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A valuation criterion for normal basis generators of Hopf-Galois extensions in characteristic $p$

par NIGEL P. BYOTT

1. Introduction

Let $L/K$ be a finite Galois extension of fields with Galois group $G = \text{Gal}(L/K)$. The Normal Basis Theorem asserts that there is an element $\rho$ of $L$ whose Galois conjugates $\{\sigma(\rho) \mid \sigma \in G\}$ form a basis for the $K$-vector space $L$. Equivalently, $L$ is a free module of rank 1 over the group algebra $K[G]$ with generator $\rho$. Such an element $\rho$ is called a normal basis generator for $L/K$. The question then arises whether there is a simple condition on elements $\rho$ of $L$ which guarantees that $\rho$ is a normal basis generator. Specifically, suppose that $L$ is equipped with a discrete valuation $v_L$. (Throughout, whenever we consider a discrete valuation $v_F$ on a field $F$, we assume it is normalised so that $v_F(F) = \mathbb{Z} \cup \{\infty\}$.) We may then ask whether there exists an integer $b$ such that any $\rho \in L$ with $v_L(\rho) = b$
is automatically a normal basis generator for \( L/K \). We shall refer to any such \( b \) as an integer certificate for normal basis generators of \( L/K \). In the case that \( K \) has characteristic \( p > 0 \), and is complete with perfect residue field, this question was recently settled by G. Elder [4]. His result can be stated as follows:

**Theorem 1** (Elder). Let \( K \) be a field of characteristic \( p > 0 \), complete with respect to the discrete valuation \( v_K \), and with perfect residue field. Let \( L \) be a finite Galois extension of \( K \) of degree \( n \) with Galois group \( G = \text{Gal}(L/K) \), let \( w = v_L(\mathcal{D}_{L/K}) \), where \( \mathcal{D}_{L/K} \) denotes the different of \( L/K \) and \( v_L \) is the valuation on \( L \), and let \( b \in \mathbb{Z} \).

(a) If \( L/K \) is totally ramified, \( n \) is a power of \( p \), and \( b \equiv -w - 1 \) (mod \( n \)), then every \( \rho \in L \) with \( v_L(\rho) = b \) is a normal basis generator for \( L/K \).

(b) The result of (a) is best possible in the sense that, if

(i) \( n \) is not a power of \( p \), or

(ii) \( L/K \) is not totally ramified, or

(iii) \( b \not\equiv -w - 1 \) (mod \( n \)),

then there is some \( \rho \in L \) with \( v_L(\rho) = b \) such that \( \rho \) is not a normal basis generator for \( L/K \).

The purpose of this paper is to show that Theorem 1, suitably interpreted, applies not just in the setting of classical Galois theory, but also in the setting of Hopf-Galois theory for separable field extensions, as developed by C. Greither and B. Pareigis [5]. A finite separable field extension \( L/K \) is said to be \( H \)-Galois, where \( H \) is a Hopf algebra over \( K \), if \( L \) is an \( H \)-module algebra and the map \( H \longrightarrow \text{End}_K(L) \) defining the action of \( H \) on \( L \) extends to an \( L \)-linear isomorphism \( L \otimes_K H \longrightarrow \text{End}_K(L) \). A Hopf-Galois structure on \( L/K \) consists of a \( K \)-Hopf algebra \( H \) and an action of \( H \) on \( L \) so that \( L \) is \( H \)-Galois. This generalises the classical notion of Galois extension: if \( L/K \) is a finite Galois extension of fields with Galois group \( G \), we can take \( H \) to be the group algebra \( K[G] \) with its standard Hopf algebra structure and its natural action on \( L \), and then \( L/K \) is \( H \)-Galois. A Galois extension may, however, admit many other Hopf-Galois structures in addition to this classical one, and many (but not all) separable extensions which are not Galois nevertheless admit one or more Hopf-Galois structures. Moreover, if \( L \) is \( H \)-Galois, then \( L \) is a free \( H \)-module of rank 1 (see the proof of [3, (2.16)]), and, by analogy with the classical case, we will shall refer to any free generator of the \( H \)-module \( L \) as a normal basis generator for \( L/K \) with respect to \( H \). Our main result is that Theorem 1 holds in this more general setting:
Theorem 2. Let $S/R$ be a finite extension of discrete valuation rings of characteristic $p > 0$, and let $L/K$ be the corresponding extension of fields of fractions. Let $n = [L : K]$, let $v_L$ be the valuation on $L$ associated to $S$, and let $w = v_L(D_{S/R})$ where $D_{S/R}$ denotes the different of $S/R$. Suppose that $L/K$ is separable, and is $H$-Galois for some $K$-Hopf algebra $H$. Let $b \in \mathbb{Z}$.

