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A GENERAL THEOREM ON TRIANGULAR FINITE $C^{(m)}$-ELEMENTS

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Communiqué par P.-A. Raviart.

Summary. — *The following theorem is proved : To achieve piecewise polynomials of class C^m on an arbitrary triangulation of a polygonal domain, the nodal parameters must include all derivatives of order less than or equal to $2m$ at the vertices of the triangles.*

For the sake of brevity we shall use the expression « triangular $C^{(m)}$ element » for a polynomial on a triangle which generates piecewise polynomial and m-times continuously differentiable functions on an arbitrary triangulation. (From this point of view the Clough-Tocher element [4, p. 84] is not a triangular $C^{(1)}$-element.)

In the last few years there were constructed various types of interpolation polynomials on a triangle (see e.g., [3, 5]). All these polynomials have two following features :

1. A general triangular $C^{(m)}$-element is constructed in such a way that at the vertices of a triangle there are prescribed at least all derivatives of order less than or equal to $2m$.

2. The lowest degree of a general triangular $C^{(m)}$-element is equal to $4m + 1$.

These two features suggest the following questiones :

(i) Which derivatives should be prescribed at the vertices of a triangle to get a triangular $C^{(m)}$-element? (In other words : Is it necessary for constructing a triangular $C^{(m)}$-element to prescribe all derivatives of order less than or equal to $2m$ at the vertices of a triangle?)

(ii) What is the lowest degree of a triangular $C^{(m)}$-element?

The aim of this paper is to prove the following theorem which gives the answers to both questiones (i) and (ii).

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Theorem 1. (i) To get a triangular $C^{(m)}$-element we must prescribe all derivatives of order less than or equal to $2m$ at the vertices of a triangle.

(ii) The lowest degree of a triangular $C^{(m)}$-element is equal to $4m + 1$.

In [4, p. 84] the first part of Theorem 1 is formulated in a little different way with reference to [6]. However, in [6] the features 1 and 2 are mentioned only.

To express ourselves in a concise form we divide the parameters uniquely determining a triangular $C^{(m)}$-element into two groups :

1. The parameters of the first kind guarantee the $C^{(m)}$-continuity of a global function on an arbitrary triangulation. These parameters are prescribed at the vertices of a triangle and at some points lying on the sides of a triangle.

In other words, the parameters of the first kind prescribed at the points of the segment P_rP_s uniquely determine the polynomials

$$(1) \qquad q_{rs,\varkappa}(\tau) = \frac{\partial^\varkappa p}{\partial \nu^\varkappa}\bigg|_l \equiv \frac{\partial^\varkappa}{\partial \nu^\varkappa} p(x_r + (x_s - x_r)\tau, y_r + (y_s - y_r)\tau)$$

where $\varkappa = 0, \dots, m$, $P_r(x_r, y_r)$, $P_s(x_s, y_s)$ are two vertices of a triangle $\bar{T}$, l is the straight line determined by the points P_r, P_s and $p(x, y)$ is a triangular $C^{(m)}$-element on the triangle $\bar{T}$.

2. The parameters of the second kind have no influence on the smoothness of a global function; they enable together with the parameters of the first kind to determine uniquely a triangular $C^{(m)}$-element. These parameters are usually prescribed in the interior T of a triangle $\bar{T}$ but they may be prescribed also at the vertices of a triangle (see, e.g., [5, Corollary of Theorem 3]) or at some points lying on the sides of a triangle.

The basic property of the parameters of the first kind can be expressed also in the following way :

Lemma 1. Let $p(x, y)$ be a triangular $C^{(m)}$-element, P_r, P_s two vertices of the triangle $\bar{T}$ and $l(P_r, P_s)$ the straight line determined by the points P_r, P_s. If all parameters of the first kind prescribed at the points of the segment P_rP_s are equal to zero then

$$(2) \qquad D^\alpha p(P) = 0 \quad , \quad |\alpha| \leqslant m \quad , \quad \forall\, P \in l(P_r, P_s).$$

In (2) and in what follows we use the following notation for derivatives :

$$D^\alpha u = \partial^{|\alpha|} u / \partial x^{\alpha_1} \partial y^{\alpha_2} \quad , \quad \alpha = (\alpha_1, \alpha_2) \quad , \quad |\alpha| = \alpha_1 + \alpha_2 .$$

The proof of Lemma 1 is very simple : If the assumption of Lemma 1 is satisfied then

$$q_{rs,\varkappa}(\tau) \equiv 0 \qquad (\varkappa = 0, \dots, m).$$

These relations imply with respect to (1)

