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A DISSIPATIVE GALERKIN METHOD APPLIED TO SOME QUASILINEAR HYPERBOLIC EQUATIONS

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Abstract. — *A nonstandard continuous-in-time Galerkin method, based on piecewise polynomial spaces, is applied to the periodic initial value problem for the equation*

$$u_t = a(x, t, u)u_x + f(x, t, u).$$

Under the condition that $a(x, t, u) \geqslant a_0 > 0$ *for the solution, optimal order* L^2 *error estimates are derived.*

1. INTRODUCTION

Consider the periodic initial value problem for $u = u(x, t)$,

$$(1.1) \qquad \begin{cases} u_t = a(x, t, u)u_x + f(x, t, u) \quad , \quad t > 0, x \in R, \\ u(x, 0) = v_0(x). \end{cases}$$

Here $a(\cdot, t, u), f(\cdot, t, u), u(\cdot, t)$ and $v_0(\cdot)$ are periodic of period 1. It is assumed that a sufficiently smooth solution exists, see Theorem 1.1, and furthermore that with a positive constant a_0,

$$(1.2) \qquad a(x, t, u(x, t)) \geqslant a_0 > 0$$

for this solution.

For the numerical treatment of the problem (1.1) we use certain piecewise polynomial spaces. Let μ and k be integers, $\mu - 1 > k \geqslant 0$, and let $\{ i \cdot h \}$, $i = 0, ..., n = h^{-1} \in Z$ be a uniform partition of $I = [0, 1]$ depending on the parameter h. Let (supressing k and h in the notation).

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$S^\mu = \{ \chi(x), x \in I$: the periodic extension of χ lies in $C^k(R)$ and $\chi \mid_{(ih, (i+1)h)}$ is a polynomial of degree $\leqslant \mu - 1$, $i = 0, ..., h^{-1} - 1 \}$.

Given $T > 0$ we define a continuous-in-time Galerkin approximation $U(x, t)$ to the solution of (1.1) as a differentiable map

$$U : [0, T] \to S^\mu$$

such that

$$(1.3) \begin{cases} (U_t, \chi - h\chi_x) = (a(x, t, U)U_x + f(x, t, U), \chi - h\chi_x), \chi \in S^\mu \quad , \quad t > 0, \\ U(0) \text{ given in } S^\mu. \end{cases}$$

Here $(g_1, g_2) = \int_I g_1(x) g_2(x) \, dx$.

For a and f continuously differentiable with respect to all arguments, U is well defined in a neighborhood of $t = 0$, and it will be established in Section 3 that given any compact subset $[0, T]$ of the lifespan of the solution to (1.1) where the solution is sufficiently smooth, U is defined on $[0, T]$ for h sufficiently small.

The result of this paper is the following asymptotic error estimate, where the notation will be defined in Section 2.

Theorem 1.1.

Assume that

$$(1.4) \qquad a \in C^1(R^3), \quad f \in C^1(R^3), \quad v_0 \in H^\mu.$$

Let $T > 0$ be given and assume that a solution $u(x, t)$ to (1.1) exists for $t \in [0, T]$ such that

$$(1.5) \qquad u_t \in L^2(0, T; H^\mu)$$

and

$$(1.2) \qquad a(x, t, u(x, t)) \geqslant a_0 > 0, x \in R, t \in [0, T].$$

Assume furthermore that $U(0)$ is given such that

$$(1.6) \qquad \| U(0) - v_0 \|_{L^2(I)} \leqslant C_1 h^\mu.$$

Then there exists a constant C, independent of C_1 and h, such that for h sufficiently small,

$$\| U - u \|_{L^\infty(0,T;L^2)} \leqslant C h^\mu (C_1 + 1).$$

We note that the conditions $v_0 \in H^\mu$ and (1.5) imply that $u \in L^\infty(0, T; H^\mu)$. Since $\mu \geqslant 2$ it is also easy to see that the conditions (1.4) and (1.5) guarantee that the solution to (1.1) is unique.