(a) If $L/K$ is totally ramified, $n$ is a power of $p$, and $b \equiv -w - 1 \pmod{n}$, then every $\rho \in L$ with $v_L(\rho) = b$ is a normal basis generator for $L/K$ with respect to $H$.

(b) The result of (a) is best possible in the sense that, if

(i) $n$ is not a power of $p$, or
(ii) $L/K$ is not totally ramified, or
(iii) $b \not\equiv -w - 1 \pmod{n}$,

then there is some $\rho \in L$ with $v_L(\rho) = b$ such that $\rho$ is not a normal basis generator for $L/K$ with respect to $H$.

In Theorem 2, we do not require $K$ to be complete with respect to the valuation $v_K$ on $K$ associated to $R$, and we do not require the residue field of $R$ to be perfect. Thus, even in the case of Galois extensions (in the classical sense), Theorem 2 is slightly stronger than Theorem 1.

We recall that the different $D_{S/R}$ is defined as the fractional $S$-ideal such that

$$D_{S/R}^{-1} = \{ x \in S \mid \text{Tr}_{L/K}(xS) \subseteq R \},$$

where $\text{Tr}_{L/K}$ is the trace from $L$ to $K$. In the case that $S/R$ is totally ramified and $L/K$ is separable, let $p(X) \in R[X]$ be the minimal polynomial over $R$ of a uniformiser $\Pi$ of $S$. Then $D_{S/R}$ is generated by $p'(\Pi)$, where $p'(T)$ denotes the derivative of $p(T)$ [6, III, Cor. 2 to Lemma 2]. (This does not require $L/K$ to be Galois, or the residue field of $K$ to be perfect.) The formulation of Theorem 1(a) in [4] is in terms of $p'(\Pi)$.

If $S$ (and hence $L$) is complete with respect to $v_L$, then $D_{S/R}$ is the same as the different $D_{L/K}$ of the extension $L/K$ of valued fields occurring in Theorem 1. Theorem 2 also applies, however, if $K$ is a global function field of dimension 1 over an arbitrary field $k$ of characteristic $p$. In particular, if $L$ is an $H$-Galois extension of $K$ of $p$-power degree, and some place $\mathfrak{p}$ of $K$ is totally ramified in $L/K$, then Theorem 2(a) gives an integer certificate for normal basis generators of $L/K$ with respect to $H$, in terms of the valuation $v_L$ on $L$ corresponding to the unique place $\mathfrak{P}$ of $L$ above $\mathfrak{p}$ and the $\mathfrak{P}$-part of $D_{L/K}$. If, on the other hand, there is more than one place $\mathfrak{P}$ of $L$ above $\mathfrak{p}$, then the integral closure of $R$ in $L$ is the intersection $S_0$ of the corresponding valuation rings $S$ of $L$ [8, III.3.5]. Any one such $S$ strictly contains $S_0$ and is therefore not integral over $R$. In particular, $S$ is not finite over $R$ and Theorem 2 does not apply in this case.
We briefly recall the background to the above results. In the (characteristic 0) situation where \( K \) is a finite extension of the field \( \mathbb{Q}_p \) of \( p \)-adic numbers, the author and Elder [2] showed the existence of integer certificates for normal basis generators in totally ramified elementary abelian extensions \( L/K \), under the assumption that \( L/K \) contains no maximally ramified subfield. This assumption is necessary, since there can be no integer certificate in the case \( L = K(\sqrt[p]{\pi}) \) with \( v_K(\pi) = 1 \): indeed, for any \( b \in \mathbb{Z} \), the element \( \pi^{b/p} \) has valuation \( b \) but is not a normal basis generator. (Here \( K \) must contain a primitive \( p \)th root of unity for \( L/K \) to be Galois.) We also raised the question of whether the corresponding result held in characteristic \( p > 0 \), where the exceptional situation of maximal ramification cannot arise. Our question was answered by L. Thomas [9], who observed that general properties of group algebras of \( p \)-groups in characteristic \( p \) allow an elegant derivation of integer certificates for arbitrary finite abelian \( p \)-groups \( G \). Her result was expressed in terms of the last break in the sequence of ramification groups of \( L/K \), but is equivalent to Theorem 1 for totally ramified abelian \( p \)-extensions \( G \). Finally, Elder [4] removed the hypothesis that \( G \) is abelian by expressing the result in terms of the valuation of the different, and also gave the converse result that no integer certificate exists if \( L/K \) is not totally ramified or is not a \( p \)-extension.

