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B. D. SHARMA
Y. D. MATHUR
J. MITTER

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BOUNDS ON THE RATE-DISTORTION FUNCTION FOR GEOMETRIC MEASURE OF DISTORTION

by B. D. SHARMA, Y. D. MATHUR and J. MITTER (1)

Abstract. — *Earlier the authors have defined the Geometric Measure of Distortion ${}_\alpha D_G$ where $\alpha(> 0)$ stands for the cost for distorsion per letter for correct transmission. In this paper we calculate the Rate Distortion Function $R({}_\alpha D_G^*)$. In Section 3, the Symmetric Measure of Distortion is defined and bounds are obtained on $R({}_\alpha D_G^*)$ and ${}_\alpha D_G^*$.*

1. INTRODUCTION

In a communication process, let $\left\{ x_i \right\}_{i=0}^{N-1}$ be the set of symbols transmitted and $\left\{ Y_j \right\}_{j=0}^{m-1}$ be the set of symbols received such that for correct transmission x_i corresponds to y_i for every i. For an independent letter source, we shall denote by p_i, the probability of transmitting x_i; and by $q_{j/i}$, the probability of receiving y_j when x_i is sent. The average mutual information is given by

$$I(P;Q) = \sum_{i=0}^{N-1} \sum_{j=0}^{M-1} p_i \cdot q_{j/i} \log \left(\frac{q_{j/i}}{\sum_l p_l \cdot q_{j/l}} \right) \tag{1.1}$$

For convenience, the logarithms are considered to the base e. For a transmission with a fidelity criterion [3], the authors [4] have introduced the geometric measure of distortion given by

$${}_\alpha D_G = \prod_{i,j} \rho_{ij}^{p_i \cdot q_{j/i}}, \tag{1.2}$$

where ρ_{ij} is the distortion (cost) of transmitting x_j and receiving y_j so that

$$\rho_{ij} > \alpha \quad \text{if} \quad i \neq j \quad \text{and} \quad \rho_{ii} = \alpha \quad \text{where} \quad \alpha > 0 \tag{1.3}$$

(1) Faculty of Mathematics University of Delhi (India).

The rate distortion function of the source relative to the given distortion measure is then defined as

$$R\,({}_{\alpha}D_G^*) = \min I\,(P\,;\,Q), \tag{1.4}$$

where the minimization is done with respect to $q_{j/i}$ under the condition that

$${}_{\alpha}D_G \leqslant {}_{\alpha}D_G^*. \tag{1.5}$$

Gallager [2]; Berger [1] and others have investigated noisy channel coding theorems with the Shannon's measure of distortion geven by

$$D_S = \sum_i \sum_j p_i \cdot q_{j/i} \cdot d_{ij}, \tag{1.6}$$

in which $\quad d_{ij} > 0 \quad$ if $\quad i \neq j \quad$ and $\quad d_{ii} = 0 \quad$ (1.7)

In this paper, we shall investigate the values of $R({}_{\alpha}D_G^*)$ and prove theorems on the symmetric measure of distortion with the geometric fidelity criterion.

It is rather obvious that $R({}_{\alpha}D_G^*)$ is non negative and a non increasing function of ${}_{\alpha}D_G^*$ for minimization in (1.4) is done over a constraint set which is enlarged as ${}_{\alpha}D_G^*$ is increased.

2. CALCULATION OF $R({}_{\alpha}D_G^*)$

Theorem 2.1 The set $\{ q_{j/i} \}$ which gives $R({}_{\alpha}D_G^*)$ i.e. min $I(P;\,Q)$ subject to the constraint ${}_{\alpha}D_G \leqslant {}_{\alpha}D_G^*$ is given by

$$q_{j/i} = \frac{q_j \cdot c_i}{p_i} \cdot \rho_{ij}^{-\lambda \cdot {}_{\alpha}D_G} \quad \text{for all } i, j \tag{2.1}$$

where $\quad \sum_i c_i \rho_{ij}^{-\lambda_{\alpha} D_G} = 1 \quad$ for all j and $\quad q_j = \sum_i p_i \cdot q_{j/i}$. (2.2)

Proof : We have to minimize (1.1) under the conditions

$${}_{\alpha}D_G = \exp\left(\sum_i \sum_j p_i \cdot q_{j/i} \cdot \log \rho_{ij}\right) \leqslant {}_{\alpha}D_G^*$$

and $\sum_j q_{j/i} = 1$ for all i.

