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## ERROR ESTIMATES FOR THE APPROXIMATION OF SOME UNILATERAL PROBLEMS (\*) (1)

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Communique par P A RAVIART

Abstract — An error estimate for the affine finite element approximation of some unilateral problems is given

### 1. INTRODUCTION

In the mechanics of Fluids through semipermeable boundary the following problem is studied :

$$-\Delta u = f \quad \text{in } \Omega; \quad u \geq \psi, \quad \frac{\partial u}{\partial n} \geq 0, \quad (u - \psi) \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma \quad (1)$$

where  $\Gamma$  is a thin membrane around the space  $\Omega$  filled by the fluid : semi-permeable means that the fluid is allowed to enter but not to escape.

$\Delta$  denotes the Laplace operator,  $u$  the pressure of the fluid in its stable condition,  $f$  the amount of the fluid that has been put in,  $\psi$  denotes the external fluid pressure on  $\Gamma$ ,  $\frac{\partial}{\partial n}$  the outer normal derivative on  $\Gamma$

(1) can be used also to sketch some problems in thermo-dynamics or electric dynamics

A system such as (1) is known in the literature as a complementarity system and can be solved when some compatibility conditions are imposed on  $f$  (see [8])

If the internal pressure  $u$  is greater than the external one  $\psi$  then the semi-permeable membrane holds the fluid and the fluid cannot escape (so  $\frac{\partial u}{\partial n} = 0$ ).

If  $\psi$  is greater or equal than  $u$ , the external fluid enters through  $\Gamma$  ( $\frac{\partial u}{\partial n} \geq 0$ ) until there is  $u = \psi$ ; in theory there is no  $u < \psi$ .

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The membrane could be divided into two parts  $\Gamma_1$  and  $\Gamma_2$

on  $\Gamma_1$  we have  $u = \psi$  (Dirichlet condition),

on  $\Gamma_2$  we have  $\frac{\partial u}{\partial n} = 0$  (Neumann condition).

but we look at (1) as a free boundary problem because we do not know  $\Gamma_1$  or  $\Gamma_2$ .

We shall study some variational inequalities (with coerciveness assumption) (*cf* [23]), we shall consider an approximation using the triangular affine elements: the solutions of the corresponding discrete complementarity systems are supposed known.

Our purpose is to estimate the distance between the exact solution  $u$  and the discrete one  $u_h$ .

## 2. THE BASIC NOTATION AND TERMINOLOGY

$\Omega$  denotes a bounded, open set of  $\mathbf{R}^2$ ,  $\Gamma$  denotes the boundary of  $\Omega$  and  $\bar{\Omega}$  the closure of  $\Omega$  so that  $\bar{\Omega} = \Omega \cup \Gamma$ :  $\Omega$  is supposed with "not too bad" a boundary.

$C^k(\bar{\Omega})$ , ( $k = 0, 1, 2, \dots$ ) is a Banach space, the elements of which are functions that are continuous in  $\bar{\Omega}$  and have continuous derivatives in  $\bar{\Omega}$  of the first  $k$  order: the norm is defined by:

$$\|u\|_{C^k(\bar{\Omega})} = \sum_{j=0}^k \sum_{|\alpha|=j} \max_{\bar{\Omega}} |D^\alpha u|$$

(if  $k = +\infty$  the functions are infinitely differentiable)

$\mathcal{D}(\Omega)$  is the space of the functions of  $C^\infty(\bar{\Omega})$  which are zero in a neighbourhood of  $\partial\Omega$ , and we put on it Schwartz's topology,  $L^p(\Omega)$ , ( $1 < p < +\infty$ ) denotes the Banach space of all functions on  $\Omega$  that are measurable and  $p$ -summable in  $\Omega$ . the norm in this space is defined by

$$\|u\|_{L^p(\Omega)} = \left( \int_{\Omega} |u|^p dx \right)^{1/p}$$

or if  $p = +\infty$  a.e. bounded in  $\Omega$ , (the elements  $L^p(\Omega)$  are the class of equivalent functions on  $\Omega$ ).

