Local $H^{-1}$ Galerkin and adjoint local $H^{-1}$ Galerkin procedures for elliptic equations

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LOCAL $H^{-1}$ GALERKIN AND ADJOINT LOCAL $H^{-1}$ GALERKIN PROCEDURES FOR ELLIPTIC EQUATIONS (*)

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Abstract. — Two essentially dual, finite element methods for approximating the solution of the boundary value problem $Lu = \nabla \cdot (a \nabla u) + b \cdot \nabla u + cu = f$ on $\Omega$, a rectangle, with $u = 0$ on $\partial \Omega$ are shown to give optimal order convergence. The local $H^{-1}$ method is based on the inner product $(u, L^* v)$ and the adjoint method on $(Lu, v)$. Discontinuous spaces can be employed for the approximate solution in the local $H^{-1}$ procedure and for the test space in the adjoint method.

1. INTRODUCTION

Consider the elliptic boundary value boundary problem

$$
(Lu)(p) = \nabla \cdot (a(p) \nabla u) + b(p) \cdot \nabla u + c(p)u = f(p), \quad p \in \Omega,
$$

$$
\begin{align*}
&u(p) = 0, \quad p \in \partial \Omega,
\end{align*}
$$

where $\Omega$ is the square $I \times I$ and $I = (0, 1)$. We assume that $a, \nabla a, b, c \in C^1(\Omega)$, that $f \in L_2(\Omega)$, and that $0 < a_0 \leq a(p) \leq a_1$, $p \in \Omega$, where $a_0$ and $a_1$ are constants. We further assume that, given $g \in L_2(\Omega)$, there exists a unique function $\varphi \in H^2(\Omega)$ satisfying $L \varphi = g$ in $\Omega$ and $\varphi = 0$ on $\partial \Omega$.

We shall use the following notation. Let $\delta : 0 = x_0 < x_1 < \ldots < x_N = 1$ be a partition of $[0, 1]$. Set $I_j = (x_{j-1}, x_j), h_j = x_j - x_{j-1}$, and $h = \max h_j$.

For $E \subset I$ let $P_r(E)$ denote the functions defined on $I$ whose restrictions to $E$ coincide with polynomials of degree at most $r$. Let

$$
\mathcal{M}(-1, r, \delta) = \bigcap_{j=1}^N P_r(I_j)
$$

and, for $k$ a non-negative integer,

$$
\begin{align*}
\mathcal{M}(k, r, \delta) &= \mathcal{M}(-1, r, \delta) \cap C^k(I), \\
\mathcal{M}^0(k, r, \delta) &= \mathcal{M}(k, r, \delta) \cap \{v \mid v(0) = v(1) = 0\}, \\
\tilde{\mathcal{M}}(k-1, r-1, \delta) &= \{v' \mid v \in \mathcal{M}^0(k, r, \delta)\}.
\end{align*}
$$

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We assume that $\delta$ is quasi-uniform and that $r \geq 1$. For brevity, we set

$$\mathcal{N} = \mathcal{M}^0(k+2, r+2, \delta) \otimes \mathcal{M}^0(k+2, r+2, \delta),$$

and

$$\mathcal{M} = \mathcal{M}(k, r, \delta) \otimes \mathcal{M}(k, r, \delta).$$

Note that $\mathcal{M}$ and $\mathcal{M}$ are the images of $\mathcal{N}$ under the maps given by $\partial^2/\partial x \partial y$ and $\partial^4/\partial x^2 \partial y^2$, respectively.

The local $H^{-1}$ Galerkin approximation is defined as the solution $U \in \mathcal{M}$ of the equations

$$(U, L^* \varphi) = (f, \varphi), \quad \varphi \in \mathcal{N}, \quad (2)$$

where the inner product is the standard $L^2(\Omega)$ one. The adjoint local $H^{-1}$ Galerkin approximation is given by $W \in \mathcal{N}$ satisfying

$$(LW, \varphi) = (f, \varphi), \quad \varphi \in \mathcal{M}. \quad (3)$$

We first show that there exists a unique $U$ and a unique $W$ satisfying (2) and (3), respectively, for $L = \Delta$. Optimal $L^2$ error estimates are also obtained for the operator $\Delta$. We then generalize our results to obtain optimal $L^2$ results for operators of the form given in (1).

