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Finite element methods for nonlinear parabolic equations

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Finite Element Methods
For Nonlinear Parabolic Equations (*)

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Communicé par P.-A. Raviart

Summary. — Linear two-step A-stable methods of the second order introduced in [15] together with finite element discretizations in space are applied for the solution of nonlinear parabolic initial-boundary value problems. These include linear problems with time dependent coefficients as a special case. The resulting schemes are algebraically linear and unconditionally stable. A priori error estimates in the $L_2$-norm of optimal order of accuracy are derived. Similar error estimates hold for linear one-step A-stable methods.

1. Introduction

We shall consider the approximate solution of the initial-boundary value problem

$$\alpha(x, t) \frac{\partial u}{\partial t} = Pu, \quad (x, t) \in \Omega \times (0, T],$$

$$u(x, t) = 0, \quad (x, t) \in \Gamma \times (0, T],$$

$$u(x, 0) = u^0(x), \quad x \in \Omega.$$  

Here $x = (x_1, \ldots, x_N)$ is a point of a bounded domain $\Omega$ lying in the $N$-dimensional Euclidean space, $\Gamma$ is its boundary and

$$Pu = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \left[ k_{ij}(x, t, u) \frac{\partial u}{\partial x_j} \right] + \text{div} f(x, t, u) + g(x, t, u),$$

$$f(x, t, u) = (f_1(x, t, u), \ldots, f_N(x, t, u))^T$$

($T$ written as a superscript means transposition of a vector or of a matrix). Concerning the coefficients and the right-hand side of (1.1), all assumptions are summed up in:

$A_1$: (i) $\alpha(x, t)$ is bounded from below and above by a positive constant and is uniformly Lipschitz continuous as a function of $t$, i.e.

$$0 < m_1 \leq \alpha(x, t) \leq m_2, \quad (x, t) \in \Omega \times (0, T];$$

$$|\alpha(x, t_1) - \alpha(x, t_2)| \leq L |t_1 - t_2|, \quad t_1, t_2 \in (0, T], \quad x \in \Omega;$$

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(ii) the matrix \( \{ k_{ij}(x, t, u) \}_{i,j=1}^N \) is uniformly positive definite and bounded, i.e.

\[
\begin{align*}
c^{-1} \sum_{i=1}^N \xi_i^2 \leq \sum_{i,j=1}^N k_{ij}(x, t, u) \xi_i \xi_j \leq c \sum_{i=1}^N \xi_i^2,
\end{align*}
\]

\[c > 0, \quad (x, t) \in \Omega \times (0, T].\]  

(iii) the coefficients \( k_{ij}(x, t, u) \) are uniformly Lipschitz continuous as functions of \( t \) and \( u \), i.e.

\[
\begin{align*}
\sum_{i,j=1}^N |k_{ij}(x, t_1, u) - k_{ij}(x, t_2, u)| &\leq L |t_1 - t_2|, \\
(0, T], \quad \infty < u < \infty, \\
\sum_{i,j=1}^N |k_{ij}(x, t, u_1) - k_{ij}(x, t, u_2)| &\leq L |u_1 - u_2|, \\
(x, t) \in \Omega \times (0, T], \quad \infty < u_1, u_2 < \infty.
\end{align*}
\]

(iv) the functions \( f \) and \( g \) are uniformly Lipschitz continuous as functions of \( u \), i.e.

\[
\begin{align*}
\sum_{i=1}^N |f_i(x, t, u_1) - f_i(x, t, u_2)| \\
(0, T], \quad \infty < u_1, u_2 < \infty.
\end{align*}
\]

Before formulating the given problem in a variational form let us introduce some notation. By \( H^m \) we denote the Sobolev space of real functions which together with their generalized derivatives up to the \( m \)-th order inclusive are square integrable over \( \Omega \). The inner product and the norm are denoted by \((.,.)_m\) and \( || \cdot ||_m \), respectively. \( H^1_0 \) is the closure in the \( H^1 \)-norm of infinitely differentiable functions having compact support contained in \( \Omega \).

Multiplying (1.1) by \( \varphi \in H^1_0 \) and using Green's theorem we come to the identity

\[
(\alpha(x, t) \dot{u}, \varphi)_0 + a(t, u; u, \varphi) = -(f(x, t, u), \nabla \varphi)_0 + (g(x, t, u), \varphi)_0, \\
\forall \varphi \in H^1_0, \quad t \in (0, T];
\]

(1.9)

here the dot means the derivative with respect to \( t \),

\[
(f, \nabla \varphi)_0 = \sum_{i=1}^N \left( f_i, \frac{\partial \varphi}{\partial x_i} \right)_0
\]

and

\[
a(t, u; u, \varphi) = \int_{\Omega} \sum_{i,j=1}^N k_{ij}(x, t, u) \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_j} dx.
\]

(1.10)
Hence the weak solution of the problem (1.1)-(1.3) (for the definition see, for instance, J. L. Lions and E. Magenes, *Problèmes aux limites non homogènes*, Dunod, Paris 1968) satisfies (1.9).

