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$L_\infty$-convergence of saddle-point approximations for second order problems

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Abstract — Let $u_0$ be the solution of the second order model problem

$$-\Delta u + qu = f \quad \text{in} \quad \Omega,
\quad u = 0 \quad \text{on} \quad \partial \Omega,$$

with $\Omega$ bounded in $\mathbb{R}^2$, $(u_0, \text{grad } u_0)$ is characterized as the saddle-point of a quadratic functional and approximated by finite elements.

1. INTRODUCTION

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded domain and $q \geq 0$ be a bounded and measurable function. We consider the second order model problem

$$-\Delta u + qu = f \quad \text{in} \quad \Omega,
\quad u = 0 \quad \text{on} \quad \partial \Omega,$$

$f \in L_2(\Omega)$; the solution will be denoted by $u_0$.

The basic idea of the mixed method is to characterize $(u_0, v_0), v_0 := \text{grad } u_0$, as the saddle-point of a quadratic functional and to approximate $(u_0, v_0)$ by elements of suitably chosen finite dimensional subspaces.

The construction of approximating finite element spaces and the $L_2$-error analysis for this problem was given by P. A. Raviart-J. M. Thomas [5]. Using the same subspaces our goal is to derive $L_\infty$-error estimates. The method of proof is based on weighted $L_2$-norms, similarly to the work of J. Nitsche [3], [4], and F. Natterer [2].

2. NOTATIONS, STATEMENT OF THE PROBLEM

If we define the operator

$$Tu := \text{grad } u$$
with \( T : D(T) = \tilde{W}^{1}_2 \subseteq L_2 \to L_2 \times L_2 \), then the dual operator is

\[
T^* v = - \text{div} v
\]

with \( T^* : D(T^*) = W^{1}_2 \times W^{1}_2 \subseteq L_2 \times L_2 \to L_2 \). (We omit the specification of the domain, if no confusion is possible.)

Later on we need the following assertions: \( T \) is a closed operator and \( R(T) \) is closed in \( L_2 \times L_2 \). Therefore by the Closed Range Theorem \( L_2 \times L_2 \) is the orthogonal sum of \( R(T) \) and \( N(T^*) \). (See, for instance, K. Yosida [7], p. 205.)

Further we define the operator \( Q : L_2 \to L_2 \) by \( Q u := qu, u \in L_2 \). Then equation (1) is equivalent to the system

\[
T^* v + Q u = f \\
T u - v = 0
\]

with the solution \((u_0, v_0), v_0 := \text{grad} u_0\).

For convenience we assume that \( \overline{\Omega} \) is a bounded polygon. Suppose \( \Gamma_h \) is a \( \kappa \)-regular triangulation of \( \Omega \), \( 0 < h \), i.e. for any \( \Delta \in \Gamma_h \) there are two circles \( K \) and \( \overline{K} \) with radii \( \rho \) and \( \overline{\rho} \) such that \( K \subseteq \Delta \subseteq \overline{K} \) and

\[
\kappa^{-1} h \leq \rho \leq \overline{\rho} \leq \kappa h.
\]

In the following let \( r \geq 1 \) be a fixed integer.

By \((W^r_p \times W^r_p)' = (W^r_p \times W^r_p)'(\Gamma_h), 2 \leq p \leq \infty, \) we denote those elements of \( L^p \times L^p \), which fulfill the following conditions:

(i) \( v \in W^r_p(\Delta) \times W^r_p(\Delta) \) for all \( \Delta \in \Gamma_h \);
(ii) for all \( u \in W^1_2(\Omega) \) we have

\[
\int_{\Omega} v \cdot \text{grad} u \, ds + \int_{\Omega} u \text{div} v \, ds = \int_{\partial\Omega} uv \cdot v \, d\sigma.
\]

(\( v \) is the exterior unit normal to \( \partial\Omega \).)

Equation (3) holds if and only if for any pairs of adjacent triangles \( \Delta_1, \Delta_2 \in \Gamma_h \) we have

\[
v|_{\Delta_1} \cdot v_1 + v|_{\Delta_2} \cdot v_2 = 0 \quad \text{on} \quad \Delta_1 \cap \Delta_2,
\]

where \( v_i \) is the outward unit normal to the boundary of \( \Delta_i, i = 1,2 \). (See P. A. Raviart-J. M. Thomas [5].)
We denote by $(\cdot, \cdot)$ the scalar product in $L^2$ as well as in $L^2 	imes L^2$. We also write $\|v\|_p^p$ instead of $\|v\|_{W^p_p \times W^p_p}$ for $v \in W^p_p \times W^p_p$. Finally we introduce in $(W^p_p \times W^p_p)'$ the norm

$$\|v\|_p^p := \left\{ \sum_{\Delta \in \mathcal{T}_h} \|v\|_{W^p_p(\Delta)}^p \right\}^{1/p}, \quad 2 \leq p < \infty$$

with the usual modification for $p = \infty$.