For the actual numerical solution of (1.1) a discretization of the time variable has to be made in (1.3); this is left to the reader. One point of the method (1.3) is that « dissipation » is introduced already at the level of the continuous-in-time method. The terminology « dissipation » here is connected with the usual sense of the word in finite difference theory in the following way, see Wahlbin [10] for details : When applying the method (1.3), or the ordinary Galerkin method, to the simple equation $u_t = u_x$ and with S^{μ} the space of smoothest splines of order $\mu (k = \mu - 2)$, then both methods can be interpreted, cf. Thomée [9], as semidiscrete finite difference methods involving the values of the approximate solution at certain meshpoints. For the ordinary Galerkin method the corresponding finite difference operator has order of accuracy 2μ and no dissipation, whereas the method (1.3) leads to a finite difference operator which is accurate of order $2\mu - 1$ and dissipative of order 2μ. We note that the result for the method (1.3) corresponds to the Kreiss stability condition, see Richtmyer and Morton [8, Section 5.4]. The results that the order of accuracy is 2μ, resp. $2\mu - 1$, imply superconvergence at meshpoints.

Theorem 1.1 shows that the method (1.3) leads to L^2-optimal error for all the spaces S^{μ} under the condition (1.6) on initial data. For the ordinary Galerkin method, L^2-optimality can be expected e.g. when S^{μ} is a space of smoothest splines, see Fix and Nassif [6], whereas, as was shown in Dupont [4], if one employs the space of Hermite cubics ($\mu = 4$, $k = 1$) the error is not optimal in L^2 but one loses one power of h in accuracy. Dendy [2] has introduced a method similar to ours-his method gives L^2 optimal error estimates if $\| U(0) - W_0 \|_{H^1} \leqslant C_2 h^{\mu}$, where W_0 is the elliptic projection of v_0 into S^{μ}.

The idea employed in this paper of comparing the Galerkin solution to a certain projection into S^{μ} of the solution to (1.1) was originated in Wheeler [11] for parabolic problems, and has been used for hyperbolic problems in e.g. Dendy [2] and Dupont [5]. A slightly new twist is required here in that the projection depends on the mesh parameter h. For this reason we give the details of Section 2.

The author thanks T. Dupont for pointing out simplifications in the analysis.

2. PRELIMINARIES

We first introduce some notation. The functions considered are real valued, and C will denote a generic constant. For J an interval, let

$$\| v \|_{L^2(J)} = \left(\int_J v^2(x) \, dx \right)^{1/2},$$

and for r an integer let

$$\| v \|_{H^r(J)} = \left(\sum_{i=0}^{r} \left\| \frac{d^i v}{dx^i} \right\|^2_{L^2(J)} \right)^{1/2}.$$

Let $I = [0, 1]$. When no confusion can arise, we write $\|v\|$ for $\|v\|_{L^2(I)}$ and $\|v\|_r$ for $\|v\|_{H^r(I)}$.

Let $H^r_{\text{loc}}(R)$ denote the set of functions such that $\|\cdot\|_{H^r(J)} < \infty$ for any compact J. Since we shall always consider functions that are 1-periodic in x, we shall say that a function belongs to H^r if it belongs to $H^r_{\text{loc}}(R)$.

Let $C^r(R)$ denote the set of functions with r continuous derivatives, and set

$$\|v\|_{C^r} = \sum_{i=0}^{r} \sup_{x \in R} \left| \frac{d^i v(x)}{dx^i} \right|.$$

Note that

(2.1) $$\|v\|_{C^0} \leqslant 2^{1/2} \|v\|_1.$$

For a function $v(x, t)$ and $B = L^2(I), H^r(I)$ or $C^r(R)$, we let

$$\|v\|_{L^p(0,T;B)} = \| \|v(\cdot, t)\|_B \|_{L^p(0,T)}.$$

We say that a function $v(x, t)$ which is one-periodic in x lies in $L^p(0, T; B)$ if $v(\cdot, t) \in B$ a.e. in t and the relevant norm is finite.

The following special notation will be employed :

(2.2) $$\tilde{v} = v - hv_x.$$

Our first lemma summarizes some well known properties of the spaces S^μ :

Lemma 2.1

(i) There exists a constant C such that given r, $1 < r \leqslant \mu$, and v a 1-periodic function in $H^r_{\text{loc}}(R)$, there exists $\chi \in S^\mu$ such that

$$\|v - \chi\| + h \|v - \chi\|_1 \leqslant Ch^r \|v\|_r.$$

(ii) Inverse property. There exists a constant C such that for any $\chi \in S^\mu$,

$$\|\chi\|_1 \leqslant Ch^{-1} \|\chi\|.$$

(iii) Given $y(x)$ 1-periodic and in $C^1(R)$ there exists a constant C, depending on $\|y\|_{C^1}$, such that for any $\chi \in S^\mu$ there exists $\psi \in S^\mu$ such that

$$\|y\chi - \psi\| + h \|y\chi - \psi\|_1 \leqslant Ch \|\chi\|.$$

For a proof of (iii) see the proof of Lemma 3.2 in Douglas, Dupont and Wahlbin [3].