Consider the function

$$\Phi = I(P\,;\,Q) + \lambda \cdot {}_{\alpha}D_G + \sum_i \mu_i \cdot \sum_j q_{j/i} \tag{2. 3}$$

where λ and μ_i are Lagrange's constants.

For a suitable choice let $\mu_i = -p_i \log \dfrac{c_i}{p_i}$. (2.4)

Replacing the set $\mu = \left\{ \mu_i \right\}_{i=0}^{N-1}$ by $c = \left\{ c_i \right\}_{i=0}^{N-1}$, (2.3) becomes

$$\Phi = \sum_{i=0}^{N-1} \sum_{j=0}^{M-1} p_i \cdot q_{j/i} \left(\log \frac{q_{j/i}}{\sum_l p_l \cdot q_{j/l}} - \log \frac{c_i}{p_i} \right) + \lambda \cdot \exp \left(\sum_i \sum_j p_{ij} \cdot \log \rho_{ij} \right) \tag{2. 5}$$

Thus the condition for $q_{j/i}$ to yield a stationary point for Φ is

$$\log \frac{q_{j/i}}{q_j} + \lambda \cdot \exp \left(\sum_i \sum_j p_i q_{j/i} \log \rho_{ij} \right) \log \rho_{ij} - \log \frac{c_i}{p_i} = 0 \tag{2. 6}$$

for every i and j

where

$$q_j = \sum_i p_i \cdot q_{j/i} \tag{2.7}$$

Next (2.6) gives

$$q_{j/i} = \frac{c_i}{p_i} \cdot q_j \cdot \rho_{ij}^{-\lambda \cdot {}_aD_G}. \tag{2. 8}$$

Multiphying (2.8) by p_i and summing over i, we get

$$\sum_i c_i \cdot \rho_{ij}^{-\lambda \cdot {}_aD_G} = 1 \qquad \text{for all } j. \tag{2. 9}$$

Again summing up (2.8) over j and using the constraint $\sum_j q_{j/i} = 1$ for every i, we obtain

$$\frac{c_i}{p_i} \cdot \sum_j q_j \cdot \rho_{ij}^{-\lambda \cdot {}_aD_G} = 1 \qquad \text{for all } i. \tag{2. 10}$$

From (2.9) we get a set of M-linear equations in the unknowns c_i and another set of M-linear equations in q_j obtained from (2.10). If $N = M$, we can usually solve the equations and then find $q_{j/i}$ from (2.8). Since $I(P\,;Q)$ is convex ∪ in Q, Φ is also convex ∪ and therefore the solution is a minimum. |

The above approach does not take into account the non-negativity of quantities $q_{j/i}$ and the resulting values of $q_{j/i}$, giving minimum of $I(P\,;Q)$ may become negative, leading to a non-feasible solution.

In the next theorem we follow an approach which always gives a feasible solution.

Now we define a function

$$\psi = \sum_i \sum_j p_i \cdot q_{j/i} \left[\log \frac{q_{j/i}}{\sum_l p \cdot q_{j/l}} \right] + \lambda \cdot {}_aD_G \tag{2. 11}$$

where $q_{j/j} > 0$.

It would be noted that since ${}_{\alpha}D_G \leqslant {}_{\alpha}D_G^*$

$$\min_{q_{j/i}} \psi - \lambda \cdot {}_{\alpha}D_G^* \leqslant R({}_{\alpha}D_G^*). \tag{2.12}$$

Theorem 2.2 For any $\lambda > 0$,

$$\min_{q_{j/i}} \psi = H(U) + \max_{c} \sum_{i=0}^{N-1} p_i \cdot \log c_i \, ; c_i > 0, \tag{2.13}$$

where $H(U)$ is the entropy of the source and $C = \left\{ c_i \right\}_{i=0}^{N-1}$ is such that

$$\sum_{i=0}^{N-1} c_i \cdot \rho_{ij}^{-\lambda \cdot {}_{\alpha}D_G} \leqslant 1 \qquad \text{where} \qquad \rho_{ij} \geqslant e. \tag{2.14}$$

Also ψ is minimized for values of c_i given by (2.8) in terms of $q_{j/i}$ and the necessary and sufficient conditions on c_i to achieve the maximum in (2.13) are that there exists an output distribution satisfying (2.10) and (2.14) with equality.