To get the approximate solution we shall first discretize (1.9) in space. We shall use only finite element spaces which are subspaces of $H^1_0(\Omega)$. This restriction means that we can consider straight elements of different kind if $\Omega$ is a polyhedron and we have to consider curved elements which match exactly curved boundaries if $\Gamma$ is curved. We denote the finite element spaces which will be used by $V_h^p$ and we postulate the following properties:

$A_2$: (i) $V_h^p$ is either a regular family of straight elements according to the definition by Ciarlet and Raviart (see [1] or [2], section 6, p. 9) or a family of curved triangular elements (see Zlámal appendix of [12] and [13]) satisfying the condition that the smallest angle of all triangles is bounded away from zero.

(ii) to any $u \in H^{p+1} \cap H^1_0$ there exists $\hat{u} \in V_h^p$ such that

$$
\| u - \hat{u} \|_0 + h \| u - \hat{u} \|_1 \leq C h^{p+1} \| u \|_{p+1};
$$

$h$ is the greatest diameter of all elements or the greatest side in case of triangles.

The discretization of (1.9) in space gives the continuous-time Galerkin solution $U(x, t)$. It is a function from $V_h^p$ such that

$$
\begin{align*}
(\alpha(x, t) \dot{U}, \varphi)_0 + a(t, U; U, \varphi) &= -(f(x, t, U), \text{grad} \varphi)_0 + (g(x, t, U), \varphi)_0, \\
\forall \varphi &\in V_h^p, \\
U(x, 0) &= \hat{u}^0(x), \quad \hat{u}^0(x) \in V_h^p.
\end{align*}
$$

$\hat{u}^0(x)$ is an approximation of $u^0(x)$ and the simplest way is to choose the interpolate of $u^0(x)$ for it.

The continuous-time Galerkin solution has no practical significance. To get a computable approximate solution we must discretize also with respect to $t$. To this end we write (1.12), which represents a system of ordinary nonlinear differential equations, in a matrix form. Let $\{v_i\}_{i=1}^d$ be a basis of $V_h^p$ (of course, in finite element spaces we do not choose an arbitrary basis; however this circumstance does not play any role in our considerations) and put $U(x, t) = a^T(t) v(x)$ where $a = (a_1, \ldots, a_d)^T$, $v = (v_1, \ldots, v_d)^T$. Setting the basis functions $v_i$ for $\varphi$ in (1.12) we get

$$
M(t) \dot{a} + K(t, a) a = F(t, a).
$$

Here

$$
\begin{align*}
M(t) &= (\alpha(x, t) v, v)_0, \quad K(t, a) = a(t, a^T v; v, v), \\
F(t, a) &= -(f(t, x, a^T v), \text{grad} v)_0 + (g(x, t, a^T v), v)_0.
\end{align*}
$$
Both matrices $M(t)$ and $K(t, a)$ are positive definite, therefore
\[ \dot{a} = -A(t, a)a + M^{-1}(t)F(t, a), \quad A(t, a) = M^{-1}(t)K(t, a). \]  
(1.15)

The system (1.15) is a stiff system and we shall use first linear two-step $A$-stable methods of the second order for its solution. If
\[ \rho(\zeta) = \sum_{s=0}^{2} \alpha_s \zeta^s \quad \text{and} \quad \sigma(\zeta) = \sum_{s=0}^{2} \beta_s \zeta^s \]
are characteristic polynomials of a linear two-step method $(\rho, \sigma)$ normalized by
\[ \sum_{s=0}^{2} \beta_s = 1, \]  
(1.16)
then $(\rho, \sigma)$ is of the second order iff
\[ \begin{aligned}
\alpha_1 &= 1 - 2\alpha_2, & \alpha_0 &= -1 + \alpha_2, \\
\beta_1 &= \frac{1}{2} + \alpha_2 - 2\beta_2, & \beta_0 &= \frac{1}{2} - \alpha_2 + \beta_2.
\end{aligned} \]  
(1.17)

The result of Liniger [9] (see also Zlámal [15], section IV) can be stated as follows: Let $(\rho, \sigma)$ satisfy (1.16), (1.17) and let $\rho$ and $\sigma$ have no common root. Then the necessary and sufficient condition that the method be Dahlquist and $A$-stable is
\[ \alpha_2 \geq \frac{1}{2}, \quad \beta_2 \geq \frac{1}{2} \alpha_2. \]  
(1.18)

Let us apply the scheme $(\rho, \sigma)$ to the solution of (1.15). The result is
\[ \begin{aligned}
\sum_{s=0}^{2} \alpha_s a^{n+s} + \Delta t \sum_{s=0}^{2} \beta_s A(t_{n+s}, a^{n+s})a^{n+s} &= \Delta t \sum_{s=0}^{2} \beta_s M^{-1}(t_{n+s})F(t_{n+s}, a^{n+s}).
\end{aligned} \]  
(1.19)

This recurrence relation is algebraically nonlinear and has no practical significance. The idea of extrapolation was used often in recent years (we mention Douglas and Dupont [4] and Dupont, Fairweather and Johnson [5]) and here the extrapolation which linearizes (1.19) will be done in the following way: if $\gamma(t) \in C^2$ and $\gamma^n = \gamma(n\Delta t)$ choose $c_0$, $c_1$ such that $\gamma^n = c_1 \gamma^{n+1} + c_0 \gamma^n$ satisfies
\[ \sum_{s=0}^{2} \beta_s \gamma^{n+s} - \gamma^n = O(\Delta t^2 \ddot{\gamma}). \]  
(1.20)
Further determine $t^n$ such that
\[ \gamma^n - \gamma(t^n) = O(\Delta t^2 \gamma). \tag{1.21} \]