$W^{k,p}(\Omega)$ , ( $k \in \mathbf{N}$ ,  $1 < p < +\infty$ ) denotes the Banach space of all elements of  $L^p(\Omega)$  that have generalized derivatives of all kinds of the first  $k$  orders that are  $p$ -summable in  $\Omega$ : the norm is defined by

$$\|u\|_{W^{k,p}(\Omega)} = \left( \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^p(\Omega)}^p \right)^{1/p}, \quad \alpha = (\alpha_1, \alpha_2) : |\alpha| = \sum_{i=1}^2 \alpha_i$$

$W_0^{k,p}(\Omega)$  denotes the closure in the space  $W^{k,p}(\Omega)$  of  $\mathcal{D}(\Omega)$ ; for further details and for the spaces  $W^{s,p}(\Omega)$ , with  $s$  real, see e. g. [15] and [20].

We shall use the following notations

$$H^k(\Omega) = W^{k,2}(\Omega) \quad ; \quad H_0^k(\Omega) = W_0^{k,2}(\Omega)$$

$$\|v\|_{k,\Omega} = \|v\|_{W^{k,2}(\Omega)} \quad |v|_{k,\Omega} = \left( \sum_{|\alpha|=k} \|D^\alpha v\|_{0,\Omega}^2 \right)^{1/2}.$$

### 3. VARIATIONAL FORMULATION OF THE PROBLEM

Let us look at the following form

$$a(u, v) = \int_{\Omega} \left( \sum_{i,j=1}^2 a_{ij}(x) \frac{\partial u}{\partial x_i} \cdot \frac{\partial v}{\partial x_j} + a_0(x)uv \right) dx$$

where  $a_{ij}(x) \in C^1(\bar{\Omega})$ , and  $a_0(x) \in L^\infty(\Omega)$

let us suppose :

$$\sum_{i,j=1}^2 a_{ij}(x)\xi_i\xi_j \geq \alpha_0 |\xi|^2 \quad \forall \xi \in \mathbf{R}^2 \quad , \quad \alpha_0 > 0 \quad \text{and} \quad a_0(x) \geq c > 0$$

so that  $\bar{a}(\dots)$  is a bilinear, continuous, coercive form on  $H^1(\Omega) \times H^1(\Omega)$   
i. e.  $a(u, u) \geq \alpha \|u\|_{1,\Omega}^2$ ;

$$|a(u, v)| \leq M \|u\|_{1,\Omega} \cdot \|v\|_{1,\Omega}, \quad \alpha, M \in \mathbf{R}; \quad \alpha > 0, M > 0$$

We shall consider the problem

$$u \in K: a(u, v - u) \geq \langle f, v - u \rangle \quad \forall v \in K \tag{2}$$

where  $K$  is a convex set :

$$K = \{ v \mid v \in H^1(\Omega): v \geq \psi \text{ on } \Gamma \}$$

$\langle, \rangle$  denotes the pairing between  $(H^1(\Omega))'$  and  $H^1(\Omega)$ ,  $f \in L^2(\Omega)$ ,  $\psi \in H^2(\Omega)$ .

In these conditions (see e. g. [3], [14]) there is one and only one solution  $u$  of the problem (2) and  $u \in H^2(\Omega)$ .

In what follows we shall use the notations :

$$Lu = - \sum_{i,j=1}^2 \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial u}{\partial x_j} \right) + a_0(x)u, \quad \frac{\partial u}{\partial \nu} = \sum_{i,j=1}^2 a_{ij}(x) \frac{\partial u}{\partial x_i} \cos(n, x_j)$$

where  $n$  is the outer normal to  $\Gamma$ .

