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A general theorem on triangular finite $C^{(m)}$-elements


<http://www.numdam.org/item?id=M2AN_1974__8_2_119_0>
A GENERAL THEOREM
ON TRIANGULAR FINITE $C^{(m)}$-ELEMENTS

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Summary. — The following theorem is proved: To achieve piecewise polynomials of class $C^m$ on an arbitrary triangulation of a polygonal domain, the nodal parameters must include all derivatives of order less than or equal to $2m$ at the vertices of the triangles.

For the sake of brevity we shall use the expression «triangular $C^{(m)}$ element» for a polynomial on a triangle which generates piecewise polynomial and $m$-times continuously differentiable functions on an arbitrary triangulation. (From this point of view the Clough-Tocher element [4, p. 84] is not a triangular $C^{(1)}$-element.)

In the last few years there were constructed various types of interpolation polynomials on a triangle (see e.g., [3, 5]). All these polynomials have two following features:

1. A general triangular $C^{(m)}$-element is constructed in such a way that at the vertices of a triangle there are prescribed at least all derivatives of order less than or equal to $2m$.

2. The lowest degree of a general triangular $C^{(m)}$-element is equal to $4m + 1$.

These two features suggest the following questiones:

(i) Which derivatives should be prescribed at the vertices of a triangle to get a triangular $C^{(m)}$-element? (In other words: Is it necessary for constructing a triangular $C^{(m)}$-element to prescribe all derivatives of order less than or equal to $2m$ at the vertices of a triangle?)

(ii) What is the lowest degree of a triangular $C^{(m)}$-element?

The aim of this paper is to prove the following theorem which gives the answers to both questiones (i) and (ii).

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Revue Française d'Automatique, Informatique et Recherche Opérationnelle n° août 1974, R-2.
Theorem 1. (i) To get a triangular $C^m$-element we must prescribe all
derivatives of order less than or equal to $2m$ at the vertices of a triangle.

(ii) The lowest degree of a triangular $C^m$-element is equal to $4m + 1$.

In [4, p. 84] the first part of Theorem 1 is formulated in a little different
way with reference to [6]. However, in [6] the features 1 and 2 are mentioned
only.

To express ourselves in a concise form we divide the parameters uniquely
determining a triangular $C^m$-element into two groups:

1. The parameters of the first kind guarantee the $C^m$-continuity of a
global function on an arbitrary triangulation. These parameters are prescribed
at the vertices of a triangle and at some points lying on the sides of a triangle.

In other words, the parameters of the first kind prescribed at the points
of the segment $P_rP_s$ uniquely determine the polynomials

$$q_{rs,\kappa}(\tau) = \frac{\partial^\nu p}{\partial\nu^\nu}(x_r + (x_s - x_r)\tau, y_r + (y_s - y_r)\tau)$$

where $\kappa = 0, ..., m$, $P_r(x_r, y_r)$, $P_s(x_s, y_s)$ are two vertices of a triangle $\bar{T}$, $l$ is
the straight line determined by the points $P_r$, $P_s$ and $p(x, y)$ is a triangular
$C^m$-element on the triangle $\bar{T}$.

2. The parameters of the second kind have no influence on the smoothness
of a global function; they enable together with the parameters of the first
kind to determine uniquely a triangular $C^m$-element. These parameters are
usually prescribed in the interior $T$ of a triangle $\bar{T}$ but they may be prescribed
also at the vertices of a triangle (see, e.g., [5, Corollary of Theorem 3]) or at
some points lying on the sides of a triangle.

The basic property of the parameters of the first kind can be expressed
also in the following way:

Lemma 1. Let $p(x, y)$ be a triangular $C^m$-element, $P_r$, $P_s$ two vertices of
the triangle $\bar{T}$ and $l(P_r, P_s)$ the straight line determined by the points $P_r$, $P_s$.
If all parameters of the first kind prescribed at the points of the segment $P_rP_s$
are equal to zero then

$$D^\alpha p(P) = 0, \quad |\alpha| \leq m, \quad \forall P \in l(P_r, P_s).$$

In (2) and in what follows we use the following notation for derivatives:

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x^{\alpha_1} \partial y^{\alpha_2}}, \quad \alpha = (\alpha_1, \alpha_2), \quad |\alpha| = \alpha_1 + \alpha_2.$$