In the remainder of this section we shall consider the projection operator taking the solution $u = u(x, t)$ of (1.1) into $Z(x, t)$, where $Z(x, t) \in S^\mu$ and is defined for each fixed t by

(2.3) $$\begin{cases} z = u - Z, \\ (z, \chi) - (z_x, \tilde{\chi}) = 0 \quad , \quad \chi \in S^\mu. \end{cases}$$

We have the following estimates for z :

Lemma 2.2

For any $l = 0, 1, \ldots$ there exists a constant C such that for each fixed t,

$$\left\|\frac{\partial^l z}{\partial t^l}\right\| + h\left\|\frac{\partial^l z}{\partial t^l}\right\|_1 \leqslant Ch^{\mu}\left\|\frac{\partial^l z}{\partial t^l}\right\|_{\mu}. \tag{2.4}$$

Proof : It is sufficient to prove (2.4) for $l = 0$, since $\partial^l z/\partial t^l$ satisfies

$$\left(\frac{\partial^l z}{\partial t^l}, \chi\right) - \left(\left(\frac{\partial^l z}{\partial t^l}\right)_x, \tilde{\chi}\right) = 0.$$

For any $\chi \in S^{\mu}$ we have

$$\|z\|^2 + h\,\|z_x\|^2 = (z, z) - (z_x, \tilde{z}) = (z, z - \chi) - (z_x, \tilde{z} - \tilde{\chi}).$$

Now, $z - \chi = u - (Z + \chi)$. If we use Lemma 2.1 (i) to choose $Z + \chi$, then

$$\|z\|^2 + h\,\|z_x\|^2 \leqslant \tfrac{1}{2}\,\|z\|^2 + \tfrac{1}{2}h\,\|z_x\|^2 + Ch^{2\mu-1}\,\|u\|_{\mu}^2.$$

Hence,

$$\|z_x\| \leqslant Ch^{\mu-1}\,\|u\|_{\mu} \quad , \quad \|z\| \leqslant Ch^{\mu-1/2}\,\|u\| . \tag{2.5}$$

It remains to use duality, Nitsche [7]. We can solve the periodic equation

$$y + y_x - hy_{xx} = z$$

by a Fourier series expansion, and then

$$\|y\|_1 + h\,\|y\|_2 \leqslant C\,\|z\|. \tag{2.6}$$

It follows that for any $\chi \in S^{\mu}$,

$$\|z\|^2 = (z, y) - (z_x, \tilde{y}) = (z, y - \chi) - (z_x, \tilde{y} - \tilde{\chi}).$$

Thus, if χ is choosen via Lemma 2.1 (i), then (2.5) and (2.6) imply that

$$\|z\|^2 \leqslant Ch^{\mu}\,\|u\|_{\mu}\,\|z\|.$$

This completes the proof of Lemma 2.2.

For the next lemma, assume that the hypotheses of Theorem 1.1 hold.

Lemma 2.3

There exists a constant $C = C(t)$, depending on $\|a(x, t, u(x, t))\|_{C^1}$, such that for each fixed t and $\chi \in S^{\mu}$,

$$|(a(x, t, u)z_x, \tilde{\chi})| \leqslant Ch^{\mu}\,\|u\|_{\mu}\,\|\chi\|. \tag{2.7}$$

Proof. Let a denote $a(x, t, u)$ and a_x the total derivative with respect to x. By (2.3) we have for any χ, ψ in S^μ,

$$(az_x, \tilde{\chi}) = (z_x, a\chi - ha\chi_x) = (z_x, \tilde{a\chi}) + h(z_x, a_x\chi)$$
$$= (z_x, \tilde{a\chi} - \tilde{\psi}) + (z, \psi) + h(z_x, a_x\chi)$$

Using Lemma 2.1 (iii) to chose ψ we obtain

$$|(az_x, \tilde{\chi})| \leqslant \|z_x\|\ Ch\,\|\chi\| + \|z\|\ \|\psi\| + h\,\|z_x\|\ C\,\|\chi\|.$$

By Lemma 2.2 this implies the desired result (2.7).

3. PROOF OF THE MAIN RESULT

Recall that $z = u - Z$ was defined by (2.3), and let

$$\mathcal{V} = U - Z. \tag{3.1}$$

Theorem 1.1 will follow from the following lemma. Here $\|\cdot\|_\infty$ denotes the maximum norm without requirements on continuity.