We end this introduction by outlining the structure of the paper. In §2, we review the facts we shall need from Hopf-Galois theory, and prove several preliminary results in the case of \( p \)-extensions. These show, in effect, that the relevant Hopf algebras behave similarly to the group algebras considered in [9]. In §3 we develop some machinery to handle extensions whose degrees are not powers of \( p \). In [4], such extensions were treated by reducing to a totally and tamely ramified extension. For Hopf-Galois extensions, it is not clear whether such a reduction is always possible. (Indeed, while a totally ramified Galois extension of local fields is always soluble, the author does not know of any reason why such an extension could not admit a Hopf-Galois structure in which the associated group \( N \), as in §2 below, is insoluble.) We therefore adopt a different approach, using a small part of the theory of modular representations. We complete the proof of Theorem 2 in §4. The ramification groups, which play an essential role in the arguments of [4] and [9], are not available in the Hopf-Galois setting, but their use can be avoided by working directly with the inverse different. Finally, in §5, we give an example of a family of extensions which are not Galois, but to which Theorem 2 applies.

2. Hopf-Galois theory for \( p \)-extensions in characteristic \( p \)

In this section, we briefly recall the description of Hopf-Galois structures on a finite separable field extension \( L/K \), and note some properties of the
Hopf algebras $H$ which arise when $[L : K]$ is a power of $p = \text{char}(K)$. We do not make any use of valuations on $K$ and $L$ in this section.

Let $E$ be a (finite or infinite) Galois extension of $K$ containing $L$. Set $G = \text{Gal}(E/K)$ and $G' = \text{Gal}(E/L)$, and let $X = G/G'$ be the set of left cosets $gG'$ of $G'$ in $G$. Then $G$ acts by left multiplication on $X$, giving a homomorphism $G \rightarrow \text{Perm}(X)$ into the group of permutations of $X$. The main result of [5] can be stated as follows: the Hopf-Galois structures on $L/K$ (up to the appropriate notion of isomorphism) correspond bijectively to the regular subgroups $N$ of $\text{Perm}(X)$ which are normalised by $G$. In the Hopf-Galois structure corresponding to $N$, the Hopf algebra acting on $L$ is $H = E[N]^G$, the fixed point algebra of the group algebra $E[N]$ under the action of $G$ simultaneously on $E$ (as field automorphisms) and on $N$ (by conjugation inside $\text{Perm}(X)$). The Hopf algebra operations on $H$ are the restrictions of the standard operations on $E[N]$. We write $1_X$ for the trivial coset $G'$ in $X$. Then there is a bijection between elements $\eta$ of $N$ and $K$-embeddings $\sigma : L \rightarrow E$, given by $\eta \mapsto \sigma_\eta$ where $\sigma_\eta(\rho) = g(\rho)$ with $\eta^{-1}(1_X) = gg'$. The action of $H$ on $L$ can be described explicitly as follows (see e.g. [1, p. 338]):

\begin{equation}
\left( \sum_{\eta \in N} \lambda_\eta \eta \right)(\rho) = \sum_{\eta \in N} \lambda_\eta \sigma_\eta(\rho) \text{ for } \sum_{\eta \in N} \lambda_\eta \eta \in H \text{ and } \rho \in L.
\end{equation}