The proof of Lemma 1 is very simple: If the assumption of Lemma 1 is
satisfied then

$$q_{rs,\kappa}(\tau) \equiv 0 \quad (\kappa = 0, ..., m).$$
These relations imply with respect to (1)

(3) \[ x^+\lambda p(P)/\partial^x\partial^\lambda = 0; \lambda = 0, \ldots, m; \forall P \in l(P_r, P_s). \]

As the derivative \( \partial^k p/\partial x^k \partial y^k (k = k_1 + k_2) \) can be written in the form of a linear combination of \( k + 1 \) derivatives

\[
\partial^p p/\partial^k, \partial^2 p/\partial^k \partial^1, \ldots, \partial^k p/\partial^k \partial^1
\]

the relations (2) follow from (3).

Theorem 1 is in the case \( m = 0 \) trivial. In the case \( m \geq 1 \) the first part of Theorem 1 is equivalent to the assertion of Lemma 2.

Lemma 2. Let \( m \geq 1, k \geq 1, l \geq 0 \) and \( \rho \geq 0 \) be given integers. It is impossible to construct a triangular \( C^{(m)} - \) element the parameters of the first kind of which prescribed at the vertices \( P_1, P_2, P_3 \) of a triangle are of the form

(4) \[ D^\alpha p(P_i), \quad \forall |\alpha| \in A \setminus B \quad (i = 1, 2, 3) \]

where the sets \( A, B \) are defined by

(5) \[ A = \{ 0, 1, \ldots, 2m + \rho \}, \]

(6) \[ B = \{ j_1, j_2, \ldots, j_k, h_1, h_2, \ldots, h_t \} \]

and the integers from the set \( B \) satisfy the inequalities

(7) \[ m < j_1 < j_2 < \ldots < j_k \leq 2m < h_1 < h_2 < \ldots < h_t \leq 2m + \rho. \]

Before proving Lemma 2 we introduce some lemmas which will be used in the proof of Lemma 2.

Lemma 3. If at every point \( P \) of the straight line \( l(P_r, P_s) \) determined by the points \( P_r(x_r, y_r), P_s(x_s, y_s) \) the relations (2) hold then the polynomial \( p(x, y) \) is divisible by the polynomial \([f_{rs}(x, y)]^{m+1}\) where

(8) \[ f_{rs}(x, y) = -(y_s - y_r)(x - x_r) + (x_s - x_r)(y - y_r). \]

The proof of Lemma 3 is a modification of one device used in the proof of [2, Theorem 1].

Lemma 4. Let \( P_1(x_1, y_1), P_2(x_2, y_2), P_3(x_3, y_3) \) be the vertices of a triangle \( \overline{T} \). Let the polynomial \( p(x, y) \) be of the form

(9) \[ p(x, y) = g(x, y)q(x, y) \]

where

(10) \[ g(x, y) = [f_{12}(x, y)f_{13}(x, y)f_{23}(x, y)]^{m+1}, \]
the linear functions \( f_{rs}(x, y) \) being defined by the relation (8). Then the conditions

\[
D^a p(P_i) = 0, \quad |a| = 2m + x \quad (x \geq 2)
\]
give at most \( x - 1 \) linearly independent homogeneous conditions for the polynomial \( q(x, y) \) which are prescribed at the vertex \( P_i \).

**Proof.** We prove Lemma 4 in the case \( i = 3 \). Let \( \bar{T}_0 \) be the triangle which lies in the Cartesian co-ordinate system \( \xi, \eta \) and has the vertices \( \bar{P}_1(0, 0), \bar{P}_2(1, 0), \bar{P}_3(0, 1) \). The transformation

\[
x = x_0(\xi, \eta) \equiv x_3 + (x_1 - x_3)\xi + (x_2 - x_3)\eta,
\]

(12)

\[
y = y_0(\xi, \eta) \equiv y_3 + (y_1 - y_3)\xi + (y_2 - y_3)\eta
\]
maps one-to-one the triangle \( \bar{T} \) on the triangle \( \bar{T}_0 \) and the vertex \( P_3 \) is mapped on the vertex \( \bar{P}_1 \). Let us define the polynomial \( \tilde{p}(\xi, \eta) \) by

\[
\tilde{p}(\xi, \eta) = p(x_0(\xi, \eta), y_0(\xi, \eta)).
\]