Let us recall (see [14]) the problem (2) is equivalent to the following system :

$$\begin{cases} Lu = f & \text{in } \Omega \\ u \geq \psi & \text{on } \Gamma \\ \frac{\partial u}{\partial \nu} \geq 0 & \text{on } \Gamma \\ (u - \psi) \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma \end{cases} \quad (3)$$

**REMARK 3.1.**

If  $a_{ij} = a_{ji}$  the problem (2) can be formulated as follows :

find  $u \in K$  such that

$$J(u) = \inf_{v \in K} J(v)$$

where  $J(v) = \frac{1}{2} a(v, v) - \langle f, v \rangle$  is a weakly lower semicontinuous, strictly convex, differentiable functional.

#### 4. A DISCRETIZATION OF THE PROBLEM BY THE FINITE ELEMENT METHOD

We shall sketch here an approximation of problem (2) by means of triangular affine elements.

We shall suppose that  $\Omega$  is a bounded convex open subset of  $\mathbf{R}^2$ , with a smooth boundary  $\Gamma$  (e. g.  $C^2$ ).

Given  $h$ ,  $0 < h < 1$ , we first inscribe a polygon  $\Omega_h$  in  $\Omega$  whose vertices belong to  $\Gamma$  and whose sides have a length which does not exceed  $h$ .

We then decompose  $\Omega_h$  into triangles in such a way that :

$$0 < l \leq h, \quad l'/l'' \leq \beta, \quad 0 < \vartheta_0 < \vartheta \leq \frac{\pi}{2}$$

where  $\beta$  and  $\vartheta_0 < \frac{\pi}{2}$  are given positive constants,  $l, l', l''$  are lengths of arbitrary sides of the triangulation,  $\vartheta$  an arbitrary angle of our triangles.

(As always, the triangulation is not permitted to place a vertex of one triangle along the edge of another, each pair of triangles shares a vertex, a whole edge, or nothing.)

We call "*regular*" (see [2], [11]) such a triangulation.

We shall denote by  $I_0 = \{ 1, \dots, N_0 \}$  the set of all indices  $i$  associated with the internal nodes  $x_i$  of the triangulation ( $x_i \in \Omega$ ) and we shall denote by  $I_1 = \{ N_0 + 1, \dots, N \}$  the set of all indices  $i$  associated with the boundary nodes of the triangulation ( $x_i \in \partial \Omega$ ): and let be  $I = I_0 \cup I_1$ .

We number the vertices in such a way that  $x_{N_0+1}, \dots, x_N$  are the boundary vertices, numbered consecutively counterclockwise around  $\Gamma$ : the curved side (lying on  $\Gamma$ ) with end points  $x_i, x_{i+1}$  is  $\Gamma_i$  (for sake of notation we shall make the identification  $x_{N+1} = x_{N_0+1}$ ) we shall denote by  $\Sigma_i$  the zone between the curved side  $\Gamma_i$  and the right side with endpoints  $x_i, x_{i+1}$ , and by  $T_i$  the triangle corresponding to the  $[x_{i+1}, x_i]$  side: finally we shall call *curved elements* the following subdomains:

$$T_i \cup \Sigma_i, \quad i \in I_1.$$

For each  $i \in I$ , we shall consider the continuous function

$$\phi_i^h(x), \quad x \in \Omega$$

which is affine in each triangular or curved (see the above position) element of the decomposition, is = 1 at  $x_i$  and = 0 in all  $x_j \neq x_i, j \in I$ .

We shall now consider the piecewise affine function  $v_h(x)$  defined by

$$v_h(x) = \sum_{i \in I} v_i^h \phi_i^h(x), \quad \{ v_i^h \}_{i \in I} \in \mathbf{R}^N \tag{4}$$

and the space:

$$H_h^1(\Omega) = \left\{ v_h : v_h = \sum_{i \in I} v_i^h \phi_i^h(x) \right\}$$

(trivially  $H_h^1(\Omega)$  is contained in the space  $H^1(\Omega)$ ).