Let $H^k(\Omega)$ be the Sobolev space of functions having $L^2(\Omega)$-derivatives through order $k$. Denote the usual norm on $H^s(\Omega)$ by $\| . \|_s$; for $s = 0$ the subscript will be omitted. We also use the norm

$$\| w \|_{-1} = \sup_{z \in H^1(\Omega)} \frac{(w, z)}{\| z \|_1}.$$

If the reader wishes to use any of the results derived below for non-integral indices, then standard interpolation theory [3] should be applied.

2. ERROR ESTIMATES FOR $L = \Delta$

First note that, since $\dim \mathcal{M} = \dim \mathcal{N}$, uniqueness implies existence.

**Lemma 1 :** Suppose that $V \in \mathcal{M}$ satisfies

$$(V, \Delta \varphi) = 0, \quad \varphi \in \mathcal{N}.$$ 

Then, $V \equiv 0$.

**Proof :** Note that there exists a unique $Q \in \mathcal{N}$ such that $Q_{xxyy} = V$. Integrating by parts, we have

$$(\nabla Q_{xy}, \nabla w) = 0, \quad w \in \mathcal{M}.$$ 

Since $Q_{xy} \in \mathcal{M}$, we note that $Q_{xxy} = 0$ and $Q_{yxx} = 0$. Thus, $V = 0$.

Since the matrix arising in (3) is the adjoint of that of (2), there exists a unique $W$ satisfying (3) for $L = \Delta$. 

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We now derive $L_2$ and negative norm error estimates for $U - u$ when $L = \Delta$. Let $Z \in \mathcal{N}$ satisfy $Z_{xxyy} = U$. Also let $z_{xxyy} = u$ in $\Omega$ and $z = 0$ on $\partial\Omega$. We observe from (1) and (2) with $\xi = Z - z$ that

$$
(\nabla \xi_{xy}, \nabla w) = 0, \quad w \in \mathcal{Q}.
$$

(4)

**Theorem 1**: Let $z$ and $Z$ be as defined above, and let $z_{xy} \in H^s(\Omega)$ for some $s$ such that $1 \leq s \leq r + 2$. Then,

$$
|| (z - Z)_{xy} || + h || (z - Z)_{xy} ||_1 \leq C || z_{xy} ||_s h^s.
$$

**Proof**: It follows from (4) that

$$
|| \nabla \xi_{xy} || = \inf_{\chi \in \mathcal{Q}} || \nabla (z_{xy} - \chi) ||.
$$

(5)

Let $T : H^1(I) \to \mathcal{M}(k+1, r+1, \delta)$ be determined by the relations

$$
\int_0^1 (g - Tg)v dx = \int_0^1 (g - Tg) dx = 0, \quad v \in \mathcal{M}(k, r, \delta).
$$

It is easy to see that $(g - Tg)(0) = (g - Tg)(1) = 0$, by taking $v = x$ or $1 - x$. Since $(Tg)'$ is the $L_2(I)$-projection of $g'$ into $\mathcal{M}(k, r, \delta)$,

$$
|| (g - Tg)' ||_{L_2(I)} \leq C || g^{(s)} ||_{L_2(I)} h^{s-1}, \quad 1 \leq s \leq r + 2.
$$

Let

$$
-\varphi'' = \xi = g - Tg, \quad x \in I,
$$

$$
\varphi'(0) = \varphi'(1) = 0,
$$

$$
\int_0^1 \varphi dx = 0.
$$

Then for $v \in \mathcal{M}(k, r, \delta)$ appropriately chosen

$$
|| \xi ||^2 = (\xi', \varphi' - v) \leq C || \xi' ||_{L_2(I)} || \xi ||_{L_2(I)} h,
$$

and

$$
|| g - Tg ||_{L_2(I)} \leq C || g^{(s)} ||_{L_2(I)} h^s, \quad 1 \leq s \leq r + 2.
$$

