and $Y = Y_2$.

The finite element method will be the following: Given finite dimensional spaces $\mathscr{P}_h \subset \mathscr{P}_2$ and $Y_h \subset Y_2$, find $(\sigma_h, u_h, U_h) \in \mathscr{P}_h \times Y_h \times \mathbf{R}$ such that

$$a(\sigma_h, \tau-\sigma_h)+(u_h, \operatorname{div}\tau-\operatorname{div}\sigma_h) + \int_\Gamma \psi(U_h-\varphi)(\tau_{22}-\sigma_{22,h})\,ds \geqq 0, \qquad \tau\in\mathscr{P}_h, \tag{4.1 a}$$

$$(v, \operatorname{div}\sigma_h) = 0, \qquad v\in Y_h, \tag{4.1 b}$$

$$-\int_\Gamma \psi\,\sigma_{22,h}\,ds = 0. \tag{4.1 c}$$

We shall now consider a particular choice of the space $\mathscr{P}_h$ and $Y_h$. For simplicity we shall assume that $\Omega$ is polygonal. Let $\{\mathscr{C}_h\}$, $0 < h < 1$, be a regular family of triangulations of $\Omega$

$$\Omega = \bigcup_{K\in\mathscr{C}_h} K,$$

indexed by the parameter $h$ denoting as usual the maximum of the diameters of the triangles $K\in\mathscr{C}_h$. We assume that nodes are placed at the endpoints of $\Gamma_0$ and $\Gamma_1$. We shall construct a finite dimensional space $\mathscr{H}_h \subset \mathscr{H}_2$ and then define $\mathscr{P}_h = \mathscr{H}_h \cap P$. The finite element method will be an equilibrium method, i. e., the spaces $\mathscr{H}_h$ and $Y_h$ will satisfy:

$$\text{If } \tau\in\mathscr{H}_h \text{ and } (\operatorname{div}\tau, v) = 0 \text{ for } v\in Y_h, \text{ then } \operatorname{div}\tau = 0 \text{ in } \Omega. \tag{4.2}$$

Methods of this type including the present one have been studied in [5] in the case of linear elasticity. To define $\mathscr{H}_h$ each triangle $K$ is divided into three subtriangles $T_k$, $k = 1, 2, 3$, by connecting the center of gravity with the three nodes of $K$. For each $K\in\mathscr{C}_h$ we introduce the finite dimensional space $H_k$ defined by

$$H_K = \{\tau = \{\tau_{ij}\}: \ \tau_{ij} = \tau_{ji} \text{ is linear on } T_k, \ k = 1, 2, 3, \ i, j = 1, 2, \text{ and } \operatorname{div}\tau\in[L^2(K)]^2\}.$$

One can prove (*see* [5]) that an element $\tau\in H_K$ is uniquely determined by the following 15 degrees of freedom:

$$\text{the value of } \tau\cdot n \text{ at two points of each side of } K, \tag{4.3}$$

$$\int_K \tau_{ij}\,dx, \qquad i, j = 1, 2, \tag{4.4}$$

where $\tau\cdot n = (\tau_{11}n_1+\tau_{12}n_2, \tau_{21}n_1+\tau_{22}n_2)$ and $n = (n_1, n_2)$ is the outward unit normal to the boundary of $K$. The space $\mathscr{H}_h$ can now be defined:

$$\mathscr{H}_h = \{\tau : \tau|_K\in H_K, K\in\mathscr{C}_h, \text{ and } \operatorname{div}\tau\in Y_2\}.$$

If $\tau|_K \in H_K$, $K \in \mathscr{C}_h$, then $\operatorname{div} \tau \in Y_2$ if and only if $\tau \cdot n$ is continuous at the interelement boundaries, i. e., if for any side $S$ common to the triangles $K$ and $K'$,

$$\tau|_K \cdot n = \tau|_{K'} \cdot n \quad \text{on } S,$$

where $n$ is a normal to $S$. Therefore the degrees of freedom for an element $\tau \in H_h$ can be chosen as follows: the value of $\tau \cdot n$ at two points at each side of $\mathscr{C}_h$ and the values given by (4.4) for $K \in \mathscr{C}_h$.