In [2] it was shown that:

$$\|u - u_I\|_{r, \Omega_h} \leq ch^{2-r} |u|_{2, \Omega_h}, \quad r = 0, 1, \quad \forall u \in H^2(\Omega)$$

where  $u_I(x)$  is the piecewise affine function which interpolates  $u$  at every vertex

$$\text{i.e.} \quad u_I(x) = \sum_{i \in I} u(x_i) \phi_i^h(x)$$

but we can modify the process of the quoted paper and using the regularity conditions of the decomposition and a continuous extension theorem in the seminorms (see [26]) we can obtain:

$$\|u - u_I\|_{r, \Omega} \leq ch^{2-r} |u|_{2, \Omega}, \quad r = 0, 1. \tag{6}$$

We shall consider the convex

$$K_h = \{ v_h(x) : v_h \in H_h^1(\Omega) / v_h(x_i) \geq \psi(x_i) \forall i \in I_1 \}$$

The approximate problem is obtained by replacing  $K$  with  $K_h$  in problem (2)

$$u_h \in K_h : a(u_h, u_h - v_h) \leq \langle f, u_h - v_h \rangle \quad \forall v_h \in K_h \quad (7)$$

Let us write the discrete problem by replacing the expression (4) of  $u_h(x)$  that is :

$$u_h(x) = \sum_{i \in I} U_i \varphi_i^h(x) \quad (8)$$

**PROPOSITION 4.1.** The problem (7) is equivalent to the following discrete system.

$$\begin{cases} M_{I_0} = A_{I_0 I} U_I - b_{I_0} = 0 \\ U_{I_1} - \Psi_{I_1} \geq 0 \\ M_{I_1} = A_{I_1 I} U_I - b_{I_1} \geq 0 \\ M_{I_1} (U_{I_1} - \Psi_{I_1}) = 0 \end{cases} \quad (9)$$

where  $A = A_{II} = \{ a_{ij} \}_{i,j \in I}$ ,  $a_{ij} = a(\varphi_j^h, \varphi_i^h)$ ;  $b_I = \{ b_i \}_{i \in I}$   
 $b_i = (f, \varphi_i^h)$ ;  $\Psi_{I_1} = \{ \Psi_i \}_{i \in I_1}$ ,  $\Psi_i = \Psi(x_i)$

**PROOF.** By choosing  $v_h = u_h \pm \varphi_i^h$  for every  $i \in I_0$  we find the first equation  
 i<sub>h</sub>) 
$$M_i = \sum_{j \in I} a_{ij} U_j - b_i = 0 \quad i \in I_0.$$

The second condition in (9) is the definition of  $K_h$

ii<sub>h</sub>) 
$$U_i \geq \Psi_i \quad i \in I_1$$

By putting  $v_h = u_h + \varphi_i^h$   $i \in I_1$  we have

iii<sub>h</sub>) 
$$M_i = \sum_{j \in I} a_{ij} U_j - b_i \geq 0 \quad i \in I_1.$$

The last condition in (9) is obtained by choosing

$$\begin{aligned} v_h(x_i) &= \varepsilon u_h(x_i) \quad i \in I_0 \\ v_h(x_i) &= \Psi(x_i) + \varepsilon(u_h - \Psi)(x_i) \quad i \in I_1 \end{aligned}$$

in fact i<sub>h</sub>) gives

$$(\varepsilon - 1) \sum_{i \in I_1} (U_i - \Psi_i) \left\{ \sum_{j \in I} a_{ij} U_j - b_i \right\} \geq 0$$

and with choices :  $\varepsilon > 1$ ,  $1 > \varepsilon > 0$  due to ii<sub>h</sub>) and iii<sub>h</sub>) we find :

iv<sub>h</sub>) 
$$M_{I_1} (U_{I_1} - \Psi_{I_1}) = 0.$$

It is easy to check, in turn, that if we take the coefficients  $U_j$  of the function (8) to be the solution of (9) then  $u_h(x)$  is the solution of (7).