Consider $(T \otimes T)z_{xy} \in \mathcal{M}(k+1, r+1, \delta) \otimes \mathcal{M}(k+1, r+1, \delta)$. It is easy to see that $(T \otimes T)z_{xy} \in \mathcal{Q}$ and that

$$
|| z_{xy} - (T \otimes T)z_{xy} ||_q \leq C || z_{xy} ||_s h^{s-q}, \quad 2 \leq s \leq r + 2, \quad 0 \leq q \leq 1,
$$

(6)

since $T \otimes T - I \otimes I = (T - I) \otimes I + I \otimes (T - I) + (T - I) \otimes (T - I)$. Thus, from (5) and (6),

$$
|| \nabla \xi_{xy} || \leq C || z_{xy} ||_s h^{s-1}, \quad 2 \leq s \leq r + 2.
$$

The inequality

$$
|| \nabla \xi_{xy} || \leq C || \nabla z_{xy} || \leq C || z_{xy} ||_1,
$$

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is obvious, and the desired result follows:

\[ \| \nabla \xi_{xy} \| \leq C \| z_{xy} \|_{s} h^{s-1}, \quad 1 \leq s \leq r+2. \]

Since \( \xi_{xy} \) has average value zero,

\[ \| \xi_{xy} \|_{1} \leq C \| z_{xy} \|_{s} h^{s-1}, \quad 1 \leq s \leq r+2. \]

To obtain the \( L_{2} (\Omega) \) estimate, first let

\[ -\Delta \varphi = \xi_{xy}, \quad (x, y) \in \Omega, \]

\[ \frac{\partial \varphi}{\partial n} = 0, \quad (x, y) \in \partial \Omega. \]

Since \( \langle \xi_{xy}, 1 \rangle = 0 \), there exists \( \varphi \) such that \( \langle \varphi, 1 \rangle = 0 \) and \( \| \varphi \|_{2} \leq C \| \xi_{xy} \| \). Then,

\[ \| \xi_{xy} \|^{2} = \langle \nabla \xi_{xy}, \nabla (\varphi - \chi) \rangle, \quad \chi \in \mathcal{D}, \]

and

\[ \| \xi_{xy} \|^{2} \leq C \| \nabla \xi_{xy} \| \inf_{\chi \in \mathcal{D}} \| \nabla (\varphi - \chi) \|. \]

The function \( \xi_{xy} \) can be expanded in a double cosine series:

\[ \xi_{xy} = \sum_{p, q=1}^{\infty} c_{pq} \cos \pi px \cos \pi qy. \]

Thus,

\[ \varphi = \frac{1}{\pi^{2}} \sum_{p, q=1}^{\infty} \frac{c_{pq}}{p^{2} + q^{2}} \cos \pi px \cos \pi qy. \]

It then follows by approximating each product of cosines in \( \mathcal{D} \) that

\[ \inf_{\chi \in \mathcal{D}} \| \nabla (\varphi - \chi) \| \leq Ch \| \xi_{xy} \|, \]

and the theorem has been proved.

Denote by \( P \) the restriction of the projection \( T \) to the subclass of \( H^{1} (I) \) consisting of functions having zero average value. Let \( \mathcal{P} = P \otimes P \).

We wish to obtain a better \( H^{1} \) estimate of \( v = \mathcal{P} z_{xy} - Z_{xy} \) than would follow from (6) and theorem 1. We deduce from (4) that

\[ (\nabla v, \nabla w) = (\nabla (\mathcal{P} z_{xy} - z_{xy}), \nabla w) = \tau_{x} + \tau_{y}, \quad w \in \mathcal{D}. \]

Using the definition of \( P \) and integration by parts, we see that, for \( w \in \mathcal{D} \),

\[ \tau_{x} = (((I \otimes P)(P \otimes I) z_{xy} - z_{xy})_{x}, w_{x}) \]

\[ = (I \otimes (P - I) z_{xy}, w_{x}) \]

\[ = - (I \otimes (P - I) z_{xxy}, w) \]

\[ + \int_{0}^{1} I \otimes (P - I) z_{xxy} (., y) w (., y) \bigg|_{0}^{1} dy. \]