According to (9), (10), (12) and (13), the polynomial \( \tilde{p}(\xi, \eta) \) is of the form

\[
\tilde{p}(\xi, \eta) = \tilde{g}(\xi, \eta)\tilde{q}(\xi, \eta)
\]
where

\[
\tilde{g}(\xi, \eta) = J^{3m+3}x^{m+1}\eta^{m+1}(\xi + \eta - 1)^{m+1},
\]

\( J \) being the Jacobian of the transformation (12), and

\[
\tilde{q}(\xi, \eta) = q(x_0(\xi, \eta), y_0(\xi, \eta))
\]

It follows from (15) that at the vertex \( \bar{P}_1(0, 0) \) the following derivatives of the function \( \tilde{g}(\xi, \eta) \) are different from zero only:

\[
\frac{\partial^{2m+2+\sigma} \tilde{g}(\bar{P}_1)}{\partial \xi^{m+1+\sigma} \partial \eta^{m+1+\rho}}, \quad \rho = 0, ..., \sigma; \quad \sigma = 0, ..., m + 1.
\]

This fact and the Leibnitz rule for differentiation of a product imply

\[
\frac{\partial^{2m+x} \tilde{p}(\bar{P}_1)}{\partial \xi^{\alpha_1} \partial \eta^{\alpha_2}} = 0, \quad \alpha_1 + \alpha_2 = 2m + x, \quad \alpha_1 \leq m \text{ or } \alpha_2 \leq m.
\]

Let

\[
\xi = \xi_0(x, y), \quad \eta = \eta_0(x, y)
\]

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be the inverse transformation to the transformation (12). The polynomial $p(x, y)$ can be written in the form
\begin{equation}
(19) \quad p(x, y) = \tilde{p}(\xi_0(x, y), \eta_0(x, y)).
\end{equation}
As the transformation (18) is linear we get from (19), according to the rule of differentiation of a composite function,
\begin{equation}
(20) \quad D^\alpha p(P_i) = \sum_{|\beta| = 2m + x} a_{\alpha\beta} D^\beta p(P_i), \quad |\alpha| = 2m + x
\end{equation}
where $a_{\alpha\beta}$ are constants.

Setting (20) into (11) we get, with respect to (17), $2m + x + 1$ homogeneous linear equations for at most $x - 1$ derivatives of order $2m + x$ of the polynomial $\tilde{p}(\xi, \eta)$ at the point $\tilde{P}_1$. Omitting the linearly dependent equations we get a system of at most $x - 1$ linearly independent equations. This system is, according to (14) and (15), a system of linear equations for derivatives of the function $\tilde{q}(\xi, \eta)$ at the point $\tilde{P}_1$. Returning to the variables $x, y$ by means of the transformation (18), we get, according to (16), a system of at most $x - 1$ linearly independent homogeneous equations for the derivatives of the polynomial $q(x, y)$ at the point $P_i$. Lemma 4 is proved.

**Proof of Lemma 2.** Lemma 2 will be proved by a contradiction. Let us suppose that the assertion of Lemma 2 is not true, i.e. that it is possible to determine uniquely a triangular $C^{(m)}$-element $p(x, y)$ the parameters of the first kind of which prescribed at the vertices of a triangle are the parameters (4) only. Let $n$ be the degree of this triangular $C^{(m)}$-element. As the triangulation is chosen quite arbitrarily the polynomials $q_{rs}(x, y)$ are polynomials of degree $n$. Thus it holds, with respect to (5) and (6),
\begin{equation}
(21) \quad n \geq 4m + 2\rho - 2k - 2l + 1.
\end{equation}

Let us set
\begin{equation}
(22) \quad d = n - (4m + 2\rho - 2k - 2l + 1).
\end{equation}

As the triangulation is quite arbitrary the polynomials $q_{rs}(\tau)$ are polynomials of degree $n - \tau$. Thus to achieve the $C^{(m)}$-continuity we must prescribe $d + \tau$ parameters of the first kind on each side $P_rP_s$ for each $\tau$ ($\tau = 0, \ldots, m$). Usually these parameters are of the form
\begin{equation}
(23) \quad \frac{\partial^{d+\tau} p(Q_{rs}^{(\lambda, d+\tau)})}{\partial v_{rs}^{\lambda}} (\lambda = 1, \ldots, d + \tau; \tau = 0, \ldots, m)
\end{equation}
where $v_{rs}$ is the normal to the segment $P_rP_s$ and $Q_{rs}^{(1,\rho)}, \ldots, Q_{rs}^{(q,a)}$ are the points dividing the segment $P_rP_s$ into $q + 1$ equal parts.