REMARK 4.1. The matrix  $A$  belongs to the class  $(P)$  that is to say, all principal minors  $A_{JJ} = (a_{h,k})_{h,k \in J}$ ,  $J \subset I$  have a positive determinant; therefore the existence and uniqueness of the solution  $U_I$  of (9) is a well known result; if  $A$  belongs, also, to the class  $(Z)$ , that is  $a_{ij} \leq 0$  ( $i \neq j$ ) (\*) then a monotone algorithm for solving system (9) can be found, (see e. g. [12], [17], [22] for other unilateral problems).

5. ERROR ESTIMATES

In order to estimate the distance between the solutions  $u$  of (2) and  $u_h$  of (7) we shall follow the procedure described in [9] for another unilateral problem; of course our problem demands some modifications of the Falk's method for the "inability" of piecewise polynomials to satisfy the conditions on a curved boundary.

Our main result is this: the error is of order  $h^{\frac{3}{2}}$  in the energy norm:

THEOREM I. We have

$$\|u - u_h\|_{1,\Omega} \leq Ch^{\frac{3}{2}} (**)$$
(10)

*Proof.* Let us now write the inequalities (2) and (7) by choosing  $v = \Psi$  in (2) and  $v = u_I$  in (7)

$$\begin{aligned} a(u, u - \Psi) &\leq \langle f, u - \Psi \rangle \\ a(u_h, u_h - u_I) &\leq \langle f, u_h - u_I \rangle \end{aligned}$$

We find

$$a(u, u) + a(u_h, u_h) \leq \langle f, u - u_I + u_h - \Psi \rangle + a(u, \Psi) + a(u_h, u_I)$$

and, by subtracting  $a(u, u_h) + a(u_h, u)$ , also

$$a(u - u_h, u - u_h) \leq \langle f, u - u_I + u_h - \Psi \rangle - a(u, u - u_I + u_h - \Psi) + a(u - u_h, u - u_I).$$

Let us write the inequality using Green's formula

$$\begin{aligned} a(u - u_h, u - u_h) &\leq (f - Lu, u - u_I + u_h - \Psi) + a(u - u_h, u - u_I) - \\ &\quad + \int_{\Gamma} (u - u_I + u_h - \Psi) \frac{\partial u}{\partial \nu} d\Gamma \end{aligned}$$

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(\*) This is true, for example, for  $Lu = -\Delta u + a_0 u$ .

(\*\*) In the sequel  $C$  will denote a generic constant not necessarily the same in any two places.



and by the first equation of the system (3)

$$a(u - u_h, u - u_h) \leq a(u - u_h, u - u_I) - \int_{\Gamma} (u - u_I + u_h - \psi) \frac{\partial u}{\partial \nu} d\Gamma \quad (11)$$

We shall split the last term into two integrals i. e.

$$- \int_{\Gamma} (u_h - \psi_I) \frac{\partial u}{\partial \nu} d\Gamma - \int_{\Gamma} ((u - \psi) - (u - \psi_I)) \frac{\partial u}{\partial \nu} d\Gamma$$

The estimation of  $\|u - u_h\|_{1,\Omega}$  is therefore reduced to study the convergence of the boundary integrals.

We begin by remarking that  $g = u_h - \psi_I$  is nonnegative at the boundary nodes, if also  $u_h - \psi_I \geq 0$  on  $\Gamma$  we could eliminate the integral  $-\int_{\Gamma} \frac{\partial u}{\partial \nu} (u_h - \psi_I)$  and so the inequality would increase (cf. systems (3) (9));

if this is not the case. let us denote by  $\tilde{\Gamma}$  the subset of  $\Gamma$  in which.

$$g < 0$$

and let us split  $\tilde{\Gamma}$  into a finite number of curved sides  $\tilde{\Gamma}_i$  with endpoints  $z_i, z_{i+1}$  such that :

$$\tilde{\Gamma} = \bigcup_{i \in I_1} \tilde{\Gamma}_i, \quad \tilde{\Gamma}_i \subset \Gamma_i, \quad g(z_i) = g(z_{i+1}) = 0 \quad (1).$$

We can now prove the following :