Lemma 3.1

Let the hypotheses of Theorem 1.1 hold, with $T > 0$ given. Assume that a solution to (1.3) exists and that

$$\|\mathcal{V}\|_\infty + \|\mathcal{V}_x\|_\infty \leqslant 1 \quad , \quad t \in [0, T]. \tag{3.2}$$

Then there exists a constant C depending on

$$\|u\|_{L^2(0,T;H^\mu)} \quad , \quad \|v_0\|_\mu \quad , \quad \|u_t\|_{L^2(0,T;H^\mu)} \quad ,$$
$$\|a(x, t, u(x, t))\|_{L^\infty(0,T;C^1)} \quad , \quad \|f(x, t, u(x, t))\|_{L^\infty(0,T;C^1)}$$

and a_0 such that

$$\|\mathcal{V}\|(t) \leqslant Ch^\mu(C_1 + 1), t \in [0, T]. \tag{3.3}$$

Before giving the proof of this lemma, we show how Theorem 1.1 follows from it.

Proof of Theorem 1.1. First note that since $\mu \geqslant 2$, the hypotheses of the theorem and (2.1) imply that the constant C in (3.3) is uniformly bounded for fixed T. Next note that the additional hypothesis (3.2) holds for a short time interval, by the hypothesis on initial data and Lemma 2.2. Also, note that the system (1.3) can be viewed as a system of ordinary differential equations for the coefficients of U in a basis for S^μ, Hence, see e.g. Birkhoff and Rota

[1, § 6.11], if the solution U ceases to exist at a point t_1, then $\|U\|_\infty$ tends to infinity as t tends to t_1. Now, by Lemma 2.2 and the inverse property,

$$(3.4) \qquad \|Z\|_\infty + \|Z_x\|_\infty \leqslant \|u\|_{L^\infty(0,T;C^1)} + Ch^{\mu-3/2}\|u\|_{L^\infty(0,T;H^\mu)} \leqslant C$$

and hence if $t_1 \in [0, T]$ the condition (3.2) must be violated. Thus existence of the Galerkin solution U follows if we can establish the a priori inequality (3.2).

Assume now that (3.2) fails for some $t \in [0, T]$ and let

$$t_0 = \inf\{\, t \in [0, T] : \qquad (3.2) \text{ fails }\}.$$

Then $t_0 > 0$. Since the inverse property implies that

$$\|\mathcal{V}\|_\infty + h\|\mathcal{V}_x\|_\infty \leqslant Ch^{-1/2}\|\mathcal{V}\|,$$

Lemma 3.1 with T replaced by t_0 establishes a contradiction for h sufficiently small.

Hence the Galerkin solution U exists on $[0, T]$ and the result (3.3) is valid. Since

$$\|U - u\|_{L^\infty(0,T;L^2)} \leqslant \|\mathcal{V}\|_{L^\infty(0,T;L^2)} + \|z\|_{L_\infty(0,T;L^2)},$$

the theorem follows from (3.3) and Lemma 2.2.

It remains to prove Lemma 3.1.

Proof of lemma 3.1. We first introduce some abbreviated notation. Let

$$\begin{aligned}
a(q) &= a(q)(x, t) = a(x, t, q(x, t)), \quad q = u, U \text{ or } Z,\\
f(q) &= f(q)(x, t) = f(x, t, q(x, t)), \quad q = u, U \text{ or } Z,\\
b(z) &= a(u) - a(Z),\\
b(\mathcal{V}) &= a(U) - a(Z),\\
g(z) &= f(u) - f(Z),\\
g(\mathcal{V}) &= f(U) - f(Z).
\end{aligned}$$

The notation $a(q)_x$ will mean the total derivative.

For simplicity we do not write out all dependence in the constants occuring, but leave it to the reader to trace it.