Let us define the quadratic functional $I: L^2 \times (W^1_2 \times W^1_2)' \to \mathbb{R}$ by

$$I(u, v) := a(u, v) - \frac{1}{2} (v, v) - (f, u) + \frac{1}{2} (Q u, u)$$

with

$$a(u, v) := - \int_\Omega u \text{ div } v \, ds.$$

The equation

$$I(u, v) - I(u_0, v_0) = - \frac{1}{2} (v - v_0, v - v_0) + \frac{1}{2} (Q(u - u_0), u - u_0)$$

implies

$$I(u_0, v) \leq I(u_0, v_0) \leq I(u, v_0) \quad (4)$$

for all $u \in L^2$, $v \in (W^1_2 \times W^1_2)'$, i.e. $(u_0, v_0)$ is a saddle-point of the functional $I$.

Given finite dimensional subspaces $U_h \subseteq L^2$ and $V_h \subseteq (W^1_2 \times W^1_2)'$, we approximate $(u_0, v_0)$ by a saddle-point $(u_h, v_h)$ of $I$ restricted to $U_h \times V_h$. From the condition

$$I(u_h, \eta) \leq I(u_h, v_h) \leq I(\xi, v_h)$$

for all $\xi \in U_h$, $\eta \in V_h$, we get

$$a(\xi, v_h) + (\xi, Q u_h) = (f, \xi)$$

$$a(u_h, \eta) - (v_h, \eta) = 0 \quad (5)$$

for all $\xi \in U_h$, $\eta \in V_h$. (5) has a unique solution if $U_h \subseteq \text{div } V_h$ holds.

Then the equation (5) can be written in the form

$$a(\xi, v_0 - v_h) + (\xi, Q(u_0 - u_h)) = 0 \quad \text{for all } \xi \in U_h$$

$$a(u_0 - u_h, \eta) - (v_0 - v_h, \eta) = 0 \quad \text{for all } \eta \in V_h \quad (5')$$

Thus, the mapping $(u_0, v_0) \to (u_h, v_h)$ may be considered as a projection operator from $L^2 \times (W^1_2 \times W^1_2)'$ onto $U_h \times V_h$.

For the sake of simplicity in the following we only regard the case $Q = 0$. 

vol. 11, n° 2, 1977
3. CONSTRUCTION OF APPROXIMATING SUBSPACES, $L_2$-ERROR ESTIMATES

Given a $\kappa$-regular triangulation $\Gamma_h$ and an integer $r \geq 1$, P. A. Raviart-J. M. Thomas [5] construct a linear subspace $V_h$ of $(W_p^{r+1} \times W_p^{r+1})'(\Gamma_h)$ in the following way:

$\eta = (\eta_1, \eta_2)$ belongs to $V_h$ if in each triangle $\Delta \in \Gamma_h$ the functions $\eta_1$ and $\eta_2$ are special polynomials of degree $r + 1$, determined by the values of

$$\int_{K_i} \sigma^j \eta \cdot v \, d\sigma \ , \ 0 \leq j \leq r \ , \ i = 1, 2, 3$$

and

$$\int_{\Delta} s_1^k s_2^l \eta \, ds \ , \ 0 \leq k, l \ , \ k + l \leq r - 1,$$

where $K_i$ denotes the sides of $\Delta$. Furthermore in each triangle $\text{div} \eta$ is a polynomial of degree $r$.

Let $U_h$ be the space of finite elements of degree $r$ for the same triangulation $\Gamma_h$ (without any boundary or continuity conditions), and denote by $P_h: L_2 \to U_h$ the orthogonal projection from $L_2$ onto $U_h$.

Then $\text{div} V_h \subseteq U_h$ and the following assertion holds. (See [5], compare also P. G. Ciarlet-P. A. Raviart [1].)