An easy calculation gives
\[ c_1 = 2\beta_2 + \beta_1, \quad c_0 = \beta_0 - \beta_2, \quad t^n = (n + c_1)\Delta t = t_n + (2\beta_2 + \beta_1)\Delta t. \]

Now replace $t_{n+s}$ and $a^{n+s}$ in nonlinear terms of (1.19) by
\[ t^n = t_n + (2\beta_2 + \beta_1)\Delta t, \quad a^n = (2\beta_2 + \beta_1)a^{n+1} + (\beta_0 - \beta_2)a^n. \tag{1.22} \]

Multiplying the resulting recurrence relation by $M(t^n)$ we get the final algebraically linear relation
\[ M^n \sum_{s=0}^{2} \alpha_s a^{n+s} + \Delta t K^n \sum_{s=0}^{2} \beta_s a^{n+s} = \Delta t F^n. \tag{1.23} \]

Here
\[ M^n = M(t^n), \quad K^n = K(t^n, a^n), \quad F^n = F(t^n, a^n). \tag{1.24} \]

Evidently, at every step we have to compute the matrices $M^n, K^n$ and to solve
\[ \text{a system of linear equations with the positive definite matrix } \alpha_2 M^n + \beta_2 \Delta t K^n. \]

Of course, we need to know the starting values $a^0, a^1$. $a^0$ is determined by the initial condition (1.13) whereas for the computation of $a^1$ a suitable one-step method can be used (see section 3).

We can come back to a variational form and write (1.23) as
\[ \begin{cases} \left( \sum_{s=0}^{2} \alpha_s U^{n+s}, \varphi \right) + \Delta t a \left( t^n, U^n; \sum_{s=0}^{2} \beta_s U^{n+s}, \varphi \right) \\ = -\Delta t \left( f^n, \text{grad } \varphi \right)_0 + \Delta t \left( g^n, \varphi \right)_0 \forall \varphi \in V^n_p, \end{cases} \tag{1.25} \]
\[ \alpha^n = \alpha(x, t^n), \quad f^n = f(x, t^n, U^n), \quad g^n = g(x, t^n, U^n). \]

Linear two-step schemes for nonlinear parabolic equations have been proposed recently by Comini, Del Guidice, Lewis and Zienkiewicz [3] and by Dupont, Fairweather and Johnson [5]. They are special cases of (1.23) and (1.25), respectively, with $\alpha_2 = 1/2, \beta_2 = 1/3$ in [3], $\alpha_2 = 1/2, \beta_2 = 0$ and $\alpha_2 = 1, \beta_2 = 1/2+\Theta$ in [5].

2. ERROR ESTIMATES

The technique for deriving error estimates used here is closely related to that of Wheeler [11] and Dupont, Fairweather, Johnson [5]. We shall decompose the exact solution in $u = \xi + \eta$, $\xi$ being the Ritz approximation defined by
\[ a(t, u; U^n) = a(t, u; \xi, \varphi), \quad \forall \varphi \in V^n_p. \tag{2.1} \]
We shall need estimates of \( \| \dot{\eta} \|_0 \) and \( \| \eta \|_0 \) of the form (4.15) in [5], i.e.
\[
\| \eta \|_0 + \| \dot{\eta} \|_0 \leq C h^{p+1} (\| u \|_{p+1} + \| \dot{u} \|_{p+1}), \quad t \in (0, T]. \tag{2.2}
\]
One can prove (2.2) exactly in the same way as Dupont, Fairweather and Johnson proved (4.15) in [5] under the following additional assumptions

\( A_3 \): (i) if \( z \in H^1_0 \) is defined by
\[
a(t, u; z, \phi) = (f, \phi)_0, \quad \forall \phi \in H^1_0
\]
then \( \| z \|_2 \leq C \| f \|_0 \) where \( C \) does not depend on \( t \) and on \( u \).

(ii) The coefficients \( k_{ij}(x, t, u) \) have partial derivatives
\[
\frac{\partial k_{ij}}{\partial t}, \quad \frac{\partial k_{ij}}{\partial u}, \quad \frac{\partial^2 k_{ij}}{\partial x_i \partial t}, \quad \frac{\partial^2 k_{ij}}{\partial x_i \partial u}
\]
and the matrices
\[
\left\{ k_{ij} + \frac{\partial k_{ij}}{\partial u} \right\}_{i, j=1}^N, \quad \left\{ \frac{\partial}{\partial x_i} \left( k_{ij} + \frac{\partial k_{ij}}{\partial u} \right) \right\}_{i, j=1}^N
\]
are bounded on \( \Omega \times (0, T] \).

**Remark:** If \( \Gamma, u \) and \( k_{ij} \) are sufficiently smooth (i) follows from (1.6) and from Theorem 37, I in Miranda [10] p. 169. However, (i) may hold even when \( \Omega \) has corners.