Proof : Consider the function

$$\Phi = \sum_i \sum_j p_i \cdot q_{j/i} \cdot \log \frac{q_{j/i}}{q_j} + \lambda \cdot {}_{\alpha}D_G - \sum_i p_i \log \frac{c_i}{p_i} \sum_j q_{j/i} \tag{2.15}$$

then

$$\Phi = \psi - H(U) - \sum_i p_i \cdot \log c_i \tag{2.16}$$

(2.15) can be put as

$$-\Phi = \sum_i \sum_j p_i q_{j/i} \log \frac{q_j \cdot c_i}{q_{j/i} \cdot p_i} + \sum_i \sum_j p_i \cdot q_{j/i} \cdot \log \rho_{ij}^{-\lambda \cdot {}_{\alpha}D_G} \cdot \log_{\rho_{ij}} e$$

$$\leqslant \sum_i \sum_j p_i \cdot q_{j/i} \log \frac{q_j \cdot c_i}{q_{j/i} \cdot p_i} + \sum_i \sum_j p_i \cdot q_{j/i} \cdot \log \rho_{ij}^{-\lambda \cdot {}_{\alpha}D_G}$$

as $\rho_{ij} \geqslant e$.

Using the inequality $\log x \leqslant x - 1$, we obtain

$$-\Phi \leqslant \sum_i \sum_j p_i q_{j/i} \left[\frac{q_j \cdot c_i \cdot \rho_{ij}^{-\lambda \cdot {}_{\alpha}D_G}}{q_{j/i} \cdot p_i} - 1 \right]$$

$$= \sum_i \sum_j q_j \cdot c_i \cdot \rho_{ij}^{-\lambda \cdot {}_{\alpha}D_G} - \sum_j q_j$$

$$\leqslant \sum_j q_j - \sum_j q_j = 0 \tag{2.17}$$

(using (2.14))

Combining (2.17) and (2.16), we get

$$\psi \geqslant H(U) + \sum_i p_i \log c_i \tag{2.18}$$

(2.18) is satisfied with equality if and only if the inequalities $\log x \leqslant x - 1$ and (2.14) are satisfied with equality, or if and only if

$$\frac{q_j \cdot c_i \cdot \rho_{ij}^{-\lambda \cdot {}_\alpha D_G}}{q_{j/i} \cdot p_i} = 1 \qquad \text{for all} \qquad q_{j/i} > 0 \tag{2.19}$$

and

$$\sum_i c_i \rho_{ij}^{-\lambda \cdot {}_\alpha D_G} = 1 \qquad \text{for all} \qquad q_j > 0 \tag{2.20}$$

The conditions in the theorem are necessary for equality in (2.18) as we obtain (2.10) from (2.19) after multiplying by $q_{j/i}$ and summing over j. Again if the output probabilities satisfy (2.10) and if (2.20) is satisfied then as already seen $q_{j/i}$ given by (2.8) is a transition assignment with output probabilities q_j. By (2.10), the choice satisfies (2.19) so that the conditions of the theorem are sufficient for equality in (2.18).

3. SYMMETRIC MEASURE OF DISTORTION

If the number of input and output symbols are same and if the cost of correct transmission is α and the cost of any incorrect transmission is β (obviously $\alpha < \beta$) so that the distortion is

$$\rho_{ij} = \begin{cases} \alpha & \text{if} \quad i = j \\ \beta & \text{if} \quad i \neq j \end{cases} \tag{3.1}$$

then we refer to this as Symmetric Measure of Distortion.

Theorem 3.1. Under symmetric measure of Distortion, we have

$$R({}_\alpha D_G^*) \geqslant H(\mathrm{U}) - \hat{H}\left(\frac{{}_\alpha D_G^* - \alpha \log \alpha}{\beta \log \beta - \alpha \log \alpha}\right) - \left(\frac{{}_\alpha D_G^* - \alpha \log \alpha}{\beta \log \beta - \alpha \log \alpha}\right) \log(N - 1) \tag{3.2}$$

where $$\hat{H}\left(\frac{{}_\alpha D_G^* - \alpha \log \alpha}{\beta \log \beta - \alpha \log \alpha}\right) = -\left(\frac{{}_\alpha D_G^* - \alpha \log \alpha}{\beta \log \beta - \alpha \log \alpha}\right) \log\left(\frac{{}_\alpha D_G^* - \alpha \log \alpha}{\beta \log \beta - \alpha \log \alpha}\right) - \left(1 - \frac{{}_\alpha D_G^* - \alpha \log \alpha}{\beta \log \beta - \alpha \log \alpha}\right) \log\left(1 - \frac{{}_\alpha D_G^* - \alpha \log \alpha}{\beta \log \beta - \alpha \log \alpha}\right) \tag{3.3}$$

with equality if

$${}_\alpha D_G^* \leqslant \alpha \log \alpha + (\beta \log \beta - \alpha \log \alpha)(N - 1)\, p_{\min}.$$

where $p_{\min}$ is the minimum of all p_i's.