(3) $$\partial^{\varkappa+\lambda}p(P)/\partial\nu^{\varkappa}\partial\tau^{\lambda} = 0\,;\ \varkappa, \lambda = 0, \ldots, m\,;\ \forall\ P \in l(P_r, P_s).$$

As the derivative $\partial^k p/\partial x^{k_1}\partial y^{k_2}(k = k_1 + k_2)$ can be written in the form of a linear combination of $k + 1$ derivatives

$$\partial^k p/\partial\nu^k,\ \partial^k p/\partial\nu^{k-1}\partial\tau, \ldots, \partial^k p/\partial\nu\partial\tau^{k-1},\ \partial^k p/\partial\tau^k$$

the relations (2) follow from (3).

Theorem 1 is in the case $m = 0$ trivial. In the case $m \geqslant 1$ the first part of Theorem 1 is equivalent to the assertion of Lemma 2.

Lemma 2. Let $m \geqslant 1$, $k \geqslant 1$, $l \geqslant 0$ and $\rho \geqslant 0$ be given integers. It is impossible to construct a triangular $C^{(m)}$-element the parameters of the first kind of which prescribed at the vertices P_1, P_2, P_3 of a triangle are of the form

(4) $$D^\alpha p(P_i) \quad , \quad \forall\ |\alpha| \in A \backslash B \qquad (i = 1, 2, 3)$$

where the sets A, B are defined by

(5) $$A = \{0, 1, \ldots, 2m + \rho\},$$

(6) $$B = \{j_1, j_2, \ldots, j_k, h_1, h_2, \ldots, h_l\}$$

and the integers from the set B satisfy the inequalities

(7) $$m < j_1 < j_2 < \ldots < j_k \leqslant 2m < h_1 < h_2 < \ldots < h_l \leqslant 2m + \rho.$$

Before proving Lemma 2 we introduce some lemmas which will be used in the proof of Lemma 2.

Lemma 3. If at every point P of the straight line $l(P_r, P_s)$ determined by the points $P_r(x_r, y_r)$, $P_s(x_s, y_s)$ the relations (2) hold then the polynomial $p(x, y)$ is divisible by the polynomial $[f_{rs}(x, y)]^{m+1}$ where

(8) $$f_{rs}(x, y) = -(y_s - y_r)(x - x_r) + (x_s - x_r)(y - y_r).$$

The proof of Lemma 3 is a modification of one device used in the proof of [2, Theorem 1].

Lemma 4. Let $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$ be the vertices of a triangle $\bar{T}$. Let the polynomial $p(x, y)$ be of the form

(9) $$p(x, y) = g(x, y)q(x, y)$$

where

(10) $$g(x, y) = [f_{12}(x, y)f_{13}(x, y)f_{23}(x, y)]^{m+1},$$

the linear functions $f_{rs}(x, y)$ being defined by the relation (8). Then the conditions

$$D^{\alpha}p(P_i) = 0 \quad , \quad |\alpha| = 2m + \varkappa \quad (\varkappa \geqslant 2) \tag{11}$$

give at most $\varkappa - 1$ linearly independent homogeneous conditions for the polynomial $q(x, y)$ which are prescribed at the vertex P_i.

Proof. We prove Lemma 4 in the case $i = 3$. Let $\bar{T}_0$ be the triangle which lies in the Cartesian co-ordinate system ξ, η and has the vertices $\tilde{P}_1(0, 0)$, $\tilde{P}_2(1, 0)$, $\tilde{P}_3(0, 1)$. The transformation

$$\begin{aligned} x = x_0(\xi, \eta) &\equiv x_3 + (x_1 - x_3)\xi + (x_2 - x_3)\eta, \\ y = y_0(\xi, \eta) &\equiv y_3 + (y_1 - y_3)\xi + (y_2 - y_3)\eta \end{aligned} \tag{12}$$

maps one-to-one the triangle $\bar{T}$ on the triangle $\bar{T}_0$ and the vertex P_3 is mapped on the vertex $\tilde{P}_1$. Let us define the polynomial $\tilde{p}(\xi, \eta)$ by

$$\tilde{p}(\xi, \eta) = p(x_0(\xi, \eta), y_0(\xi, \eta)). \tag{13}$$

According to (9), (10), (12) and (13), the polynomial $\tilde{p}(\xi, \eta)$ is of the form

$$\tilde{p}(\xi, \eta) = \tilde{g}(\xi, \eta)\tilde{q}(\xi, \eta) \tag{14}$$

where

$$\tilde{g}(\xi, \eta) = J^{3m+3}\xi^{m+1}\eta^{m+1}(\xi + \eta - 1)^{m+1}, \tag{15}$$