**Theorem:** Let the assumptions \( A_1, A_2, A_3 \) be satisfied. Let the scheme \((p, \sigma)\) normalized by (1.16) satisfy (1.17) and (1.18). Finally, let the exact solution \( u \) be such that \( \text{grad } u \) is bounded in the maximum norm, \( \partial^3 u / \partial t^3 \) is continuous for \((x, t) \in \bar{\Omega} \times [0, T] \) and \( \| u \|_{p+1} + \| \dot{u} \|_{p+1} \leq C, t \in [0, T] \). Then for arbitrary \( h, \Delta t \)
\[
\max_{2 \leq n \leq T/\Delta t} \| u^n - U^n \|_0 \leq C \left[ \sum_{i=0}^1 \| u^i - U^i \|_0 + h^{p+1} + \Delta t^2 \right] \tag{2.3}
\]
here \( u^n = u(x, n\Delta t) \), \( U^n \) is defined by (1.25) and the constant \( C \) does not depend on \( h \) and \( \Delta t \).

**Proof:** a) Set
\[
u^n - U^n = \nu^n - \xi^n - \xi^n - U^n = \eta^n + \varepsilon^n, \quad \varepsilon^n = \xi^n - U^n \in V^p_h.
\]
With respect to (2.2) it is sufficient to find a bound for \( \| \varepsilon^n \|_0 \).

For further purpose we prove now what we shall need later, namely
\[
\max_{\bar{n}} \| \text{grad } \xi \| \leq C, \quad t \in (0, T] \tag{2.4}
\]

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[$\xi$ is defined by (2.1)]. We restrict ourselves to the case that $V^p_h$ is formed by curved triangular elements. The proof for straight elements is analogous. If we prove that $\max |\text{grad} \eta| \leq Ch^{p-1} ||u||_{p+1}$ then (2.4) follows because

$$\max |\text{grad} \eta| \leq \max |\text{grad} u| + \max |\text{grad} \eta| \leq C$$

(notice that $p \geq 1$). Set $\eta = u - u_t + u_t - \xi$ where $u_t$ is the interpolate of $u$, i.e. that function from $V^p_h$ which has the same nodal parameters as $u$. Standard arguments give $\max |\text{grad} (u - u_t)| \leq Ch^p ||u||_{p+1}$ (see [12], Th. 2; here polynomials of the degree $p = 2n - 1$, $n = 1, 2, \ldots$ are considered, however the generalization is immediate—see appendix of [13]). Therefore what we need to prove is

$$\max |\text{grad} (u_t - \xi)| \leq C h^{p-1}.$$
(see [12], equations (8) and (7)) we get

\[
\left( \frac{\partial r}{\partial s} \right)^2 + \left( \frac{\partial r}{\partial t} \right)^2 \leq C \int \left\{ \left[ \frac{\partial}{\partial x_1} (u_t - \xi) \right]^2 + \left[ \frac{\partial}{\partial x_2} (u_t - \xi) \right]^2 \right\} \, dx_1 \, dx_2
\]

\[
\leq C (||u - u_t||^2 + ||u - \xi||^2).
\]

The bound \( ||u - \xi||_1 \leq C h^p ||u||_{p+1} \) follows by standard arguments and by (1.6), hence \( M_i \leq C h^{p-1} \).

b) Here we want to prove that \( \varepsilon^n \) satisfies a recurrent relation of the form

\[
\left( \alpha^n \sum_{s=0}^{2} \alpha_s \varepsilon^{n+s}, \varphi \right)_0 + \Delta t \left( t^n, U^n; \sum_{s=0}^{2} \beta_s \varepsilon^{n+s}, \varphi \right) = \Delta t (\psi^n, \varphi)_1,
\]

\( \forall \varphi \in V^p \)

where \( \psi^n \) is a function such that

\[
||\psi^n||_1 \leq C (9 + ||\varepsilon^n||_0), \quad 9 = h^{p+1} + \Delta t^2.
\] (2.6)

The left-hand side of (2.5) differs from the left-hand side of (1.25) in that \( \varepsilon^{n+s} \) stands in place of \( \varepsilon^{n+s} \). As \( \varepsilon^{n+s} = \xi^{n+s} - U^{n+s} \) we shall try to express

\[
\left( \alpha^n \sum_{s=0}^{2} \alpha_s \xi^{n+s}, \varphi \right)_0 + \Delta t \left( t^n, U^n; \sum_{s=0}^{2} \beta_s \xi^{n+s}, \varphi \right)
\]

in a suitable way. We shall find that

\[
\left( \alpha^n \sum_{s=0}^{2} \alpha_s \xi^{n+s}, \varphi \right)_0 + \Delta t \left( t^n, U^n; \sum_{s=0}^{2} \beta_s \xi^{n+s}, \varphi \right)
\]

\[
= -\Delta t \left( f(x, t^n, \xi^n), \text{grad} \varphi \right)_0 + \Delta t \left( g(x, t^n, \xi^n), \varphi \right)_0 + \Delta t (\psi^n, \varphi)_1, \quad \psi^n \text{ satisfies (2.6)}.\]

Subtract (1.25) from (2.7). The left-hand side of this difference is that of (2.5). The right-hand side is equal to \( \Delta t (\kappa^n + \psi^n, \varphi)_1 \) where \( \kappa^n \) is the function from \( V^p \) defined uniquely by

\[
(\kappa^n, \varphi)_1 = -(f(x, t^n, \xi^n) - f(x, t^n, U^n), \text{grad} \varphi)_0 + (g(x, t^n, \xi^n) - g(x, t^n, \xi^n), \varphi)_0, \quad \forall \varphi \in V^p.\]

(2.8)

Setting \( \varphi = \kappa^n \) in (2.8) and using (1.8) you obtain \( ||\kappa^n||_1 \leq C ||\varepsilon^n||_0 \).