Let the symbols $V$ and $S$ denote the numbers of the parameters of the first kind prescribed at one vertex and on one side, respectively. It follows
from (4)-(7), (22) and (23) that the total number of the parameters of the first kind is given by the relation

\[(24)\quad 3(V + S) = 3(m + 1)n - 9m(m + 1)/2 + 3\rho(\rho - 1)/2 + 6(m + 1)(k + l) - 3(k + l + j + h)\]

where

\[(25)\quad j = j_1 + j_2 + ... + j_k,\]
\[(26)\quad h = h_1 + h_2 + ... + h_i.\]

The polynomial \(p(x, y)\) has \(N\) coefficients where

\[(27)\quad N = (n + 1)(n + 2)/2.\]

The integers \(N, S, V\) must satisfy the inequality

\[(28)\quad R = N - 3(V + S) \geq 0\]

which expresses the fact that the total number of the parameters of the first kind cannot be greater than \(N\).

Let us set

\[(29)\quad G = 48(m + 1)(k + l) + 12\rho(\rho - 1) - 24(k + l + j + h) + 1.\]

If we put (24) and (27) in (28) we get a quadratic inequality in \(n\). It follows from this inequality that

\[(30)\quad n \geq n_1 = (6m + 3 + G^{1/2})/2\]

where \(n_1\) is the first root of the quadratic polynomial in \(n\) on the left-hand side of the inequality (28). The second formal possibility \(n \leq n_2\) does not suit because in this case, according to (22) and (33),

\[d \leq \max d_2 = \max n_2 - (4m + 2\rho - 2k - 2l + 1) < 0.\]

It holds, according to (7), (25) and (26),

\[(31)\quad \max j = 2mk - k(k - 1)/2,\]
\[(32)\quad \max h = 2ml + \rho l - l(l - 1)/2.\]

Thus

\[(33)\quad \min G = 12k(k + 1) + 12(\rho - l - 1)(\rho - l) + 1.\]

As \(\rho \geq l, k \geq 1\) the relations (30) and (33) imply

\[(34)\quad n > 3m + 3.\]
The integer $R$ defined by (28) is the number of the parameters of the second kind. Let us prescribe these parameters quite arbitrarily and set all $N$ parameters equal to zero. Then, according to Lemmas 1 and 3, the polynomial $p(x, y)$ is of the form (9). The relations (10) and (34) imply that in this case the polynomial $q(x, y)$ is at least a polynomial of the first degree. Let the symbol $M$ denote the total number of the coefficients of the polynomial $q(x, y)$. It is easy to find that

$$M = N - 3(m + 1)n + 9m(m + 1)/2.$$  

The relations (24), (28) and (35) imply

$$M - R = 6(m + 1)(k + l) - 3(k + l + j + h) + 3\rho(\rho - 1)/2.$$  

It holds with respect to (31) and (36)

$$M - R > Q$$  

where

$$Q = 3k(k + 1)/2 + 6(m + 1)l - 3(l + h) + 3\rho(\rho - 1)/2.$$  

Each integer $h_s$ can be expressed in the form

$$h_s = 2m + r_s (s = 1, ..., l).$$  

Using (26) and (39) we can write

$$h = 2ml + (r_1 + ... + r_l).$$  

Putting (40) in (38) we find

$$Q = 3k(k + 1)/2 + H$$  

where

$$H = 3l - 3(r_1 + ... + r_l) + 3\rho(\rho - 1)/2.$$  

According to (5)-(7), (39) and Lemma 4, the conditions

$$D^\alpha p(P_i) = 0, \ |\alpha| > 2m + 2, \ |\alpha| \in A \setminus B (i = 1, 2, 3)$$  

give $H_1$ linearly independent homogeneous conditions for the polynomial $q(x, y)$ where

$$H_1 \leq 3 \left(1 + 2 + ... + (r_1 - 2) + \sum_{s=1}^{l-1} [r_s + (r_s + 1) + ... + (r_{s+1} - 2)] + r_1 + (r_1 + 1) + ... + \rho - 1 \right).$$  