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Note that $z$ has the representation

$$z(x, y) = \int_0^x \int_0^y (x-\alpha)(y-\beta) u(\alpha, \beta) \, d\alpha \, d\beta$$

$$-x \int_0^x \int_0^1 (1-\alpha)(y-\beta) u(\alpha, \beta) \, d\alpha \, d\beta$$

$$-y \int_0^1 \int_0^x (x-\alpha)(1-\beta) u(\alpha, \beta) \, d\alpha \, d\beta$$

$$+xy \int_0^1 \int_0^1 (1-\alpha)(1-\beta) u(\alpha, \beta) \, d\alpha \, d\beta.$$ 

(9)

One can easily verify from (9) that the boundary terms in (8) are zero since $z_{xxy}(0, y) = 0$ and $z_{xxy}(1, y) = 0$. We also observe that

$$\int_0^1 z_{xxy} \, dy = z_{xxx}(x, 1) - z_{xxx}(x, 0) = 0,$$

since $z$ vanishes on the boundary. Similarly, $\int_0^1 z_{yyyy} \, dx = 0$. Thus, we see that

$$\|\nabla\psi\|_1 \leq C \|\psi\|_{-1},$$

(10)

where

$$\psi = I \otimes (I - P)(z_{xxy}) + (I - P) \otimes I(z_{yyyy}).$$

(11)

It follows that

$$\|\psi\|_{-1} \leq \left( \int_0^1 \left\| I \otimes (I - P) \frac{\partial^4 z}{\partial x^3 \partial y} (x, \cdot) \right\|_{H^{-1}(l)}^2 \, dx \right)^{1/2}$$

$$+ \left( \int_0^1 \left\| (I - P) \otimes I \frac{\partial^4 z}{\partial x \partial y^3} (\cdot, y) \right\|_{H^{-1}(l)}^2 \, dy \right)^{1/2}.$$ 

It is easy to show that

$$\|(I - P)f\|_{H^{-1}(l)} \leq C \|f^{(s)}\|_{L^2(l)} h^{s+1},$$

provided that

$$\int_0^1 f \, dx = 0,$$

by using the auxiliary problem

$$-\varphi'' = g - \int_0^1 g \, dx, \quad x \in l,$$

$$\varphi'(0) = \varphi'(1) = 0,$$
where \( g \in H^1(I) \). Thus,
\[
\| \psi \|_{-1} \leq C \left\{ \left\| \frac{\partial^{s+1} u}{\partial x^s \partial y} \right\| + \left\| \frac{\partial^{s+1} u}{\partial x \partial y^s} \right\| \right\} h^{s+2} \tag{12}
\]
for \( 0 \leq s \leq r+1 \).

**Theorem 2:** Let \( u \) be the solution to (1) with \( L = \Delta \), and let \( U \in \mathcal{M} \) satisfy (2). Let \( \hat{U} \) be the \( L_2 \) projection of \( u \) into \( \mathcal{M} \). Then,
\[
\| U - \hat{U} \| \leq C \left\{ \left\| \frac{\partial^{s+1} u}{\partial x^s \partial y} \right\| + \left\| \frac{\partial^{s+1} u}{\partial x \partial y^s} \right\| \right\} h^{s+1} \tag{13}
\]
for \( 0 \leq s \leq r+1 \).

**Proof:** Since \( \hat{U} \) satisfies
\[
(U - \hat{U}, v) = 0, \quad v \in \mathcal{M},
\]
one can easily verify that
\[
\hat{U} = (\mathcal{P} z_{xy})_{xy}.
\]
Thus, (13) follows from (10), (12), and the quasi-uniformity hypothesis on the partition \( \delta \).