Finally, defining

$$Y_h = \{ v \in Y_2 : v \text{ is linear on } K,\ K \in \mathscr{C}_h \},$$

the particular case of the finite element method (4.1) we want to consider has been fully described.

In addition to (4.2) the spaces $\mathscr{H}_h$ and $Y_h$ have the following property (*see* [5]) important for the analysis: There is an interpolation operator $\pi_h : \mathscr{H}_2 \to \mathscr{H}_h$ such that

$$(\operatorname{div} \pi_h \tau, v) = (\operatorname{div} \tau, v), \qquad v \in Y_h, \tag{4.5}$$

and

$$\|\tau - \pi_h \tau\| \leqq C h^2 \|\tau\|_{2,2}.$$

Given $\tau \in \mathscr{H}_2$, sufficiently regular e. g. $\tau \in [W_1(\Omega)]^4$, the interpolant $\pi_h \tau$ is defined to be the unique element in $H_h$ satisfying for any side $S$ of $\mathscr{C}_h$ with normal $n$,

$$\int_K ((\tau - \pi_h \tau) \cdot n) \cdot v \, ds = 0 \qquad \text{for} \quad v \text{ linear},$$

and for any $K \in \mathscr{C}_h$,

$$\cancel{\int_K (\tau_{ij} - (\pi_h \tau)_{ij}) \, ds = 0, \qquad i, j = 1, 2.}$$

We observe that if $\operatorname{div} \tau = 0$ in $\Omega$ then by (4.2) also $\operatorname{div} \pi_h \tau = 0$ in $\Omega$.

Existence of a solution of the finite element problem can be proved under the following "discrete safe load hypothesis":

$$\left.\begin{array}{c} \text{There exists } \delta > 0 \text{ and } \chi_h \in \mathscr{P}_h \cap E \text{ such that} \\ \operatorname{dist}(\chi_h(x), \partial D) \geqq \delta, \qquad x \in \Omega, \\ \|\chi_h\| \leqq C, \end{array}\right\} \tag{4.6}$$

where $C$ and $\delta$ are independent of $h$. We note that if the $\chi$ in the safe load hypothesis (1.14) for the continuous problem is sufficiently smooth (e. g. if $\chi$ is continuous), then (4.6) will be true for $h$ sufficiently small of we choose $\chi_h = \pi_h \chi$.

Let us now consider the convergence of the finite element method (4.1). We have the following result on weak convergence:

THEOREM 3: *If* (4.6) *holds, then for any* $p$, $1 \leqq p < 2$, *there exists* $(\sigma, u, U) \in \mathscr{P}_q \times Y_p \times R$ *satisfying* (1.13) *with* $\mathscr{P} = \mathscr{P}_q$ *and* $Y = Y_p$, *where* $(1/q)+(1/p) = 1$, *and a sequence* $\{ h_i \}$ *tending to zero, such that*

$$\sigma_h \to \sigma \quad \text{weakly in } H \quad \text{as} \quad h \to 0,$$

$$u_{h_i} \to u \quad \text{weak star in } Y_p,$$

$$U_{h_i} \to U.$$

*Proof:* The theorem follows easily from the following *a priori* estimates:

$$\| \sigma_h \| \leqq C, \tag{4.7}$$

$$\| u_h \| \leqq C, \tag{4.8}$$

$$| U_h | \leqq C. \tag{4.9}$$

The estimate (4.7) follows directly by taking $\tau = \chi_h$ in (4.1 *a*), where $\chi_h$ is given by (4.6).

Next, (4.9) follows by choosing $\tau = \chi_h + \pi_h \bar{\chi}$ in (4.1 *a*), where $\bar{\chi} \in C^\infty(\Omega) \cap H$ satisfies

$$\operatorname{div} \bar{\chi} = 0 \quad \text{in } \Omega,$$

$$\int_{\Gamma_1} \bar{\chi}_{22} \, ds \neq 0,$$

$$\| \bar{\chi} \|_\infty \leqq \frac{\delta}{2}.$$

Such a $\bar{\chi}$ can easily be constructed by solving a suitable linear elastic problem.

