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On the index of degeneracy of a CM abelian variety

par HIROMICHI YANAI

Abstract. We consider a degenerate abelian variety \( A \) of CM type. Then there exists \( m > 0 \) such that the ring of Hodge cycles on \( A^m \) is not generated by the divisor classes. We call the minimum of such \( m \) the index of degeneracy of \( A \).

In this paper, we determine the index of degeneracy for a certain type of CM abelian varieties. This supplements a former result of H. W. Lenstra, Jr.

1. Introduction

For an abelian variety \( A \) defined over the complex number field \( \mathbb{C} \), we call the elements of \( H^{2p}(A, \mathbb{Q}) \cap H^{p,p} \) the Hodge cycles (of codimension \( p \)), where \( H^{p,q} = H^q(A, \Omega^p) \). It is conjectured that the Hodge cycles are algebraic (this is the Hodge conjecture; for a survey see [1]). Since the divisor classes (the Hodge cycles of codimension 1) are algebraic, we are interested in Hodge cycles that are not generated by divisor classes.

We call such cycles exceptional (in [9] they are called sporadic). Exceptional Hodge cycles cause certain degeneration on various arithmetic objects (cf. [10]). Moreover, constructing abelian varieties with exceptional Hodge cycles is related to various combinatorial or group theoretic concepts (cf. [4, 5]).

Let \( A \) be a CM abelian variety of dimension \( d \). By definition, \( \text{End}A \otimes \mathbb{Q} \) contains a CM field \( K \) of degree \( 2d \). Let \( S \subset \text{Hom}(K, \mathbb{C}) \) be the CM type of \( A \); the representation of \( K \) on \( H^0(A, \Omega^1) \) is isomorphic to \( \bigoplus_{\sigma \in S} \sigma \). We say
that $A$ is of CM type $(K, S)$. Let $MT(A)$ be the Mumford-Tate group of $A$ (cf. [1]). $MT(A)$ is an algebraic subtorus of $\text{Res}_{K/Q}\mathbb{G}_m$. The dimension of $MT(A)$ is called the rank of $A$ (or the rank of the CM type $(K, S)$) in certain contexts (cf. [7]).

When $A$ is simple, the next two conditions (i) and (ii) are equivalent (cf. [1]).

(i) For each positive integer $m$, the ring of the Hodge cycles on $A^m$ is generated by the divisor classes.

(ii) $\dim MT(A) = d + 1$.

When $A$ satisfies (one of) these conditions, $A$ is called stably nondegenerate and the Hodge conjecture holds for every power of $A$ (cf. [2]). If $A$ is not stably nondegenerate (we simply say $A$ is degenerate) then there exists an integer $m > 0$ such that $A^m$ holds an exceptional Hodge cycle. Following F. Hazama [3], we call the minimum of such $m$ the index of degeneracy of $A$.

From this point on, we assume that $K$ is an abelian extension over $\mathbb{Q}$ with the Galois group $G = \text{Gal}(K/\mathbb{Q})$. Then $\dim MT(A)$ is equal to the number of characters $\chi$ of $G$ satisfying $\chi(S) = \sum_{\sigma \in S} \chi(\sigma) \neq 0$ (see [1] for a reference). Note that $\chi(S) = 0$ for every nontrivial even character $\chi$ and $\chi_0(S) = d \neq 0$ for the trivial character $\chi_0$. Hence, in this case, the above conditions (i) and (ii) are equivalent to the next (iii).

(iii) For each odd character $\chi$ of $G$, one has $\chi(S) \neq 0$.

In this paper, we prove that if there exists an odd character $\chi$ satisfying $\chi(S) = 0$ and the order of $\chi$ is a power of 2, then the index of degeneracy of $A$ is equal to 1 (see Theorem 4.1).

**Remark.** In the CM case, one can provide an example of variety $A$ having index of degeneracy strictly bigger than 1, which was obtained by S. P. White [9].

### 2. Abelian varieties of Weil type

Let $A$ be a CM abelian variety of type $(K, S)$. We assume that $K$ is an abelian extension over $\mathbb{Q}$ and $K$ contains a proper sub CM field $k$. $H$ denotes the subgroup of $G = \text{Gal}(K/\mathbb{Q})$ corresponding to $k$. Put $r = \#H = [K : k]$. For $\tau \in G/H = \text{Gal}(k/\mathbb{Q})$, put

$$H_{\tau}^{1,0} = \{ \omega \in H^{1,0} \mid \forall a \in k, a(\omega) = a^\tau \omega \},$$

$$n_{\tau} = \dim H_{\tau}^{1,0} = \# \{ \sigma \in S \mid \sigma|_k = \tau \}.$$

If $n_{\tau}$ does not depend on $\tau \in \text{Gal}(k/\mathbb{Q})$, we say that $A$ is of Weil type. It is easy to see that the condition is equivalent to $n_{\tau} = n_{\tau \rho} = \frac{r}{2}$ for each $\tau$, where $\rho$ denotes the complex conjugation. For such $A$, $\chi(S) = 0$ for the odd characters $\chi$ of $G$ which are trivial on $H$. If this is the case, $A$ is degenerate.
and there exists an exceptional Hodge cycle of codimension $\frac{r}{2}$ (cf. [6]). In particular, the index of degeneracy of $A$ is equal to 1.