Proof : The constraint equations (2.14) under symmetric measure of distortion take the form

$$c_j \cdot \alpha^{-\lambda\alpha} + \left(\sum_{i=0}^{N-1} c_i - c_j\right) \beta^{-\lambda\beta} \leqslant 1 \tag{3.5}$$
$$0 \leqslant j \leqslant M-1.$$

These are all symmetric and can be made to hold with equality by taking $c_i = c_0$ for each i. Then,

$$c_0 = \alpha^{\lambda\alpha} \cdot [1 + (N-1) \cdot \alpha^{\lambda\alpha} \beta^{-\lambda\beta}]^{-1} \tag{3.6}$$

From (2.13) and (3.6), we have

$$\min_{q_{j/i}} \psi \geqslant H(U) + \lambda \cdot \alpha \log \alpha - \log\ [1 + (N-1)\, \alpha^{\lambda\alpha} \cdot \beta^{-\lambda\beta}] \tag{3.7}$$

Invoking the relation (2.12) we get for all $\lambda > 0$,

$$R({}_\alpha D_G^*) \geqslant -\lambda \cdot {}_\alpha D_G^* + H(U) + \lambda \cdot \alpha \log \alpha - \log [1 + (N-1)\, \alpha^{\lambda\alpha} \cdot \beta^{-\lambda\beta}]. \tag{3.8}$$

Now if we maximize the right hand side with respect to λ, we get

$${}_\alpha D_G^* = \alpha \log \alpha + \frac{(\beta \log \beta - \alpha \log \alpha)(N-1)}{\beta^{\lambda\beta} \cdot \alpha^{-\lambda\alpha} + (N-1)} \tag{3.9}$$

therefore

$$\lambda = \frac{1}{\beta \log \beta - \alpha \log \alpha} \log \left(\frac{\beta \log \beta - {}_\alpha D_G^*}{{}_\alpha D_G^* - \alpha \log \alpha}\right)(N-1). \tag{3.10}$$

(3.2) follows by substituting (3.10) into (3.8).

Now by theorem 2.2 (3.7) would hold with equality if we can find a solution of (2.10) such that $q_j \geqslant 0$. Under the symmetric measure of distortion defined by (3.1), (2.10) gives

$$q_j = \frac{(p_{i/ci})\, \alpha^{\lambda\alpha} \cdot \beta^{\lambda\beta} - \alpha^{\lambda\alpha}}{\beta^{\lambda\beta} - \alpha^{\lambda\alpha}} \tag{3.11}$$

$$= \frac{p_i[\beta^{\lambda\beta} + (N-1)\, \alpha^{\lambda\alpha}] - \alpha^{\lambda\alpha}}{\beta^{\lambda\beta} - \alpha^{\lambda\alpha}} \tag{3.12}$$

for values of $c_i = c_0$ given in (3.6).

All q_j's will be non negative if

$$p_i \geqslant \frac{1}{\beta^{\lambda\beta} \cdot \alpha^{-\lambda\alpha} + (N-1)} \qquad \text{for every } i \tag{3.13}$$

If λ is sufficiently large (3.13) holds normally and (3.7) would hold with equality.

Now combining (3.9) and (3.13), we get

$$R({}_\alpha D_G^*) = H(U) - \hat{H}\left(\frac{{}_\alpha D_G^* - \alpha \log \alpha}{\beta \log \beta - \alpha \log \alpha}\right) - \left(\frac{{}_\alpha D_G^* - \alpha \log \alpha}{\beta \log \beta - \alpha \log \alpha}\right) \log (N - 1)$$

for

$${}_\alpha D_G^* \leqslant \alpha \log \alpha + (\beta \log \beta - \alpha \log \alpha)(N - 1) p_{\min}.$$