**Corollary:** The error \( U - u \) satisfies the following bounds:
\[
\| U - u \| \leq C \| u \|_s h^s, \quad 1 \leq s \leq r+1,
\]
\[
\| U - u \|_{L^\infty(\Omega)} \leq C \left\{ \| u \|_{W^s_{\infty}(\Omega)} + \left\| \frac{\partial^{s+1} u}{\partial x^s \partial y} \right\| + \left\| \frac{\partial^{s+1} u}{\partial x \partial y^s} \right\| \right\} h^s. \tag{14}
\]

**Proof:** The \( L_2(\Omega) \)-estimate is a trivial consequence of (13). To obtain the \( L^\infty(\Omega) \)-estimate, note first that (13) and the quasi-uniformity of \( \delta \) imply that, for \( 0 \leq s \leq r+1 \),
\[
\| U - \hat{U} \|_{L^\infty(\Omega)} \leq \| v \|_{W^s_{\infty}(\Omega)} \leq C \left\{ \left\| \frac{\partial^{s+1} u}{\partial x^s \partial y} \right\| + \left\| \frac{\partial^{s+1} u}{\partial x \partial y^s} \right\| \right\} h^s.
\]

It follows from inequality (28) of [2] or from [1] that
\[
\| u - \hat{U} \|_{L^\infty(\Omega)} \leq C \| u \|_{W^s_{\infty}(\Omega)} h^s, \quad 0 \leq s \leq r+1.
\]

We now wish to consider the adjoint local \( H^{-1} \) Galerkin procedure for \( L = \Delta \). As noted earlier, there exists a unique \( W \in \mathcal{N} \) satisfying
\[
(\Delta W, v) = (f, v), \quad v \in \mathcal{M}. \tag{14}
\]

**Theorem 3:** Let \( u \) be the solution to (1) with \( L = \Delta \) and assume that \( u_{xy} \in H^s(\Omega) \), \( 1 \leq s \leq r+2 \). Let \( W \in \mathcal{N} \) be defined by (14). Then,
\[
\| (W - u)_{xy} \| + h \| (W - u)_{xy} \|_1 \leq C \| u_{xy} \|_s h^s.
\]
Proof: Just as in (4),

\[(\nabla (W-u)_{xy}, \nabla w_{xy}) = 0, \quad w \in \mathcal{N}\]

Since \(w_{xy}\) represents an arbitrary element of \(\mathcal{S}\), the theorem follows from the analysis of (4) given in the proof of theorem 1.

Next, we shall derive an \(H^1(\Omega)\)-estimate of the error \(W-u\). Note that

\[
||\nabla (W-u)||^2 = -(\Delta (W-u), W-u) = -(\Delta (W-u), W-u-\chi), \quad \chi \in \mathcal{M}. \tag{15}
\]

We choose \(\chi \in \mathcal{M}\) as the local \(H^{-1}\) Galerkin approximation to \(W-u\); i.e.,

\[
(W-u-\chi, \Delta \phi) = 0, \quad \phi \in \mathcal{N}. \tag{16}
\]

By the corollary to theorem 2,

\[
||W-u-\chi|| \leq C ||W-u||_1 h.
\]

From (15) and (16), we see that

\[
||\nabla (W-u)||^2 \leq Ch \inf_{\mu \in \mathcal{N}} ||u-\mu||_2^1 \quad \text{hence,}
\]

\[
||\nabla (W-u)||^2 \leq Ch \inf_{\mu \in \mathcal{N}} ||u-\mu||_2^1 \leq Ch^{s+1} \quad 0 \leq s \leq r+1.
\]

Since the boundary values of \(u\) were imposed strongly on the elements of \(\mathcal{N}\), the \(L_2(\Omega)\)-norm of the \(\nabla (W-u)\) is equivalent to the \(H^1(\Omega)\)-norm of \(W-u\); thus,

\[
||W-u||_1 \leq C ||u||_{s+2} h^{s+1}, \quad 0 \leq s \leq r+1.
\]

As a result of the quasi-uniformity of \(\delta\), it follows easily that

\[
||W-u||_2 \leq C ||u||_{s+2} h^s, \quad 0 \leq s \leq r+1. \tag{17}
\]