J being the Jacobian of the transformation (12), and

$$\tilde{q}(\xi, \eta) = q(x_0(\xi, \eta), y_0(\xi, \eta)). \tag{16}$$

It follows from (15) that at the vertex $\tilde{P}_1(0,0)$ the following derivatives of the function $\tilde{g}(\xi, \eta)$ are different from zero only :

$$\frac{\partial^{2m+2+\sigma}\,\tilde{g}(\tilde{P}_1)}{\partial\xi^{m+1+\sigma-\rho}\,\partial\eta^{m+1+\rho}} \quad , \quad \rho = 0, \ldots, \sigma \quad ; \quad \sigma = 0, \ldots, m + 1.$$

This fact and the Leibnitz rule for differentiation of a product imply

$$\frac{\partial^{2m+\varkappa}\tilde{p}(\tilde{P}_1)}{\partial\xi^{\alpha_1}\partial\eta^{\alpha_2}} = 0, \ \alpha_1 + \alpha_2 = 2m + \varkappa, \ \alpha_1 \leqslant m \text{ or } \alpha_2 \leqslant m. \tag{17}$$

Let

$$\xi = \xi_0(x, y), \qquad \eta = \eta_0(x, y) \tag{18}$$

be the inverse transformation to the transformation (12). The polynomial $p(x, y)$ can be written in the form

$$p(x, y) = \tilde{p}(\xi_0(x, y), \eta_0(x, y)). \tag{19}$$

As the transformation (18) is linear we get from (19), according to the rule of differentiation of a composite function,

$$D^\alpha p(P_i) = \sum_{|\beta| = 2m+\varkappa} a_{\alpha\beta} D^\beta \tilde{p}(\tilde{P}_1) \quad , \quad |\alpha| = 2m + \varkappa \tag{20}$$

where $a_{\alpha\beta}$ are constants.

Setting (20) into (11) we get, with respect to (17), $2m + \varkappa + 1$ homogeneous linear equations for at most $\varkappa - 1$ derivatives of order $2m + \varkappa$ of the polynomial $\tilde{p}(\xi, \eta)$ at the point $\tilde{P}_1$. Omitting the linearly dependent equations we get a system of at most $\varkappa - 1$ linearly independent equations. This system is, according to (14) and (15), a system of linear equations for derivatives of the function $\tilde{q}(\xi, \eta)$ at the point $\tilde{P}_1$. Returning to the variables x, y by means of the transformation (18), we get, according to (16), a system of at most $\varkappa - 1$ linearly independent homogeneous equations for the derivatives of the polynomial $q(x, y)$ at the point P_i. Lemma 4 is proved.

Proof of Lemma 2. Lemma 2 will be proved by a contradiction. Let us suppose that the assertion of Lemma 2 is not true, i.e. that it is possible to determine uniquely a triangular $C^{(m)}$-element $p(x, y)$ the parameters of the first kind of which prescribed at the vertices of a triangle are the parameters (4) only. Let n be the degree of this triangular $C^{(m)}$-element. As the triangulation is chosen quite arbitrarily the polynomials $q_{rs,0}(\tau)$ (see (1)) are also polynomials of degree n. Thus it holds, with respect to (5) and (6),

$$n \geqslant 4m + 2\rho - 2k - 2l + 1. \tag{21}$$

Let us set

$$d = n - (4m + 2\rho - 2k - 2l + 1). \tag{22}$$

As the triangulation is quite arbitrary the polynomials $q_{rs,\varkappa}(\tau)$ are polynomials of degree $n - \varkappa$. Thus to achieve the $C^{(m)}$-continuity we must prescribe $d + \varkappa$ parameters of the first kind on each side $P_r P_s$ for each $\varkappa$ ($\varkappa = 0, \ldots, m$). Usually these parameters are of the form

$$\partial^\varkappa p(Q_{rs}^{(\lambda, d+\varkappa)})/\partial \nu_{rs}^\varkappa \ (\lambda = 1, \ldots, d + \varkappa\, ; \varkappa = 0, \ldots, m) \tag{23}$$

where ν_{rs} is the normal to the segment $P_r P_s$ and $Q_{rs}^{(1,q)}, \ldots, Q_{rs}^{(q,q)}$ are the points dividing the segment $P_r P_s$ into $q + 1$ equal parts.