**Lemma 1**: There exists a linear projection operator $\Pi_h: (W_p^1 \times W_p^1)' \to V_h$, $2 \leq p \leq \infty$, with the following properties:

(i) for all $v \in (W_p^1 \times W_p^1)'$ the relation

$$\text{div} \Pi_h v = P_h \text{div} v$$

is valid;

(ii) for all $v \in (W_p^{l+1} \times W_p^{l+1})'$ the estimate

$$\|v - \Pi_h v\|_{W_p^k} \leq C h^{l+1-k} \|v\|_{W_p^{l+1}}, \ 0 \leq k, l, k \leq l + 1 \leq r + 1$$

holds.

The following Lemma shows $U_h \subseteq \text{div} V_h$; therefore the equation (5) has a unique solution.

**Lemma 2**: For each $\xi \in U_h$ there is an element $\eta \in V_h$ with $\text{div} \eta = \xi$.

**Proof**: For an arbitrary element $\xi \in U_h$ let $w$ be the element of $\hat{W}_2^1 \cap W_2^2$ with $\Delta w = \xi$. Defining $\eta = \Pi_h \text{grad} w$, relation (6) shows

$$\text{div} \eta = P_h \text{grad} w$$

$$\text{div} \eta = P_h \xi = \xi.$$
Now let \((u_h, v_h) \in U_h \times V_h\) be the saddle-point approximation of \((u_0, v_0)\) defined by (5). The following approximation theorem was obtained by P. A. Raviart-J. M. Thomas \([5, \text{Theorem 5}]\) :

If \(u_0 \in \tilde{W}_2^1 \cap W_2^{r+2}\) and \(\Delta u_0 \in W_2^{r+1}\), then
\[
\|u_0 - u_h\|_{L_2} + \|v_0 - v_h\|_{L_2} + \|\text{div} (v_0 - v_h)\|_{L_2} \leq C h^{r+1} (\|u_0\|_{W_2^{r+2}} + \|\Delta u_0\|_{W_2^{r+1}}).
\]

For our purpose we need an "uncoupled" estimate.

**Lemma 3:** Suppose \(u_0 \in \tilde{W}_2^1 \cap W_2^{r+2}\). Then
\[
\|v_0 - v_h\|_{W_2^k} \leq C h^{r+1-k} \|u_0\|_{W_2^{r+2}}, \quad 0 \leq k \leq r + 1,
\]
where \(C\) is independent of \(u_0\) and \(h\).

**Proof:** Define \(\xi_h := P_h u_0\) and \(\eta_h := \Pi_h v_0\). Using (5') we find
\[
\|v_h - \eta_h\|_{L_2} = (v_h - \xi_h, v_h - \eta_h) - a(u_h - \xi_h, v_h - \eta_h) + a(u_h - \xi_h, v_h - \eta_h)
= (v_0 - \eta_h, v_h - \eta_h) - a(u_0 - \xi_h, v_h - \eta_h) + a(u_h - \xi_h, v_h - \eta_h).
\]

From \(\text{div} V_h = U_h\) and relation (6) we obtain
\[
a(u_0 - \xi_h, v_h - \eta_h) = -(u_0 - \xi_h, \text{div} (v_h - \eta_h)) = 0
\]
and
\[
a(u_h - \xi_h, v_0 - \eta_h) = -(u_h - \xi_h, \text{div} v_0 - P_h \text{div} v_0) = 0.
\]
Therefore we get
\[
\|v_h - \eta_h\|_{L_2}^2 \leq \|v_0 - \eta_h\|_{L_2} \|v_h - \eta_h\|_{L_2};
\]
with the help of (7) the estimate (8) follows for the case \(k = 0\). For \(1 \leq k \leq r + 1\), (8) is obtained by inverse inequalities, obviously valid for the elements of \(V_h\).

**Remark:** If only \(u_0 \in \tilde{W}_2^1 \cap W_2^{r+1}\) is presumed for the solution of (1), with the same proof and by application of duality arguments (see R. Scholz \([6]\)) we can show the error estimate
\[
\|u_0 - u_h\|_{L_2} + h \|v_0 - v_h\|_{L_2} \leq C h^{r+1} \|u_0\|_{W_2^{r+1}},
\]
where \(C\) is independent of \(u_0\) and \(h\).
Our main result is the following theorem.