Now, we shall seek an estimate of the error in \(L_2(\Omega)\). Consider

\[
\Delta \phi = W-u \quad \text{on} \quad \Omega,
\]

\[
\phi = 0 \quad \text{on} \quad \partial \Omega.
\]

Then,

\[
||W-u||^2 = (W-u, \Delta \phi) = (\phi, \Delta (W-u)) = (\phi-\phi^*, \Delta (W-u)), \quad \phi^* \in \mathcal{M}.
\]
Thus, choosing an appropriate $\phi^*$, we obtain the inequality
\[
|| W-u ||^2 \leq C || \phi ||_2 h^2 || \Delta ( W-u ) ||
\leq C || W-u || || \Delta ( W-u ) || h^2;
\]
therefore,
\[
|| W-u || \leq C || u ||_{s+2} h^{s+2}, \quad 0 \leq s \leq r+1.
\]
Summarizing the above results, we have proved the following theorem.

**Theorem 4:** Let $u$ be the solution to (1) with $L = \Delta$ and assume that $u \in H^s(\Omega)$, $2 \leq s \leq r+3$. Then, if $W$ is defined by (14),
\[
|| W-u ||_q \leq C || u ||_{s-q} h^{s-q}, \quad 0 \leq q \leq 2.
\]
If $k \geq 0$, then the range on $q$ in theorem 4 can be extended to $0 \leq q \leq \min{(k+3, s)}$ by repeated use of quasi-uniformity to obtain the analogue of (17) in $H^{k+3}(\Omega)$.

3. THE GENERAL CASE

Let $U \in \mathcal{M}$ be determined as the solution of (2), and introduce an auxiliary function $U_1 \in \mathcal{M}$ as the solution of
\[
(U_1 - u, \Delta v) = 0, \quad v \in \mathcal{N}.
\]
Let $\xi = U - U_1$, and let $\psi$ be given by the Dirichlet problem
\[
L^* \psi = \xi \quad \text{on } \Omega,
\]
\[
\psi = 0 \quad \text{on } \partial \Omega.
\]
Then, if $\psi^* \in \mathcal{N}$,
\[
|| | \xi ||^2 = (\xi, L^* \psi)
= (\xi, L^*(\psi - \psi^*)) + (\xi, L^* \psi^*)
= (\xi, L^*(\psi - \psi^*)) + (\eta, L^* \psi^*),
\]
where $\eta = u - U_1$. We choose $\psi^* \in \mathcal{M}$ to satisfy
\[
(\Delta (\psi - \psi^*), v) = 0, \quad v \in \mathcal{M}.
\]
Thus, with $\delta$ and $\bar{c}$ indicating the lower order coefficients of $L^*$,
\[
|| \xi ||^2 = (a \xi, \Delta (\psi - \psi^*)) + (\xi, \bar{b} \cdot \nabla (\psi - \psi^*))
+ (\xi, \bar{c} (\psi - \psi^*)) + (\eta, L^* \psi^*)
= (a \xi - \chi, \Delta (\psi - \psi^*)) + (\xi, \bar{b} \cdot \nabla (\psi - \psi^*))
+ (\xi, \bar{c} (\psi - \psi^*)) + (\eta, L^* \psi^*), \quad \chi \in \mathcal{M}.
\]

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It is well-known that, since \( a \in C^1 (\Omega) \),
\[
\inf_{\chi \in \mathcal{A}} \| a \xi - \chi \| \leq C \| \xi \| h.
\]
Replacing \( u \) by \( \psi \) and \( W \) by \( \psi^* \) in theorem 4, we observe that
\[
\| \psi - \psi^* \|_q \leq C \| \psi \|_2 h^{2-q}, \quad 0 \leq q \leq 2.
\]
Since \( \| \psi \|_2 \leq C \| \xi \| \)
\[
\| \xi \|^2 \leq C ( \| h \| \| \xi \|^2 + \| \eta \| \| \xi \| )
\]
Hence, for \( h \) sufficiently small,
\[
\| \xi \| \leq C \| \eta \|.
\]
Consequently, we have the following theorem.