**Remark.** In [5], $E$ is taken to be the the Galois closure $E_0$ of $L$ over $K$. In this case, the action of $G$ on $X$ is faithful. However, it is clear that one may take a larger field $E$ as above: all that changes is that $G$ need no longer act faithfully on $X$. (Indeed, the action of $G$ on both $X$ and $L$ factors through $\text{Gal}(E/E_0)$.) In the proof of Lemma 3.1 below, it will be convenient to take $E$ to be a finite extension of $E_0$.

Let $L/K$ be $H$-Galois, where the Hopf algebra $H$ corresponds to $N$ as above. We define

$$t_H = \sum_{\eta \in N} \eta \in E[N].$$

We now show that $t_H$ behaves like the trace element in a group algebra:

**Proposition 2.1.** We have $t_H \in H$ and, for any $h \in H$,

$$ht_H = t_H h = \epsilon(h)t_H,$$

where $\epsilon : H \rightarrow K$ is the augmentation. In particular, writing $I_H$ for the augmentation ideal $\ker \epsilon$ of $H$, we have

$$I_H t_H = t_H I_H = 0.$$

Also, $t_H(\rho) = \text{Tr}_{L/K}(\rho)$ for any $\rho \in L$. 

Proof. Since $N$ is normalised by $G$, each $g \in G$ permutes the elements of $N$. Hence $t_H \in E[N]^G = H$. For any $h = \sum_{\nu \in N} \lambda_{\nu} \nu \in H$, we have

$$ht_H = \sum_{\nu, \eta} \lambda_{\nu} \nu \eta = \left( \sum_{\nu} \lambda_{\nu} \right) \left( \sum_{\eta} \eta \right) = \epsilon(h) t_H.$$ 

In particular, if $h \in I_H$ then $ht_H = \epsilon(h) t_H = 0$, so $I_H t_H = 0$. Similarly $t_H h = \epsilon(h) t_H$ and $t_H I_H = 0$. Finally, for $\rho \in L$ we have

$$t_H(\rho) = \sum_{\eta \in N} \sigma_{\eta}(\rho) = \text{Tr}_{L/K}(\rho).$$

Remark. Proposition 2.1 shows that $K \cdot t_H$ is the ideal of (left or right) integrals of $H$.

Corollary 2.2. If $\text{Tr}_{L/K}(\rho) = 0$ then $\rho$ cannot be a normal basis generator for $L/K$ with respect to $H$.

Proof. If $\rho$ is a free generator for $L$ over $H$, then the annihilator of $\rho$ in $H$ must be trivial. But if $\text{Tr}_{L/K}(\rho) = 0$ then $\rho$ is annihilated by $t_H \neq 0$. □

We next show that [9, Proposition 7] still holds in our setting:

Lemma 2.3. If $[L : K] = p^m$ for some integer $m$, then any $\rho \in L$ with $\text{Tr}_{L/K}(\rho) \neq 0$ is a normal basis generator for $L/K$ with respect to $H$.

Proof. We first observe that the augmentation ideal $I_H$ is a nilpotent ideal of $H$, since $I_H = I_{E[N]} \cap H$ and the augmentation ideal $I_{E[N]}$ of $E[N]$ is a nilpotent ideal of $E[N]$ because $|N| = [L : K] = p^m$. Thus $I_H$ is contained in (and in fact equals) the Jacobson radical $J_H$ of $H$.