**Theorem:** Assume the solution $u_0$ of problem (1) fulfills the regularity condition $u_0 \in \hat{W}^{1,2}_2 \cap W^{r+2}_\infty \cap W^{r+1}_\infty$, and let $(u_h, v_h) \in U_h \times V_h$ be the saddle-point approximation of $(u_0, v_0)$ defined by (5). Then the following error estimate holds:

$$
\|u_0 - u_h\|_{L^\infty} + h \|v_0 - v_h\|_{L^\infty} \leq C h^{r+1} \left\{ \|u_0\|_{W^{r+1}_\infty} + \|u_0\|_{W^{r+2}_2} \right\},
$$

where the constant $C$ is independent of $u_0$ and $h$.

In order to prove (9) we use “weighted” $L_2$-norms.

Let $s_0$ be any point of $\Omega$. For $\rho > 0$ we define with $\mu : = \mu(s) : = |s - s_0|^2 + \rho^2$ for each $\alpha \in \mathbb{R}$

$$
\|u\|_\alpha := \|\mu^{-\alpha/2} u\|_{L^2}, \quad u \in L^2.
$$

($|s - s_0|$ denotes the Euclidean distance between the points $s$ and $s_0 \in \mathbb{R}^2$.)

Between $L^\infty_\omega$- and weighted $L_2$-norms we have the following relations:

(i) if $u \in L^\infty_\omega$ and $\alpha > 1$, then

$$
\|u\|_\alpha \leq C \rho^{-\alpha+1} \|u\|_{L^\infty};
$$

(ii) for $\xi \in U_h$ and the special choice of $s_0 \in \Omega$ such that $|\xi(s_0)| = \|\xi\|_{L^\infty}$ we have

$$
\|\xi\|_{L^\infty} \leq C \gamma^{-\alpha} h^{\alpha-1} \|\xi\|_\alpha, \quad h = \gamma \rho.
$$

The constants $C$ in (10) and (11) do not depend on $\rho$ respectively $h$ and the special point $s_0 \in \Omega$. For a proof see J. Nitsche [3], [4].

The weighted norms in $L_2 \times L_2$ are defined in an analogous manner.

**Proof of the Theorem:** For convenience we write $u$ and $v$ instead of $u_0$ and $v_0$. Since the operator $(u, v) \rightarrow (u_h, v_h)$ is a projection, it suffices to prove

$$
\|u_h\|_{L^\infty} + h \|v_h\|_{L^\infty} \leq C \left\{ \|u\|_{L^\infty} + h \|v\|_{L^\infty} + \sum_{k=0}^{r+1} h^k \|v - v_h\|_{W^k_2} \right\}.
$$

First we show the estimate for $u_h$. Let $s_0 \in \Omega$ be chosen such that

$$
|u_h(s_0)| = \|u_h\|_{L^\infty}.
$$

For $\alpha > 1$ we have

$$
\|u_h\|_\alpha^2 = \langle u_h, \mu^{-\alpha} u_h - \xi \rangle = \langle u, \mu^{-\alpha} u_h - \xi \rangle + \langle u, \mu^{-\alpha} u_h \rangle - \langle u - u_h, \xi \rangle
$$

with $\xi := P_h \mu^{-\alpha} u_h$. With the same arguments as in J. Nitsche [3] we find for $h = \gamma \rho$, $\gamma$ suitably chosen,