Writing \( \psi^n \) instead of \( \kappa^n + \psi^n \) you get (2.5) with \( \psi^n \) satisfying (2.6).

To prove (2.7) we first remark that for the operator

\[
Lu^n = \sum_{s=0}^{2} \alpha_s u^{n+s} - \Delta t \sum_{s=0}^{2} \beta_s \dot{u}^{n+s}
\]
it holds \( |Lu^n| \leq C\Delta t^3 \) (see Henrici [6], Lemma 5.7, p. 247). It follows on basis of (1.20), (1.21) and (1.5) that
\[
\left( \alpha^n \sum_{s=0}^{2} \alpha_s u^{n+s} - \Delta t \alpha^n \dot{u}(x, t^n), \varphi \right)_0 = (\omega^n, \varphi)_0, \quad ||\omega^n||_0 \leq C\Delta t^3.
\]

We set for \( \alpha^n \dot{u}(x, t^n) \) from (1.1) and we easily derive
\[
\left( \alpha^n \sum_{s=0}^{2} \alpha_s u^{n+s}, \varphi \right)_0 + \Delta t a(t^n, u(x, t^n); u(x, t^n), \varphi)
= -\Delta t (f(x, t^n, u(x, t^n)), \text{grad} \varphi)_0
+ \Delta t (g(x, t^n, u(x, t^n)), \varphi)_0 + (\omega^n, \varphi)_0.
\]

The above equation can be written as
\[
\left( \alpha^n \sum_{s=0}^{2} \alpha_s u^{n+s}, \varphi \right)_0 + \Delta t a(t^n, u(x, t^n); u(x, t^n), \varphi)
= -\Delta t (f(x, t^n, \tilde{\xi^n}), \text{grad} \varphi)_0 + \Delta t (g(x, t^n, \tilde{\xi^n}), \varphi)_0 + (\omega^n, \varphi)_0
- \left( \alpha^n \sum_{s=0}^{2} \alpha_s \eta^{n+s}, \varphi \right)_0
+ \Delta t (g(x, t^n, u(x, t^n)) - g(x, t^n, \tilde{\eta^n}), \varphi)_0.
\]

We have
\[
\sum_{s=0}^{2} \alpha_s \eta^{n+s} = \alpha_2 (\eta^{n+2} - \eta^n) + \alpha_1 (\eta^{n+1} - \eta^n)
\]
(from the consistency condition it follows
\[
(1) = \sum_{s=0}^{2} \alpha_s = 0.
\]

Using (2.2) we get
\[
||\alpha^n \sum_{s=0}^{2} \alpha_s \eta^{n+s}||_0 \leq C \Delta th^{n+1}.
\]

Further, the last two terms in (2.9) are easy to estimate when we use (1.8)
Therefore, (2.9) can be written as
\[
\left( \alpha^n \sum_{s=0}^{2} \alpha_s \xi^{n+s}, \varphi \right)_0 + \Delta t a(t^n, u(x, t^n); u(x, t^n), \varphi)
= -(f(x, t^n, \tilde{\xi^n}), \text{grad} \varphi)_0 + (g(x, t^n, \tilde{\xi^n}), \varphi)_0 + \Delta t (\psi^n, \varphi)_1,
\]
\[
||\psi^n||_1 \leq C 9.
\]
If we prove that
\[ a(t^n, U^n; \sum_{s=0}^2 \beta_s \xi^{n+s}, \varphi) - a(t^n, u(x, t^n); u(x, t^n), \varphi) = (\psi^n, \varphi), \]
\[ \forall \varphi \in V_h^p \]
with \( \psi^n \) satisfying (2.6) then multiplying (2.11) by \( \Delta t \) and adding to (2.10) we get (2.7).

(2.11) defines a unique \( \psi^n \in V_h^p \). We can write
\[ (\psi^n, \varphi)_1 = a(t^n, U^n; \sum_{s=0}^2 \beta_s \xi^{n+s}, \varphi) - a(t^n, u(x, t^n); \sum_{s=0}^2 \beta_s \xi^{n+s}, \varphi) \]
\[ + a(t^n, u(x, t^n); \sum_{s=0}^2 \beta_s u^{n+s} - u(x, t^n), \varphi) \]
\[ - a(t^n, u(x, t^n); \sum_{s=0}^2 \beta_s \eta^{n+s}, \varphi). \]

From (1.7) (taking into account the form of the functional (1.10)), further from (2.4), (2.2) and (1.21) there follow the estimates
\[ |a(t^n, U^n; \sum_{s=0}^2 \beta_s \xi^{n+s}, \varphi) - a(t^n, u(x, t^n); \sum_{s=0}^2 \beta_s \xi^{n+s}, \varphi)| \]
\[ \leq C \| U^n - u(x, t^n) \|_0 \| \varphi \|_1 \]
\[ \leq C \| U^n - \xi^n + \eta^n + u^n - u(x, t^n) \|_0 \| \varphi \|_1 \]
\[ \leq C (\| e^n \|_0 + \theta) \| \varphi \|_1. \]