By the hypothesis (3.2) and by (3.4) we may assume that a and f are uniformly Lipschitz continuous in the variable q. Thus

$$(3.5) \qquad \|b(z)\| \leqslant C\|z\|,$$

and analogous inequalities hold for $b(\mathcal{V})$, $g(z)$ and $g(\mathcal{V})$. Also, (3.2) and (3.4) imply that

$$(3.6) \qquad \|U_x\|_\infty \leqslant C.$$

A simple calculation establishes that

$$(3.7) \qquad (\mathcal{V}_t, \tilde{\chi})$$
$$= (a(u)\mathcal{V}_x - a(u)z_x + (b(\mathcal{V}) - b(z))U_x + g(\mathcal{V}) - g(z) + z_t, \tilde{\chi}).$$

Take $\chi = \mathcal{V}$ in (3.7). The fact that

$$(a(u)\mathcal{V}_x, \mathcal{V}) = -\tfrac{1}{2}(a(u)_x\mathcal{V}, \mathcal{V}),$$

Lemma 2.1 (ii), Lemma 2.2, Lemma 2.3, (3.5) and (3.6) imply that

$$(3.8) \qquad \tfrac{1}{2}\frac{d}{dt}\|\mathcal{V}\|^2 - h(\mathcal{V}_t, \mathcal{V}_x) + h(a(u)\mathcal{V}_x, \mathcal{V}_x)$$
$$\leqslant C\{\|\mathcal{V}\|^2 + h^{2\mu}(\|u\|_\mu^2 + \|u_t\|_\mu^2)\}.$$

Next take χ in (3.7) to be

$$\chi = \frac{\mathcal{V}_t}{a(u)} + R,$$

where, by Lemma 2.1 (iii), we may assume that

$$\|R\| + h\|R\|_1 \leqslant Ch\|\mathcal{V}_t\|.$$

We obtain in the same way as above the inequality

$$(3.9) \qquad \tfrac{1}{2}h\frac{d}{dt}\|\mathcal{V}_x\|^2 + \left(\frac{\mathcal{V}_t}{a(u)}, \mathcal{V}_t\right) - (\mathcal{V}_x, \mathcal{V}_t)$$
$$\leqslant C\{h\|\mathcal{V}_t\|^2 + h\|\mathcal{V}_x\|^2 + h^{-1}\|\mathcal{V}\|^2 + h^{2\mu-1}(\|u\|_\mu^2 + \|u_t\|_\mu^2)\}.$$

From (3.9) it follows in particular that for h small enough,

$$\|\mathcal{V}_t\|^2 \leqslant C\left\{-h\frac{d}{dt}\|\mathcal{V}_x\|^2 + \|\mathcal{V}_x\|^2 + h^{-1}\|\mathcal{V}\|^2 \right.$$
$$\left. + h^{2\mu-1}(\|u\|_\mu^2 + \|u_t\|_\mu^2)\right\}$$

If this result is inserted on the right hand side of (3.9), we get

$$(\tfrac{1}{2}h + Ch^2)\frac{d}{dt}\|\mathcal{V}_x\|^2 + \left(\frac{\mathcal{V}_t}{a(u)}, \mathcal{V}_t\right) - (\mathcal{V}_x, \mathcal{V}_t)$$
$$\leqslant C\{h\|\mathcal{V}_x\|^2 + h^{-1}\|\mathcal{V}\|^2 + h^{2\mu-1}(\|u\|_\mu^2 + \|u_t\|_\mu^2)\}.$$

If this relation is multiplied by h and added to (3.8), we obtain

$$(3.10) \qquad \tfrac{1}{2}\frac{d}{dt}(\|\mathcal{V}\|^2 + (h^2 + Ch^3)\|\mathcal{V}_x\|^2)$$
$$+ h\left[(a(u)\mathcal{V}_x, \mathcal{V}_x) + \left(\frac{\mathcal{V}_t}{a(u)}, \mathcal{V}_t\right) - 2(\mathcal{V}_x, \mathcal{V}_t)\right]$$
$$\leqslant C\{\|\mathcal{V}\|^2 + h^2\|\mathcal{V}_x\|^2 + h^{2\mu}(\|u\|_\mu^2 + \|u_t\|_\mu^2)\}.$$

Note that the term in square brackets is non negative.

At $t = 0$ the hypothesis (1.6), Lemma 2.2 and the triangle inequality imply that

$$\|\mathcal{V}\|^2(0) + (h^2 + Ch^3)\,\|\mathcal{V}_x\|^2(0) \leqslant C\,\|\mathcal{V}\|^2(0) \leqslant Ch^{2\mu}(C_1 + \|v_0\|_\mu^2).$$

We obtain

$$\|\mathcal{V}\|^2(t) \leqslant Ch^{2\mu}(C_1 + \|v_0\|_\mu^2) + Ch^{2\mu}\int_0^t (\|u\|_\mu^2 + \|u_t\|_\mu^2)\,d\tau$$

from (3.10) and Gronwall's lemma. This proves the desired result (3.3).

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