PROPOSITION 5.1. : We have :

$$- \int_{\Gamma} g \frac{\partial u}{\partial \nu} d\Gamma \leq Ch^{\frac{3}{2}} \|u\|_{2,\Omega} \quad (12)$$

*Proof.* By the third inequality of (3) we find :

$$- \int_{\Gamma} g \frac{\partial u}{\partial \nu} d\Gamma \leq - \int_{\tilde{\Gamma}} g \frac{\partial u}{\partial \nu} d\Gamma \leq \|g\|_{0,\tilde{\Gamma}} \left\| \frac{\partial u}{\partial \nu} \right\|_{\frac{1}{2},\tilde{\Gamma}}$$

We may increase the inequality by replacing  $\left\| \frac{\partial u}{\partial \nu} \right\|_{\frac{1}{2},\tilde{\Gamma}}$  with  $\left\| \frac{\partial u}{\partial \nu} \right\|_{\frac{1}{2},\Gamma}$  and

by a trace theorem (see e. g. [15] [20]) with  $C \|u\|_{2,\Omega}$ . We denote by  $z$  the intersection between the side  $[z_i, z_{i+1}]$  and its normal from  $x$ ; by  $\tilde{\Sigma}_i$  the zone between the curved side  $\tilde{\Gamma}_i$  and the right side  $[z_{i+1}, z_i]$  (see the figure 1)

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(1)  $\tilde{\Gamma}_i$  may be empty for some  $i$ .

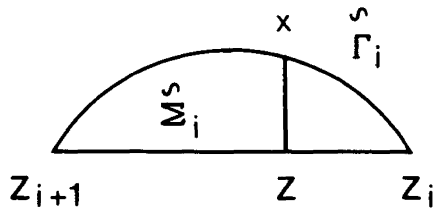


Figure 1.

The following inequalities can be obtained using Schwartz inequality and the smoothness of the boundary (cfr. [26])

$$\left\{ \int_{\tilde{\Gamma}_i} |g(x)|^2 d\Gamma \leq \int_{\tilde{\Gamma}_i} \left| \int_z^x \left| \frac{\partial g}{\partial n} \right| dn \right|^2 d\Gamma \right. \\ \left. \leq \int_{\tilde{\Gamma}_i} |z - x| \int_z^x \left| \frac{\partial g}{\partial n} \right|^2 dn d\Gamma \leq Ch^2 |g|_{1, \tilde{\Gamma}_i}^2 \right. \quad (13)$$

finally we apply Berger's ideas (see [26]) for piecewise polynomial functions: i. e.

$$\int_{\tilde{\Gamma}_i} |D^\alpha g|^2 dx \leq Ch \int_{\Gamma_i} |D^\alpha g|^2 dx \quad \forall |\alpha| \leq 1$$

and combining the relations obtained above we have (12). Now we return to (11) and we prove the:

PROPOSITION 5.2. We have

$$- \int_{\Gamma} ((u - \psi) - (u - \psi)_I) \frac{\partial u}{\partial \nu} d\Gamma \leq Ch^{\frac{3}{2}} \quad (14)$$

*Proof.*

Let us write (see proposition 5.1 and figure 2) for  $v = (u - \psi) - (u - \psi)_I$

$$- \int_{\Gamma} v \frac{\partial u}{\partial \nu} d\Gamma \leq C \|u\|_{2, \Omega} \|v\|_{0, \Gamma} \quad \text{and} \quad \|v\|_{0, \Gamma}^2 = \sum_{i=N_0+1}^{N-1} \|v\|_{0, \Gamma_i}^2$$

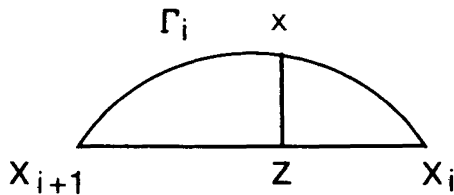


Figure 2.