3. Lenstra’s result

When a CM abelian variety $A$ is degenerate, we want to determine the index of degeneracy of $A$. H. W. Lenstra, Jr. gives an answer for some cases. Here we recall his argument briefly. For more information, see [9]. (The expressions in [9] are slightly different from ours.)

Let $A$ be a simple CM abelian variety of type $(K, S)$. Let us assume that $K$ is an abelian extension over $\mathbb{Q}$ and that there exists an odd character $\chi$ of $G = \text{Gal}(K/\mathbb{Q})$ with $\chi(S) = 0$; hence $A$ is degenerate.

When $\chi$ is faithful, $G$ is cyclic. We denote a generator of $G$ by $\gamma$ and the order of $\gamma$ by $2t$. Put

$$h = \prod_{p|t} (\epsilon + \rho \gamma^p),$$

where $p$ runs over the odd primes dividing $t$ and $\epsilon$ is the unit element of $G$.

The element $h$ lies in the group ring $\mathbb{Z}[G]$. We can see that the coefficients in $h$ are 1 or 0, hence $h$ is naturally regarded as a subset of $G$. For each $\sigma \in G$, we take a nonzero $\omega_\sigma \in H^1(A, \mathbb{C})$ such that $a(\omega_\sigma) = a^\sigma \omega_\sigma$ for each $a \in K$. Such $\omega_\sigma$ is unique up to a constant multiple; we have $H^1(A, \mathbb{C}) = \bigoplus_{\sigma \in G} \mathbb{C}\omega_\sigma$.

Then the element

$$\bigwedge_{\sigma \in h} \omega_\sigma$$

(one implicitly fixes an order for taking this wedge product) is an exceptional Hodge cycle on $A$. More precisely, it is an element of $(H^{2q}(A, \mathbb{Q}) \cap H^{2q}) \otimes \mathbb{C}$ for some $q$ and is not generated by divisor classes. This implies that the index of degeneracy of $A$ is equal to 1.

When $\chi$ is not faithful, we take the above $h$ for $G/\text{Ker}\chi$, then its pullback to $G$ works.

In the definition of $h$, we have used the odd prime factors of $t$. When $t$ is a power of 2, such a prime number does not exist and the above argument doesn’t work.

4. The 2-power case

In this section, we consider the case where $t$ is a power of 2. In this case, the index of degeneracy of $A$ is also 1, but the reason is different from that of Lenstra’s case.
Theorem 4.1. Let $A$ be a CM abelian variety of dimension $d$ as in Section 3. Assume that there exists an odd character $\chi$ of $G = \text{Gal}(K/Q)$ with $\chi(S) = 0$ and the order of $\chi (= 2t)$ is a power of 2. Then $A$ is of Weil type; the index of degeneracy of $A$ is equal to 1.

Proof. Put $t = 2^{s-1}$ with $s > 0$. Let $H$ be the kernel of $\chi$ and $k$ be the subfield of $K$ corresponding to $H$. The quotient $G/H = \text{Gal}(k/Q)$ is a cyclic group of order $2t$. Let us denote by $\gamma$ (a representative of) a generator of $G/H$. Then $\zeta = \chi(\gamma)$ is a primitive $2^s$-th root of unity. Let $G = \bigcup_{i=0}^{2t-1} H\gamma^i$ be the coset decomposition of $G$ by $H$, where $H\gamma^i$ is the coset of the complex conjugation $\rho$. For each $i$, put $n_i = #(S \cap H\gamma^i) = \dim H^{1,0}_{\tau}$, where $\tau \in \text{Gal}(k/Q)$ is corresponding to $H\gamma^i$. Then $n_i + n_{t+i} = #H = \frac{q}{2}$ ($0 \leq i \leq t - 1$).

We have

$$0 = \sum_{\sigma \in S} \chi(\sigma) = \sum_{i=0}^{2t-1} n_i \chi(\gamma^i) = \sum_{i=0}^{t-1} (n_i - n_{t+i}) \zeta^i.$$ 

Since $\zeta$ is a primitive $2^s$-th root of unity, its degree over $Q$ is $2^{s-1} = t$. This implies $n_i = n_{t+i}$ ($0 \leq i \leq t - 1$). If $H$ is trivial (i.e. $k = K$), then $n_i = 0$ or 1 for each $i$, but this is impossible because $n_i + n_{t+i} = 1$. So $k$ is a proper CM subfield of $K$ and $A$ is of Weil type with respect to $k$; the index of degeneracy of $A$ is equal to 1. \qed

5. Example: CM Jacobians

In this section, we give examples of abelian varieties satisfying the condition in the previous section. These are CM Jacobian varieties treated in [8]. For these abelian varieties, we can determine the dimensions of the Mumford-Tate groups.