**Theorem 5:** There exists \( h_0 = h_0 (L) > 0 \) such that a unique solution \( U \in \mathcal{M} \) of (2) exists for \( h \leq h_0 \); moreover, if \( 1 \leq s \leq r+1 \) and if \( u \in H^s (\Omega) \) is the solution of (1), then
\[
\| U - u \| \leq C \| u \|_s h^s.
\]

We shall now consider error estimates for the adjoint local \( H^{-1} \) Galerkin procedure. Note that the ellipticity of \( L \) implies a Gårding inequality of the form
\[
C_0 \| \phi \|_1^2 \leq -(L \phi, \phi) + C_1 \| \phi \|^2
\]
for \( \phi \in H^2 (\Omega) \) such that \( \phi = 0 \) on \( \partial \Omega \), where \( C_0 \) is some positive constant. Since (1) and (3) imply that \( (L (W - u), \psi) = 0 \) for \( \psi \in \mathcal{M} \),
\[
C_0 \| W - u \|_1^2 - C_1 \| W - u \|^2 \leq -(L (W - u), W - u - \psi), \quad \psi \in \mathcal{M}.
\]
For \( h \) sufficiently small, theorem 5 when applied to the operator \( L^* \) instead of \( L \) implies the existence of \( \psi \in \mathcal{M} \) such that
\[
(L v, W - u - \psi) = 0, \quad v \in \mathcal{N},
\]
and
\[
\| W - u - \psi \| \leq C \| W - u \|_1 h.
\]
Thus, for any \( \theta \in \mathcal{N} \):
\[
C_0 \| W - u \|_1^2 - C_1 \| W - u \|^2 \leq -(L (\theta - u), W - u - \psi) \leq C \| u - \theta \|^2 \| W - u \|_1 h.
\]
By noting that \( \| W - u \|^2 \leq \| W - u \|_1 \| W - u \| \), we see that
\[
\| W - u \|_1 \leq C ( \| u \|_{s+2} h^{s+1} + \| W - u \|), \quad 0 \leq s \leq r + 1.
\]
Again by the quasi-uniformity of $\delta$,

$$\left\| W-u \right\|_2 \leq C \left\| u \right\|_{s+2} h^s + h^{-1} \left\| W-u \right\|, \quad 0 \leq s \leq r+1.$$ 

In order to obtain an $L_2 (\Omega)$-estimate, we now consider the auxiliary Dirichlet problem given by

$$L^* \phi = W-u \quad \text{on} \quad \Omega,$$

$$\phi = 0 \quad \text{on} \quad \partial \Omega.$$ 

Then,

$$\left\| W-u \right\|^2 = (W-u, L^* \phi) = (L(W-u), \phi)$$

$$= (L(W-u), \varphi - \varphi^*), \quad \varphi^* \in \mathcal{M}.$$ 

Thus, choosing an appropriate $\varphi^*$, we obtain the inequality

$$\left\| W-u \right\|^2 \leq C \left\| W-u \right\|_2 \left\| \varphi \right\|_2 h^2$$

$$\leq C \left\| W-u \right\|_2 \left\| W-u \right\| h^2,$$

and

$$\left\| W-u \right\| \leq C \left\| W-u \right\|_2 h^2$$

$$\leq C \left( \left\| u \right\|_{s+2} h^{s+2} + \left\| W-u \right\| h \right), \quad 0 \leq s \leq r+1.$$ 

Hence, we have proved the following theorem.

**Theorem 6:** There exists $h_0 = h_0 (L) > 0$ such that there exists a unique solution $W \in \mathcal{N}$ of (3), and if $2 \leq s \leq r+3$ and if the solution $u$ of (1) belongs to $H^s (\Omega)$, then

$$\left\| W-u \right\|_{eq} \leq C \left\| u \right\|_{s} h^{s-q}, \quad 0 \leq q \leq 2.$$ 

The range on $q$ can be extended just as for theorem 4.

**REFERENCES**