Hence the theorem. |

An Extension of Theorem 3.1

We shall now calculate $R({}_\alpha D_G^*)$ for large values of ${}_\alpha D_G^*$ Without any loss of generality we can assume that the source letters are ordered in decreasing order of probabilities that is

$$p_0 \geqslant p_1 \geqslant \dots \geqslant p_{N-1}. \tag{3.14}$$

Next suppose that there is an integer m, $0 < m < N - 1$ such that

$$q_j \begin{cases} = 0 & \text{if} \quad j \geqslant m \\ > 0 & \text{if} \quad j \leqslant m - 1. \end{cases} \tag{3.15}$$

For $j \leqslant m$, (3.11) then gives

$$p_i = c_i \, \beta^{-\lambda\beta}. \tag{3.16}$$

(3.5) must be satisfied with equality for $j \leqslant m - 1$, therefore for all $j \leqslant m - 1$, all the c_j must be the same say c_0 and $c_j \leqslant c_0$ for $j \geqslant m$.

The constraint equations (2.14) for $j = 0$ gives

$$c_0 \, \alpha^{-\lambda\alpha} + \left(\sum_{i=0}^{m-1} c_i + \sum_{i=m}^{N-1} c_i - c_0\right) \cdot \beta^{-\lambda\beta} = 1,$$

$$\text{or} \quad c_0 \, \alpha^{-\lambda\alpha} + (m c_0 - c_0) \, \beta^{-\lambda\beta} + \sum_{i=m}^{N-1} c_i \cdot \beta^{-\lambda\beta} = 1, \tag{3.17}$$

$$\text{or} \quad c_0 \left[\alpha^{-\lambda\alpha} + (m - 1) \, \beta^{-\lambda\beta}\right] = \sum_{i=0}^{m-1} p_i = \sigma_m (say)$$

(using (3.16))

$$\text{or} \quad c_i = c_0 = \frac{\sigma_m \cdot \alpha^{\lambda\alpha}}{1 + (m - 1) \, \alpha^{\lambda\alpha} \cdot \beta^{-\lambda\beta}} \tag{3.18}$$

It is clear from (3.16) that $c_m \geqslant c_{m+1} \geqslant \dots \geqslant c_{N-1}$ and for $j \geqslant m$; $c_j \leqslant c_0$ will hold if

$$p_m \leqslant \frac{\sigma_m \cdot \alpha^{\lambda\alpha} \cdot \beta^{-\lambda\beta}}{1 + (m-1)\,\alpha^{\lambda\alpha} \cdot \beta^{-\lambda\beta}} \tag{3.19}$$

Now $\sum_i p_i \log c_i$ will be maximized for c given by (3.16) and (3.18), if all the q_j' s given by (3.11) are non negative. This requires from (3.11) that

$$p_{m-1} \geqslant \frac{\sigma_m \cdot \alpha^{\lambda\alpha} \cdot \beta^{-\lambda\beta}}{1 + (m-1)\,\alpha^{\lambda\alpha} \cdot \beta^{-\lambda\beta}} \tag{3.20}$$

since from (3.11) and (3.14) it is obvious that

$$q_0 \geqslant q_1 \geqslant \dots \geqslant q_{m-1}.$$

Thus for the values of λ for which (3.19) and (3.20) are satisfied, the given c yields

$$\min_{q_{j/i}} \psi = H(U) + \sum_{i=0}^{m-1} p_i \log \frac{\sigma_m \cdot \alpha^{\lambda\alpha}}{1 + (m-1)\,\alpha^{\lambda\alpha} \cdot \beta^{-\lambda\beta}} + \sum_{i=m}^{N-1} p_i \log(p_i \cdot \beta^{\lambda\beta}). \tag{3.21}$$

The min ψ over a range of λ specifies $R({}_\alpha D_G^*)$ over the corresponding range of λ. The parameter λ is related to ${}_\alpha D_G^*$ by

$${}_\alpha D_G^* = \frac{\partial}{\partial\lambda}[\min \psi] = \sigma_m \left[\frac{\alpha \log \alpha + (\beta \log \beta)(m-1)\,\alpha^{\lambda\alpha}\,\beta^{-\lambda\beta}}{1 + (m-1)\,\alpha^{\lambda\alpha} \cdot \beta^{-\lambda\beta}}\right] + (\beta \log \beta)(1 - \sigma_m). \tag{3.22}$$

Therefore

$$\lambda = \log \left[\frac{(m-1)(\beta \log \beta - {}_\alpha D_G^*)}{{}_\alpha D_G^* - \beta \log \beta + (\beta \log \beta - \alpha \log \alpha)\sigma_m}\right]^{1/(\beta\log\beta - \alpha\log\alpha)} \tag{3.23}$$