it is now easy to prove the following inequalities

$$\left\{ \int_{\Gamma_i} |v(x)|^2 d\Gamma = \int_{\Gamma_i} \left[ \int_{x_i}^z \frac{\partial v}{\partial \lambda} d\lambda + \int_z^x \frac{\partial v}{\partial n} dn \right]^2 d\Gamma \right. \\ \left. \leq \int_{\Gamma_i} 2 \left[ |z - x_i| \int_{x_i}^{x_{i+1}} \left| \frac{\partial v}{\partial \lambda} \right|^2 d\lambda + 2|x - z| \int_z^x |\text{grad } v|^2 dn \right] d\Gamma. \right. \quad (15)$$

We shall apply Bramble and Zlamal's method (cfr. [2]) and use the trace theorems (see [20]) and the regularity assumptions, to yield (14).

We are finally ready to prove our theorem I (\*): we shall replace in (11) the results of the propositions 5.1 and 5.2 and use the coerciveness and continuity assumption.

6. REMARKS

Zlámál introduced in the finite element method the "curved elements" (see e. g. [29]), by introducing Zlámál curved elements to our problem, we find again the same rate of convergence for the approximation error.

Namely we shall consider a triangulation of the given domain  $\Omega$  into triangles completed along the boundary  $\Gamma$  by curved elements, so that the union of their closures is  $\bar{\Omega}$  (the usual regularity conditions are supposed satisfied): we construct a finite-dimensional subspace ( $V_h$ ) of trial functions belonging to  $H^1(\Omega)$ :

$$v_h(x^1, x^2) = r_h(\xi(x^1, x^2), \eta(x^1, x^2)) \quad (16)$$

When  $\xi(x^1, x^2), \eta(x^1, x^2)$  is the inverse mapping of

$$\begin{cases} x^1 = x^1(\xi, \eta) \\ x^2 = x^2(\xi, \eta) \end{cases} \quad (17)$$

which maps the unit triangle  $T_1$  with vertices  $R_1 \equiv (0, 0), R_2 = (1, 0), R_3 = (0, 1)$  in the  $\xi$ - $\eta$ -plane one-to-one into the triangle  $T$  (which may be a curved one) with vertices  $(x_i^1, x_i^2), (x_{i+1}^1, x_{i+1}^2), (x_j^1, x_j^2)$  (see [29]) and  $r(\xi, \eta)$

(\*) In order to obtain the optimal error estimate :  
i.e.  $\|u - u_h\|_{1,\Omega} \leq ch$

we should need the following results

$$(+ ) \quad \|g\|_{-\frac{1}{2}, \bar{\Gamma}} \leq ch^{1/2} \|g\|_{0, \bar{\Gamma}} \quad \text{where } g = u_h - \psi_I,$$

$$(+ + ) \quad \|v\|_{-\frac{1}{2}, \bar{\Gamma}} \leq ch^{1/2} \|v\|_{0, \bar{\Gamma}} \quad \text{where } v = (u - \psi) - (u_h - \psi)_I,$$

which at the moment we are not able to prove.

is an affine function in  $T_1$ , We shall also use the Zlámal theorem 2 to estimate the difference  $u - \tilde{u}_T$  where  $\tilde{u}_T$  is the "interpolate of  $u$ " i. e.: the function from  $V_h$  such that

$$\tilde{u}_T(x_i) = u(x_i) \quad \forall i \in I.$$

We shall choose  $\tilde{K}_h$  as the convex set of the all function of  $V_h$  such that :

$$v_h(x_i) \geq \psi(x_i) \quad \forall i \in I_1$$

(and then also  $v_h(x) \geq \psi(x) \quad \forall x \in \Gamma$ ).

$\tilde{u}_h$  denotes the solution of the discrete problem (7) corresponding to the convex set  $\tilde{K}_h$ . We can now repeat our theorem I replacing  $u_h$  with  $\tilde{u}_h$  to obtain the error bounds

$$\|u - \tilde{u}_h\|_1 \leq Ch^{\frac{3}{4}}.$$

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