Let the symbols V and S denote the numbers of the parameters of the first kind prescribed at one vertex and on one side, respectively. It follows from (4)-(7), (22) and (23) that the total number of the parameters of the first kind is given by the relation

$$3(V+S) = 3(m+1)n - 9m(m+1)/2 + 3\rho(\rho-1)/2 + 6(m+1)(k+l) - 3(k+l+j+h) \tag{24}$$

where

$$j = j_1 + j_2 + \dots + j_k, \tag{25}$$

$$h = h_1 + h_2 + \dots + h_l. \tag{26}$$

The polynomial $p(x, y)$ has N coefficients where

$$N = (n+1)(n+2)/2. \tag{27}$$

The integers N, S, V must satisfy the inequality

$$R \equiv N - 3(V+S) \geqslant 0 \tag{28}$$

which expresses the fact that the total number of the parameters of the first kind cannot be greater than N.

Let us set

$$G = 48(m+1)(k+l) + 12\rho(\rho-1) - 24(k+l+j+h) + 1. \tag{29}$$

If we put (24) and (27) in (28) we get a quadratic inequality in n. It follows from this inequality that

$$n \geqslant n_1 = (6m + 3 + G^{1/2})/2 \tag{30}$$

where n_1 is the first root of the quadratic polynomial in n on the left-hand side of the inequality (28). The second formal possibility $n \leqslant n_2$ does not suit because in this case, according to (22) and (33),

$$d \leqslant \max d_2 = \max n_2 - (4m + 2\rho - 2k - 2l + 1) < 0.$$

It holds, according to (7), (25) and (26),

$$\max j = 2mk - k(k-1)/2, \tag{31}$$

$$\max h = 2ml + \rho l - l(l-1)/2. \tag{32}$$

Thus

$$\min G = 12k(k+1) + 12(\rho - l - 1)(\rho - l) + 1. \tag{33}$$

As $\rho \geqslant l$, $k \geqslant 1$ the relations (30) and (33) imply

$$n > 3m + 3. \tag{34}$$

The interger R defined by (28) is the number of the parameters of the second kind. Let us prescribe these parameters quite arbitrarily and set all N parameters equal to zero. Then, according to Lemmas 1 and 3, the polynomial $p(x, y)$ is of the form (9). The relations (10) and (34) imply that in this case the polynomial $q(x, y)$ is at least a polynomial of the first degree. Let the symbol M denote the total number of the coefficients of the polynomial $q(x, y)$. It is easy to find that

$$M = N - 3(m + 1)n + 9m(m + 1)/2. \tag{35}$$

The relations (24), (28) and (35) imply

$$M - R = 6(m + 1)(k + l) - 3(k + l + j + h) + 3\rho(\rho - 1)/2. \tag{36}$$

It holds with respect to (31) and (36)

$$M - R \geqslant Q \tag{37}$$

where

$$Q = 3k(k + 1)/2 + 6(m + 1)l - 3(l + h) + 3\rho(\rho - 1)/2. \tag{38}$$

Each integer h_s can be expressed in the form

$$h_s = 2m + r_s \ (s = 1, \dots, l). \tag{39}$$

Using (26) and (39) we can write

$$h = 2ml + (r_1 + \dots + r_l). \tag{40}$$

Putting (40) in (38) we find

$$Q = 3k(k + 1)/2 + H \tag{41}$$

where

$$H = 3l - 3(r_1 + \dots + r_l) + 3\rho(\rho - 1)/2. \tag{42}$$

According to (5)-(7), (39) and Lemma 4, the conditions

$$D^\alpha p(P_i) = 0 \quad , \quad |\alpha| \geqslant 2m + 2 \quad , \quad |\alpha| \in A \backslash B \ (i = 1, 2, 3) \tag{43}$$

give H_1 linearly independent homogeneous conditions for the polynomial $q(x, y)$ where

$$H_1 \leqslant 3\Big(1 + 2 + \dots + (r_1 - 2) + \sum_{s=1}^{l-1} [r_s + (r_s + 1) + \dots + (r_{s+1} - 2)] + r_l + (r_l + 1) + \dots + \rho - 1\Big). \tag{44}$$

The right-hand side of the inequality (44) is equal to H. Thus

$$H_1 \leqslant H. \tag{45}$$

As, according to (8) and (10), the relations

$$D^\alpha g(P) = 0 \quad , \quad |\alpha| \leqslant m \quad , \quad \forall\, P \in \partial T = \bar{T} \backslash T,$$

$$D^\alpha g(P_i) = 0 \quad , \quad |\alpha| \leqslant 2m + 1 \quad (i = 1, 2, 3)$$

hold the parameters of the first kind except for the parameters (43) give no conditions for the polynomial $q(x, y)$.