The third term on the right-hand side of (2.12) is easy to estimate using (1.20) and (1.21). The result is
\[ |a(t^n, u(x, t^n); \sum_{s=0}^2 \beta_s u^{n+s} - u(x, t^n), \varphi)| \leq C \Delta t^2 \| \varphi \|_1. \]

Concerning the last term notice first that \( a(t_n, u(x, t_n); \eta^n, \varphi) = 0, \forall \varphi \in V_h^p \).
Therefore, we have
\[ a(t^n, u(x, t^n); \sum_{s=0}^2 \beta_s \eta^{n+s}, \varphi) \]
\[ = \sum_{s=0}^2 \beta_s [a(t^n, u(x, t^n); \eta^{n+s}, \varphi) - a(t^{n+s}, u(x, t^{n+s}); \eta^{n+s}, \varphi)]. \]

Every term of the sum on the right-hand side is bounded by
\[ C \Delta t \| \eta^{n+s} \|_1 \| \varphi \|_1 \leq C \Delta t h^p \| \varphi \|_1 \]

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[it follows by means of (1.7)]. As $2 \Delta th^p \leq h^{2p} + \Delta t^2 \leq \delta$ (if $h \leq 1$) we see that $(\psi^n, \varphi)_1 \leq C (\delta + \| \varphi_n \|_0) \| \varphi \|_1$, $\forall \varphi \in V_h$, hence $\psi^n$ satisfies (2.6). This completes the proof of (2.5).

c) Setting

$$\varphi = \sum_{s=0}^{2} \beta_s e^{n+s}$$

in (2.5), using (1.6) and the inequality $|ab| \leq (1/2) \gamma a^2 + (1/2) \gamma^{-1} b^2$ we get

$$c_1 \Delta t \left| \sum_{s=0}^{2} \beta_s e^{n+s} \right|_1 \leq \frac{1}{2} \Delta t \left[ \gamma \| \psi^n \|_1^2 + \gamma^{-1} \left( \sum_{s=0}^{2} \beta_s e^{n+s} \right)^2 \right]_1, \quad c_1 = \text{const.} > 0.$$ 

Choosing $\gamma = 1/(2 c_1)$ and taking into account that $\psi^n$ satisfies (2.6) we see that

$$\left( \sum_{s=0}^{2} \alpha_s e^{n+s} + \sum_{s=0}^{2} \beta_s e^{n+s} \right)_0 \leq C \Delta t (\delta^2 + \| \varphi_n \|_0^2). \quad (2.13)$$

We write (2.13) for $n = 0, 1, \ldots, m-2$, $m \leq (T/\Delta t)$, and we sum. As $\varphi_n$ is a linear combination of $e^{n+1}$ and $e^n$ (see 1.22) we obtain

$$\sum_{n=0}^{m-2} \left( \sum_{s=0}^{2} \alpha_s e^{n+s} + \sum_{s=0}^{2} \beta_s e^{n+s} \right)_0 \leq C \delta^2 + C \Delta t \sum_{n=0}^{m-1} \| e^n \|_0^2. \quad (2.14)$$

We need to estimate from below $\sum_{n=0}^{m-2} \alpha_n S^n$ where

$$S^n = \sum_{s=0}^{2} \alpha_s e^{n+s} + \sum_{s=0}^{2} \beta_s e^{n+s}.$$

Let us write for the moment $\varepsilon_n$ instead of $\varepsilon^n$. The coefficients $\alpha_2, \beta_2$ satisfy (1.18). Therefore $\beta_2 = (1/2) \alpha_2 + \delta$, $\delta > 0$. Using (1.17) we find by inspection that

$$S_n = \frac{1}{2} (\alpha^2 + \delta) \varepsilon^2_{n+2} - \left( \alpha_2 - \frac{1}{2} \right) \varepsilon^2_{n+1} - \frac{1}{2} [(\alpha_2 - 1)^2 + \delta] \varepsilon^2_n - [\alpha_2 (\alpha_2 - 1) + \delta] (\varepsilon_{n+2} \varepsilon_{n+1} - \varepsilon_{n+1} \varepsilon_n) + \delta \left( \alpha_2 - \frac{1}{2} \right) (\varepsilon_{n+2} - 2 \varepsilon_{n+1} + \varepsilon_n)^2.$$