$$
\|u_h\|_\alpha^2 \leq C \left( \|u\|_\alpha^2 + |(u - u_h, \xi)| \right).
$$
Now let \( w \in \hat{W}^1 \cap W^2_2 \) be the solution of the auxiliary problem
\[
- \Delta w = \xi \quad \text{in} \quad \Omega \\
w = 0 \quad \text{on} \quad \partial \Omega,
\]
and define \( \omega := \text{grad} \ w \). An easy computation gives \( \text{div} \ \Pi_h \omega = - \xi = \text{div} \ \omega; \)
hence we have \( \omega - \Pi_h \omega \in N(T^*) \). With the help of (5) therefore
\[
(u - u_h, \xi) = a(u - u_h, \omega) \\
= a(u - u_h, \Pi_h \omega) \\
= (v - v_h, \Pi_h \omega) \quad (15)
\]
From the Closed Range Theorem we get \( v - v_h = v - \tilde{v}_h - \hat{v}_h \) with \( v - \tilde{v}_h \in R(T) \) and \( - \hat{v}_h \in N(T^*) \). Using \( \omega \in R(T) \), \( \omega - \Pi_h \omega \in N(T^*) \), and (5'), we find
\[
|(v - \tilde{v}_h, \Pi_h \omega)| = |(v - v_h, \omega)| \\
= |(v - v_h, \omega)| \\
= |a(w, v - v_h)| \\
= |a(w - P_h v, v - v_h)| \\
\leq C h^2 \|w\|_{W^2_2} \|\text{div} \ (v - v_h)\|_{L^2} \\
\leq C h^{\rho - \alpha} \|u_h\|_a \|v - v_h\|_{W^2_2},
\]
and
\[
|-(\hat{v}_h, \Pi_h \omega)| = |(\hat{v}_h, \omega - \Pi_h \omega)| \\
= |(v - v_h, \omega - \Pi_h \omega)| \\
\leq C h \|w\|_{W^2_2} \|v - v_h\|_{L^2} \\
\leq C h^{\rho - \alpha} \|u_h\|_a \|v - v_h\|_{L^2}.
\]
Combining these inequalities with (13), (14), and (15) we get
\[
\|u_h\|_a \leq C \left( \|u\|_a + h^{\rho - \alpha} \|v - v_h\|_{L^2} + h^2 \rho^{-\alpha} \|v - v_h\|_{W^2_2} \right).
\]
Hence, the estimate (12) for \( u_h \) follows by (10) and (11).

Next let \( s_0 \in \bar{\Omega} \) be such that
\[
\|v_{h,i}(s_0)\|_a = \|v_{h,i}\|_{L^\infty} = \|v_i\|_{L^\infty},
\]
i = 1 or i = 2. We find for \( \alpha > 1 \)
\[
\|v - v_h\|_a^2 = (v - v_h, (I - \Pi_h)\mu^{-\alpha}(v - v_h)) + (v - v_h, \Pi_h\mu^{-\alpha}(v - v_h)),
\]
where \( I \) denotes the identity. Using the approximation properties of the space \( V_h \) and
\[
|D^k \mu^{-\alpha}(s)| \leq C \rho^{-k} \mu^{-\alpha}(s), \quad k \geq 1,
\]
the first term can be estimated by
\[ |(v - v_h, (I - \Pi_h) \mu^{-\alpha}(v - v_h))| \]
\[ \leq C h^{r+1} \|v - v_h\|_{L_2} \|\mu^{-\alpha}(v - v_h)\|_{W^{r+1}_2} \]
\[ \leq C h^{r+1} \|v - v_h\|_{L_2} \sum_{k=0}^{r+1} \rho^{k-r-1-2\alpha} \|v - v_h\|_{W^{r+1}_2} \quad (17) \]
\[ \leq C \rho^{-2\alpha} \sum_{k=0}^{r+1} \rho^{2k} \|v - v_h\|_{W^{r+1}_2}. \]

Further, because of (5'_2) we can write
\[ |(v - v_h, \Pi_h \mu^{-\alpha}(v - v_h))| \]
\[ = |a(u - u_h, \Pi_h \mu^{-\alpha}(v - v_h))| \]
\[ = |(u - u_h, P_h \text{div} \mu^{-\alpha}(v - v_h))| \]
\[ = |(P_h (u - u_h), \text{div} \mu^{-\alpha}(v - v_h))| \]
\[ \leq \|P_h (u - u_h)\|_{a+1} \|\text{div} \mu^{-\alpha}(v - v_h)\|_{a-1} \]
\[ \leq C \|u - u_h\|_{a+1}^{2} + \|\text{div} \mu^{-\alpha}(v - v_h)\|_{a-1}^{2}. \]

(Here we used the boundedness of \(P_h\) in weighted norms; see J. Nitsche [3].)

An elementary computation gives
\[ \|\text{div} \mu^{-\alpha}(v - v_h)\|_{a-1}^2 \leq C (\rho^{-2(a-1)} \|\text{div} (v - v_h)\|_{L_2}^2 + \rho^{-2\alpha} \|v - v_h\|_{L_2}^2). \]
Thus,
\[ \|v_h\|_a \leq \|v\|_a + \|v - v_h\|_a \]
\[ \leq C \left( \|u - u_h\|_{a+1} + \|v\|_a + \rho^{-a} \sum_{k=0}^{r+1} \rho^{2k} \|v - v_h\|_{W^{r+1}_2} \right). \]

Finally, using the relation (10) and (11) once more, we obtain the desired estimate (12) for \(v_h\), and the proof is complete.

REFERENCES