For λ and ${}_\alpha D_G^*$ related by (3.22).

$$R({}_\alpha D_G^*) = \min_{q_{j/i}} \psi - \lambda \cdot {}_\alpha D_G^* \tag{3.24}$$

using (3.21) and (3.23); simplifying and rearranging the terms, (3.24) becomes

$$R({}_\alpha D_G^*) = \sigma_m \Bigg[H(U_m) + \left\{ \frac{{}_\alpha D_G^* - \beta \log \beta + (\beta \log \beta - \alpha \log \alpha)\sigma_m}{(\beta \log \beta - \alpha \log \alpha)\sigma_m} \right\} \times \log \left\{ \frac{{}_\alpha D_G^* - \beta \log \beta + (\beta \log \beta - \alpha \log \alpha)\sigma_m}{(\beta \log \beta - \alpha \log \alpha)\sigma_m} \right\}$$

$$+ \left\{ \frac{\beta \log \beta - {}_{\alpha}D_G^*}{(\beta \log \beta - \alpha \log \alpha)\sigma_m} \right\} \log \left\{ \frac{\beta \log \beta - {}_{\alpha}D_G}{(\beta \log \beta - \alpha \log \alpha)\sigma_m} \right\}$$

$$\left. - \left\{ \frac{{}_{\alpha}D_G^* - \beta \log \beta + (\beta \log \beta - \alpha \log \alpha)\sigma_m}{(\beta \log \beta - \alpha \log \alpha)\sigma_m} \right\} \log(m-1) \right]$$

This can be equivalently expressed as

$$R({}_{\alpha}D_G^*) = \sigma_m[H(U_m) - \hat{H}(\Delta) - \Delta \log(m-1)]$$

where $H(U_m)$ is the entropy of a reduced ensemble with probabilities

$$p_0/\sigma_m \quad , \quad p_1/\sigma_m, \ldots, p_{m-1}/\sigma_m,$$

$$\Delta = \frac{{}_{\alpha}D_G^* - \beta \log \beta + (\beta \log \beta - \alpha \log \alpha)\sigma_m}{(\beta \log \beta - \alpha \log \alpha)\sigma_m}$$

and

$$\hat{H}(\Delta) = -\Delta \log \Delta - (1-\Delta) \log (1-\Delta).$$

Substituting (3.23) into (3.19) and (3.20) we obtain the bounds of ${}_{\alpha}D_G^*$ given by

$$(\beta \log \beta - \alpha \log \alpha \left(mp_m - \sum_{i=0}^{m} p_i \right) + \beta \log \beta \leqslant {}_{\alpha}D_G^* \leqslant (\beta \log \beta - \alpha \log \alpha) \times \left[(m-1)p_{m-1} - \sum_{i=0}^{m-1} p_i \right] + \beta \log \beta.$$

When $m = N - 1$

$$(\beta \log \beta - \alpha \log \alpha)\left(mp_m - \sum_{i=0}^{m} p_i \right) = \alpha \log \alpha + (\beta \log \beta - \alpha \log \alpha)(N-1)p_{\min}.$$

which is the same as upper limt in (3.4). |

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APPENDIX

Shannon introduced ρ_{ij} as the single letter distortion when x_i is sent and y_j is received. As there is always some cost even for correct transmission, we take $\rho_{ij} > \alpha$ for $i \neq j$; $\alpha > 0$ and $\rho_{ij} = \alpha$ (where α is zero in Shannon's case). Since any measure of distortion is an average of per letter distorsions ρ_{ij}'s, the measure in its most-generalized form is taken as

$$ {}_{\alpha}D_{\psi}^{f} = \psi^{-1}\left(\frac{\sum_i \sum_j f(p_{ij})\,\psi(\rho_{ij})}{\sum_i \sum_j f(p_{ij})}\right) $$

where (i) ψ is strictly monotonic and continuous function defined for non negative values.

and (ii) f is positive valued and bounded weight function in [0, 1]

By setting $f(x) = x$ and $\psi(x) = \log x$ in (A) we get

$$ {}_{\alpha}D_G = \exp\left(\sum_i \sum_j p_{ij} \cdot \log \rho_{ij}\right) = \prod_{i,j} \rho_{ij}^{p_i q_j / t} \qquad \text{where} \qquad \sum_i \sum_j p_{ij} = 1 $$

(*) For relevant matter of [4] refer to Appendix.