Therefore

$$S^n \geq \frac{1}{2} (\alpha^2 + \delta) \varepsilon^2_{n+2} - \left( \alpha_2 - \frac{1}{2} \right) \varepsilon^2_{n+1} - \frac{1}{2} [(\alpha_2 - 1)^2 + \delta] \varepsilon^2_n - [\alpha_2 (\alpha_2 - 1) + \delta] (\varepsilon_{n+2} \varepsilon_{n+1} - \varepsilon_{n+1} \varepsilon_n). \quad (2.15)$$
Hence
\[ \sum_{n=0}^{m-2} \alpha^n S^n \geq \frac{1}{2} (\alpha_2^2 + \delta) \sum_{n=2}^{m} \alpha^{n-2} e_n^2 - \frac{1}{2} \sum_{n=2}^{m-1} \alpha^{n-2} e_n^2 \]
\[ - \frac{1}{2} [(\alpha_2 - 1)^2 + \delta] \sum_{n=2}^{m-2} \alpha^{n-2} e_n^2 - \frac{1}{2} \sum_{n=2}^{m-1} \alpha^{n-2} e_n e_{n-1} \]
\[ + \left[ 2 (\alpha_2 - 1) + \delta \right] \sum_{n=2}^{m-2} \alpha^{n-2} e_n e_{n-1} + \left( \frac{1}{2} \right) \sum_{n=2}^{m-1} \alpha^{n-2} - \alpha^{n-1} e_n^2 \]
\[ + \frac{1}{2} [(\alpha_2 - 1)^2 + \delta] \sum_{n=2}^{m-2} (\alpha^{n-2} - \alpha^n) e_n^2 \]
\[ + \left[ 2 (\alpha_2 - 1) + \delta \right] \sum_{n=2}^{m-2} (\alpha^{n-2} - \alpha^n) e_n e_{n-1} - \sum_{n=2}^{m-1} (\alpha^{n-2} - \alpha^n) e_n e_{n-1} - C(e_0^2 + e_1^2). \]

The terms containing \( e_0^2, e_{m-1}^2, e_m e_{m-1} \) give a form \( \frac{1}{2} \alpha^{m-2} Q \) where
\[ Q = (\alpha_2^2 + \delta) e_m^2 + [(\alpha_2 - 1)^2 + \delta] e_{m-1}^2 - 2 \left[ \alpha_2 (\alpha_2 - 1) + \delta \right] e_m e_{m-1}. \]

The remaining terms are easy to estimate by means of (1.5). The result is
\[ \sum_{n=0}^{m-2} \alpha^n S^n \geq \frac{1}{2} \alpha^{m-2} Q - C(e_0^2 + e_1^2) - C \Delta t \sum_{n=2}^{m-1} e_n^2. \]  

(2.16)

Assume first that \( \alpha_2 (\alpha_2 - 1) + \delta = 0 \). Then \( Q \geq (\alpha_2^2 + \delta) e_m^2 \). Now let \( \alpha_2 (\alpha_2 - 1) + \delta \neq 0 \). Then using the inequality \( |a b| \leq (1/2) \gamma a^2 + 1/2 \gamma^{-1} b^2 \) with \( \gamma^{-1} = [(\alpha_2 - 1)^2 + \delta]/(\alpha_2 (\alpha_2 - 1) + \delta) \) we have
\[ Q \geq \left\{ \alpha_2^2 + \delta - \left[ \frac{\alpha_2 (\alpha_2 - 1) + \delta}{\alpha_2 (\alpha_2 - 1) + \delta} \right]^2 \right\} e_m^2 \]
\[ = (\alpha_2^2 + \delta) \left\{ 1 - \frac{\alpha_2 (\alpha_2 - 1) + \delta}{\alpha_2 (\alpha_2 - 1) + \delta} \right\} e_m^2. \]

In both cases it holds \( Q \geq c_2 e_m^2, \ c_2 = \text{const.} > 0. \)

As \( \alpha \geq m_1 \) we see from (2.16) that
\[ \sum_{n=0}^{m-2} a^n S^n \geq c_3 e_m^2 - C(e_0^2 + e_1^2) - C \Delta t \sum_{n=2}^{m-1} e_n^2, \ c_3 > 0, \]

(2.17)

hence
\[ \sum_{n=0}^{m-2} \left( \alpha^n \sum_{s=0}^{2} \alpha_s e^{n+s}, \sum_{s=0}^{2} \beta_s e^{n+s} \right) \]
\[ \geq c_3 \| e^m \|^2_0 - C(\| e^0 \|^2_0 + \| e^1 \|^2_0) - C \Delta t \sum_{n=2}^{m-1} \| e^n \|^2_0 \]

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and from (2.14)
\[ \| e^m \|^2_0 \leq C (\| e^0 \|^2_0 + \| e^1 \|^2_0 + 9^2) + C \Delta t \sum_{n=2}^{m-1} \| e^n \|^2_0, \quad m \geq 2. \] (2.18)

The discrete analogue of Gronwall's inequality (see Lees [8] or [5], Lemma 2.1) gives \[ \| e^m \|^2_0 \leq C (\| e^0 \|^2_0 + \| e^1 \|^2_0 + 9^2) \] for \( 2 \leq m \leq T/\Delta t. \) It easily follows
\[ \| e^m \|^2_0 \leq C (\| u^0 - U^0 \|^2_0 + \| u^1 - U^1 \|^2_0 + h^{p+1} + \Delta t^2) \]
which completes the proof of (2.3).