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The right-hand side of the inequality (44) is equal to $H$. Thus

(45) \[ H_1 \leq H. \]

As, according to (8) and (10), the relations

\[ D^x g(P) = 0, \quad |x| \leq m, \quad \forall P \in \partial T = \bar{T} \setminus T, \]
\[ D^x g(P_i) = 0, \quad |x| \leq 2m + 1 \quad (i = 1, 2, 3) \]

hold the parameters of the first kind except for the parameters (43) give no conditions for the polynomial $q(x, y)$.

The parameters of the second kind prescribed for the polynomial $p(x, y)$ give $R_1$ linearly independent homogeneous conditions for the polynomial $q(x, y)$ where

(46) \[ R_1 \leq R. \]

Thus we get $H_1 + R_1$ linearly independent homogeneous equations for the coefficients of the polynomial $q(x, y)$.

As it holds, according to (37), (41), (45) and (46),

(47) \[ M - R_1 - H_1 \geq 3k(k + 1)/2 > 0 \]

we can complete these $H_1 + R_1$ homogeneous equations by such $M - R_1 - H_1$ non-homogeneous equations that we get $M$ linearly independent equations for $M$ coefficients of a polynomial $q(x, y)$ for which it holds

(48) \[ q(x, y) \neq 0. \]

According to (9), (10) and (48), we get a polynomial $p(x, y)$ which satisfies prescribed $N$ homogeneous conditions and is not identically equal to zero. This is a contradiction. Lemma 2 is proved.

The proof of the second part of Theorem 1 is now now very simple: It follows from the first part of Theorem 1 that the lowest degree of a triangular $C^{(m)}$-element is greater than or equal to $4m + 1$. This fact and the result of [5] prove the second part of Theorem 1.

The assertion of the following theorem is well-known [2, 5]:

**Theorem 2.** A triangular $C^{(m)}$-element of degree $4m + 1$ can be uniquely determined by the parameters

(49) \[ D^x p(P_i), \quad |x| \leq 2m \quad (i = 1, 2, 3) \]
(50) \[ \partial^x p(Q_{rs}^{(r,s)})/\partial r^x, \quad r = 1, 2, s = 2, 3 \quad (r < s) \]
\[ \lambda = 1, ..., x; x = 0, ..., m \]
(51) \[ D^x p(P_0), \quad |x| \leq m - 2 \]
where $P_0$ is the centre of gravity of the triangle $\bar{T}$ and the meaning of other symbols is the same as in the preceding text.

Generalizing Bell's device [1], the number of independent parameters can be reduced by imposing on $p(x, y)$ the condition that the derivatives $\partial^n p/\partial y^x$ be polynomials of degree $n-2x$ along the corresponding sides of the triangle. Then the parameters (50) prescribed on the side $P_r P_s$ are linear combinations of the parameters (49) prescribed at the vertices $P, P_s$.

Setting $k = 0$ in the proof of Lemma 2 we get no contradiction. This suggests to construct triangular $C^{(m)}$-elements with $\rho > 0$ and $l > 0$. However, these polynomials are not useful for applications because their degrees are too high. Only one exception can be mentioned: A triangular $C^{(0)}$-element of the fourth degree can be determined by the parameters

\[ (52) \quad D^\alpha p(P_i) \quad , \quad |\alpha| = 0,2 \quad ; \quad p(Q_i) \quad (i = 1, 2, 3) \]

where $Q_1, Q_2, Q_3$ are the mid-points of the sides of a triangle. This element can be used when we do not need the first derivatives and want to get from some reasons continuous second derivatives at the nodal points of a triangulation.

**Remark.** A family of triangular $C^{(m)}$-elements with arbitrary $\rho > 0$ and $l = 0$ is studied in [3].

**REFERENCES**