The parameters of the second kind prescribed for the polynomial $p(x, y)$ give R_1 linearly independent homogeneous conditions for the polynomial $q(x, y)$ where

$$R_1 \leqslant R. \tag{46}$$

Thus we get $H_1 + R_1$ linearly independent homogeneous equations for the coefficients of the polynomial $q(x, y)$.

As it holds, according to (37), (41), (45) and (46),

$$M - R_1 - H_1 \geqslant 3k(k + 1)/2 > 0 \tag{47}$$

we can complete these $H_1 + R_1$ homogeneous equations by such $M - R_1 - H_1$ non-homogeneous equations that we get M linearly independent equations for M coefficients of a polynomial $q(x, y)$ for which it holds

$$q(x, y) \not\equiv 0. \tag{48}$$

According to (9), (10) and (48), we get a polynomial $p(x, y)$ which satisfies prescribed N homogeneous conditions and is not identically equal to zero. This is a contradiction. Lemma 2 is proved.

The proof of the second part of Theorem 1 is now now very simple : It follows from the first part of Theorem 1 that the lowest degree of a triangular $C^{(m)}$-element is greater than or equal to $4m + 1$. This fact and the result of [5] prove the second part of Theorem 1.

The assertion of the following theorem is well-known [2, 5] :

Theorem 2. A triangular $C^{(m)}$-element of degree $4m + 1$ can be uniquely determined by the parameters

$$D^\alpha p(P_i) \quad , \quad |\alpha| \leqslant 2m \quad (i = 1, 2, 3) \tag{49}$$

$$\partial^\varkappa p(Q_{rs}^{(\lambda,\varkappa)})/\partial v_{rs}^\varkappa \quad , \quad r = 1, 2\,, s = 2, 3\ (r < s) \tag{50}$$

$$\lambda = 1, \dots, \varkappa\,;\ \varkappa = 0, \dots, m$$

$$D^\alpha p(P_0) \quad , \quad |\alpha| \leqslant m - 2 \tag{51}$$

where P_0 is the centre of gravity of the triangle $\bar{T}$ and the meaning of other symbols is the same as in the preceding text.

Generalizing Bell's device [1], the number of independent parameters can be reduced by imposing on $p(x, y)$ the condition that the derivatives $\partial^\varkappa p/\partial \nu^\varkappa$ be polynomials of degree $n - 2\varkappa$ along the corresponding sides of the triangle. Then the parameters (50) prescribed on the side $P_r P_s$ are linear combinations of the parameters (49) prescribed at the vertices P_r, P_s.

Setting $k = 0$ in the proof of Lemma 2 we get no contradiction. This suggests to construct triangular $C^{(m)}$-elements with $\rho > 0$ and $l > 0$. However, these polynomials are not useful for applications because their degrees are too high. Only one exception can be mentioned : A triangular $C^{(0)}$-element of the fourth degree can be determined by the parameters

$$(52) \qquad D^\alpha p(P_i) \quad , \quad |\alpha| = 0,2 \quad ; \quad p(Q_i) \quad (i = 1, 2, 3)$$

where Q_1, Q_2, Q_3 are the mid-points of the sides of a triangle. This element can be used when we do not need the first derivatives and want to get from some reasons continuous second derivatives at the nodal points of a triangulation.

REMARK. A family of triangular $C^{(m)}$-elements with arbitrary $\rho > 0$ and $l = 0$ is studied in [3].

REFERENCES

[1] BELL K., *A refined triangular plate bending finite element.* Int. J. Num. Meth. Engng., 1969, *1*, 101-122.

[2] BRAMBLE J. H. and ZLÁMAL M., *Triangular elements in the finite element method.* Math. Comp., 1970, *24*, 809-820.

[3] KOUKAL S., *Piecewise polynomial interpolations in the finite element method.* Apl. Mat., 1973, *18*, 146-160.

[4] STRANG G. and FIX G., *An analysis of the finite element method.* Prentice-Hall Inc., Englewood Cliffs, N. J., 1973.

[5] ŽENÍŠEK A., *Interpolation polynomials on the triangle.* Numer. Math., 1970, *15*, 283-296.

[6] ŽENÍŠEK A., *Polynomial approximation on tetrahedrons in the finite element method.* J. Approx. Theory, 1973, *7*, 334-351.