REMARK: In case that the vector \( f(x, t, u) \) is of the form \( f = b(x, t, u) u \) where \( b = (b_1(x, t, u), \ldots, b_N(x, t, u))^T \) we can assume (instead of \( f_i \) being uniformly Lipschitz continuous as functions of \( u \)) that the functions \( b_i \) are uniformly Lipschitz continuous as functions of \( u \) and bounded as functions of all arguments \( x, t, u. \) We have namely used the assumption (1.8) in two places, in (2.8) and (2.9). In the first case, it means to estimate \( b_i(x, t_n, \xi^n) \bar{\xi}^n - b_i(x, t_{n+1}, \bar{U}^n) \) \( \bar{U}^n. \) Now \( \bar{\xi} \) is bounded in the maximum norm because of (2.4) and \( \xi |^x = 0. \) Therefore
\[ | b_i(x, t_n, \xi^n) \bar{\xi}^n - b_i(x, t_{n+1}, \bar{U}^n) | \]
\[ = | b_i(x, t_n, U^n)(\xi^n - \bar{U}^n) + [b_i(x, t_n, \xi^n) - b_i(x, t_{n+1}, \bar{U}^n)] \xi^n | \]
\[ \leq C | \xi^n - \bar{U}^n | + C L \xi^n - \bar{U}^n \leq C | \xi^n - \bar{U}^n |. \]

The same argument applies in the other case.

3. A-STABLE LINEAR ONE-STEP METHODS

We will briefly show that error estimates for linear one-step \( A \)-stable methods are easy to derive in the same way as for linear two-step \( A \)-stable methods (the first such estimates were given by Douglas and Dupont [4] and Wheeler [11]). All linear one-step \( A \)-stable methods correspond to
\[ \rho(\zeta) = \zeta - 1, \quad \sigma(\zeta) = (1 - \Theta) \zeta + \Theta, \quad \Theta \leq \frac{1}{2}. \] (3.1)

(3.1) is often referred to as the “\( \Theta \)-method” (see Lambert [7], p. 240). If \( \Theta < 1/2 \) the method is of the first order, if \( \Theta = 1/2 \) we have the trapezoidal rule which is of the second order. Instead of (1.22) we choose
\[ t_{n+1} = t_n + \frac{1}{2} \Delta t, \quad U^n = \frac{3}{2} U^n - \frac{1}{2} U^{n-1}, \quad \Theta = \frac{1}{2}, \]
\[ U^n = U^n, \quad \Theta < \frac{1}{2}. \] (3.2)
The approximate solution $U^n$ is defined by
\[
\begin{align*}
(\alpha^n [U^{n+1} - U^n], \varphi)_0 + \Delta t a(t_n, U^n; (1 - \Theta) U^{n+1} + \Theta U^n, \varphi) = -\Delta t (f^n, \text{grad}\varphi)_0 + \Delta t (g^n, \varphi)_0, \quad \forall \varphi \in V_h^p.
\end{align*}
\tag{3.3}
\]

The matrix form of (3.3) is
\[
\begin{align*}
[M^n + (1 - \Theta) \Delta t K^n] a^{n+1} = (M^n - \Theta \Delta t K^n) a^n + \Delta t F^n
\end{align*}
\tag{3.4}
\]
(for $\Theta = 1/2$ (3.3) and (3.4), respectively, represent the Crank-Nicolson-Galerkin scheme). We easily derive that
\[
(\alpha^n [\varepsilon^{n+1} - \varepsilon^n], \varphi)_0 + \Delta t a(t_n, U^n; (1 - \Theta) \varepsilon^{n+1} + \Theta \varepsilon^n, \varphi) = \Delta t (\psi^n, \varphi)_1,
\quad \forall \varphi \in V_h^p,
\]
where
\[
\begin{align*}
||\psi^n||_1 &\leq C (h^{p+1} + \Delta t^2 + ||\varepsilon||_0), \quad \Theta = \frac{1}{2}, \\
||\psi^n||_1 &\leq C (h^{p+1} + \Delta t + ||\varepsilon||_0), \quad \Theta < \frac{1}{2}.
\end{align*}
\]

Instead of (2.15) we immediately find
\[
S^n \equiv (\varepsilon_{n+1} - \varepsilon_n) [(1 - \Theta) \varepsilon_{n+1} + \Theta \varepsilon_n] \geq \frac{1}{2} (\varepsilon_{n+1}^2 - \varepsilon_n^2)
\]
from which we easily get
\[
\begin{align*}
\sum_{n=0}^{m-1} (\alpha^n [\varepsilon^{n+1} - \varepsilon^n], (1 - \Theta) \varepsilon^{n+1} + \Theta \varepsilon^n)_0 &\geq c_2 \sum_{n=0}^{m-1} ||\varepsilon^n||_0^2 - C \Delta t \sum_{n=0}^{m-1} ||\varepsilon^n||_0^2.
\end{align*}
\tag{3.5}
\]

Assuming that we choose $\hat{u}^0$ such that
\[
||u^0 - \hat{u}^0||_0 \leq C h^{p+1}
\]
the final estimates are
\[
\begin{align*}
\max_{2 \leq n \leq T/\Delta t} ||u^n - U^n||_0 &\leq C (||u^1 - U^1||_0 + h^{p+1} + \Delta t^2), \quad \Theta = \frac{1}{2}, \\
\max_{1 \leq n \leq T/\Delta t} ||u^n - U^n||_0 &\leq C (h^{p+1} + \Delta t), \quad \Theta < \frac{1}{2}.
\end{align*}
\tag{3.6}
\]

We have to require the same assumptions as in Theorem with exception of (1.16)-(1.18) and in case of $\Theta < 1/2$ with exception that it is sufficient to assume $\partial^2 u/\partial t^2$ to be continuous for $(x, t) \in \overline{\Omega} \times [0, T]$. 

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