Let $p > 5$ be a prime number and $\zeta_p$ be a primitive $p$-th root of unity. We denote the minimal polynomial of $-(\zeta_p + \zeta_p^{-1})$ by $g(x) \in \mathbb{Z}[x]$. Then the Jacobian variety $A$ of the algebraic curve $y^2 = x \cdot g(x^2 - 2)$ is a simple CM abelian variety of dimension $\frac{p-1}{2}$. Put $K = \mathbb{Q}(\sqrt{-1}, \zeta_p + \zeta_p^{-1})$ then $G = \text{Gal}(K/Q) \cong \mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/p\mathbb{Z})^\times /\{\pm 1\}$ and $\text{End} A \otimes \mathbb{Q} \cong K$.

According to [8], the CM type $S$ of $A$ is:

when $p \equiv 1$ (mod 4), $S = \{(0, 1), (1, 2), (0, 3), \ldots, (1, \frac{p-1}{2})\},$

when $p \equiv 3$ (mod 4), $S = \{(0, 1), (1, 2), (0, 3), \ldots, (0, \frac{p-1}{2})\}.$

Let $\psi$ be the nontrivial character of $(\mathbb{Z}/4\mathbb{Z})^\times \cong \mathbb{Z}/2\mathbb{Z}$ and $\xi_0$ be the trivial character of $(\mathbb{Z}/p\mathbb{Z})^\times$. Then the product $\psi \times \xi_0$ can be viewed as an odd character of $G$; its order is 2.
When \( p \equiv 1 \pmod{4} \), one has \((\psi \times \xi_0)(S) = 1 - 1 + 1 - \cdots - 1 = 0\). Hence by Theorem 4.1, the abelian variety \( A \) is of Weil type with \( k = \mathbb{Q}(\sqrt{-1}) \) and the index of degeneracy of \( A \) is equal to 1.

Moreover, representing an odd character \( \chi \) of \( G \) by \( \chi = \psi \times \xi \) (where \( \xi \) is an even character \( \pmod{p} \)), we can describe the value \( \chi(S) \) by the generalized Bernoulli numbers \( B_{1,\chi} = \frac{1}{4p} \sum_{a=1}^{4p} \chi(a)a \). Here we are regarding \( \chi \) as a character \( \pmod{4}p \).

In fact, for \( \xi \neq \xi_0 \), we can deduce:

- when \( p \equiv 1 \pmod{4} \),
  \[ \chi(S) = \xi(1) - \xi(2) + \xi(3) - \cdots - \xi\left(\frac{p-1}{2}\right) = \tilde{\xi}(2)B_{1,\chi}, \]
- when \( p \equiv 3 \pmod{4} \),
  \[ \chi(S) = \xi(1) - \xi(2) + \xi(3) - \cdots + \xi\left(\frac{p-1}{2}\right) = -\tilde{\xi}(2)B_{1,\chi}. \]

We know \( B_{1,\chi} \neq 0 \). When \( p \equiv 3 \pmod{4} \), one has \((\psi \times \xi_0)(S) \neq 0\). Hence the dimension of the Mumford-Tate group \( \text{MT}(A) \) of \( A \) is:

- when \( p \equiv 1 \pmod{4} \), \( \dim \text{MT}(A) = \dim A \),
- when \( p \equiv 3 \pmod{4} \), \( \dim \text{MT}(A) = \dim A + 1 \).

In particular, when \( p \equiv 3 \pmod{4} \), \( A \) is stably nondegenerate; the Hodge conjecture holds for every power of \( A \).

**Remark.** When \( p \equiv 1 \pmod{4} \) and when \( \xi(*) = \left(\frac{*}{p}\right) \) (the quadratic residue symbol), the above calculations imply the following (probably well known) equalities concerning the class number \( h_p \) of the imaginary quadratic field \( \mathbb{Q}(\sqrt{-p}) \):

\[
\left(\frac{1}{p}\right) - \left(\frac{2}{p}\right) + \left(\frac{3}{p}\right) - \cdots - \left(\frac{\frac{p-1}{2}}{p}\right) = -\left(\frac{2}{p}\right) h_p,
\]

\[
\left(\frac{1}{p}\right) - \left(\frac{2}{p}\right) - \left(\frac{3}{p}\right) + \left(\frac{4}{p}\right) + \left(\frac{5}{p}\right) - \cdots + (-1)^{\frac{p-1}{4}} \left(\frac{\frac{p-1}{2}}{p}\right) = h_p.
\]

**References**


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