Finally, (4.8) will follow by choosing $\tau = \chi_h + \pi_h \tilde{\chi}$ in (4.1 *a*), where $\tilde{\chi} \in \mathscr{H}_2$ satisfies

$$\operatorname{div} \tilde{\chi} = g,$$

where $g$ ranges over the ball $\{ g \in Y_q : \| g \|_q \leqq \mu \}$, with $\mu > 0$ sufficiently small. For instance one can choose $\tilde{\chi} = \varepsilon(\tilde{u})$, where $\tilde{u}$ satisfies

$$\operatorname{div}(\varepsilon(\tilde{u})) = g \quad \text{in } \bar{\Omega},$$

$$\varepsilon(\tilde{u}) \cdot n = 0 \quad \text{on } \partial\bar{\Omega},$$

and $\bar{\Omega}$ is a domain with smooth boundary such that $\Omega \subset \bar{\Omega}$, $\Gamma_1 \subset \partial\bar{\Omega}$ and $g$ is suitably extended outside $\Omega$. To see that $\tau = \chi_h + \pi_h \chi \in \mathscr{P}_h$ for $\mu$ sufficiently small, we note that by elliptic regularity (*see* [6]) one has

$$\|\tilde{\chi}\|_{1,q} \leqq C\|g\|_q,$$

which by Sobolev's imbedding theorem implies that $\tilde{\chi}$ is continuous and

$$\|\tilde{\chi}\|_\infty \leqq C\|g\|_q.$$

Now, (4.1 *a*) and (4.5) together with the previously obtained estimates (4.7) and (4.9) show that

$$(u_h, g) = (u_h, \operatorname{div}\chi) = (u_h, \operatorname{div}\pi_h\tilde{\chi}) \leqq C,$$

if $\|g\|_q \leqq \mu$, which proves (4.8). This completes the proof.

Finally, we shall obtain an estimate for $\sigma-\sigma_h$ in terms of the quantity

$$\alpha = \inf\{\beta : \exists\tau_h \in \mathscr{H}_h \cap E \text{ such that } (1-\beta)\tau_h \in P \text{ and } \|\sigma-\tau_h\| \leqq \beta\}.$$

If $\sigma$ is sufficiently regular then by choosing $\tau_h = \pi_h\sigma$ we see that $\alpha \to 0$ as $h \to 0$.

THEOREM 4: *If* (4.6) *holds then there exists a constant C independant of h such that for* $\alpha < 1$,

$$\|\sigma-\sigma_h\| \leqq C\sqrt{\alpha}.$$

*Proof:* Choosing $\tau = (1-2\alpha)\tau_h$ in (4.1 *a*) where $\tau_h \in \mathscr{H}_h \cap E$ satisfies $\|\sigma-\tau_h\| \leqq 2\alpha$, and $\tau = \sigma_h$ in (1.13 *a*) and adding, we easily find that

$$\frac{1}{C}\|\sigma-\sigma_h\|^2 \leqq \|\sigma-\sigma_h\|_a^2 \leqq a(\sigma_h, (1-2\alpha)\tau_h-\sigma).$$

$$~~+2\alpha U_h F + \int_{\Gamma_1} \varphi(\sigma_{22}-\tau_{22,h})\,ds + 2\alpha\int_{\Gamma_1} \varphi\tau_{22,h}\,ds.~~$$

Thus, using the estimates (4.7), (4.9) and (1.24) we obtain the desired estimate.

## REFERENCES


1. G. DUVAUT, Problèmes de contact entre corps solides deformables, *Applications of Methods of Functional Analysis to Problems in Mechanics*, Joint Symposium IUTAM/IMU, Marseille, 1975, Lecture Notes in Mathematics, Springer Verlag, Berlin-Heidelberg, 1976.
2. G. DUVAUT and L.-J. LIONS, *Les inéquations en Mecanique et en Physique*, Dunod, Paris, 1972.
3. C. JOHNSON, *Existence Theorems for Plasticity Problems*, J. Math. pures et appl., 55, 1976, pp. 431-444.

4. C. Johnson, *On Plasticity with Hardening* (to appear in Mathematical Analysis and Applications).
5. C. Johnson and B. Mercier, *Some Equilibrium Finite Element Methods for Two-Dimensional Elasticity Problems*, Research report 77.08, Computer Sciences Dept., Chalmers Univ. of Techn., Göteborg.
6. J. Necas, *Les Méthodes Directes en Theorie des Équations Elliptiques*, Masson, Paris, 1976.