Now consider the $H$-submodule $M = H \cdot \rho + I_H \cdot L$ of $L$. Since $L$ is a free $H$-module of rank 1, and $H/I_H \cong K$, the $K$-subspace $I_H L$ of $L$ has codimension 1. But $\rho \notin I_H L$ since $\text{Tr}_{L/K}(I_H L) = (t_H I_H)L = 0$ by Proposition 2.1, so $M = L$. Since $I_H \subseteq J_H$, Nakayama’s Lemma shows that $H \cdot \rho = L$, and, comparing dimensions over $K$, we see that $\rho$ is a free generator for the $H$-module $L$. □

The next result is immediate from Corollary 2.2 and Lemma 2.3

Corollary 2.4. If $[L : K] = p^m$ then $\rho \in L$ is a normal basis generator for $L/K$ with respect to $H$ if and only if $\text{Tr}_{L/K}(\rho) \neq 0$. In particular, the set of normal basis generators is the same for all Hopf-Galois structures on $L/K$. 
3. The non-$p$-power case

As in Theorem 2, let $S/R$ be a finite extension of discrete valuation rings, such that the corresponding extension $L/K$ of their fields of fractions is $H$-Galois for some Hopf algebra $H$. We do not require $S$ and $R$ to be complete. Let $v_L$, $v_K$ be the corresponding valuations on $L$, $K$.

**Lemma 3.1.** Suppose that $[L : K]$ is not a power of $p$. Then $H$ contains nonzero orthogonal idempotents $e_1, e_2$ with $e_1 + e_2 = 1$, such that

$$v_L(e_j \rho) \geq v_L(\rho) \text{ for all } \rho \in L \text{ and } j = 1, 2.$$  

**Proof.** Let $[L : K] = p^m r$ where $m \geq 0$ and where $r \geq 2$ is prime to $p$. We have $H = E[N]^G$ where $G = \text{Gal}(E/K)$ and, in view of the remark before Proposition 2.1, we may take $E$ to be a finite Galois extension of $K$, containing $L$ and also containing a primitive $r$th root of unity $\zeta_r$. Let $k'$ be the algebraic closure in $E$ of the prime subfield $\mathbb{F}_p$. Thus $\zeta_r \in k'$.

Now let $t$ be the number of conjugacy classes in $N$ consisting of elements whose order is prime to $p$. As $|N| = [L : K]$ is not a power of $p$, we have $t \geq 2$. For any field $F$ of characteristic $p$ containing $\zeta_r$, the group algebra $A = F[N]$ has exactly $t$ nonisomorphic simple modules [7, §18.2, Corollary 3]. Let $J_A$ denote the Jacobson radical of $A$. Then the semisimple algebra $A/J_A$ has exactly $t$ Wedderburn components, and therefore has exactly $t$ primitive central idempotents. Since $A$ is a finite-dimensional $F$-algebra, we may lift these idempotents from $A/J_A$ to $A$. Thus $A$ has exactly $t$ primitive central idempotents, $\phi_1, \ldots, \phi_t$ say, and hence has $t$ maximal 2-sided ideals. One of these, say the ideal $(1 - \phi_1)A$ associated to $\phi_1$, is the augmentation ideal $I_A$.

Taking $F = k'$ in the previous paragraph, we obtain orthogonal idempotents $\phi_1, \ldots, \phi_t \in k'[N]$. But $k' \subset E$, and taking $F = E$, we find that $\phi_1, \ldots, \phi_t$ are again the primitive central idempotents in $E[N]$. The action of $G$ on $E[N]$ permutes these idempotents, and fixes $\phi_1$ since it fixes the augmentation ideal of $E[N]$. Hence $\phi_1 \in H$. Let $e_1 = \phi_1$ and $e_2 = 1 - \phi_1$. Then $e_1, e_2$ are orthogonal idempotents in $H \cap k'[N]$ with $e_1 + e_2 = 1$. Moreover $e_1 \neq 0$ by definition and $e_2 \neq 0$ since $t \geq 2$.

We now show that $v_L(e_j \rho) \geq v_L(\rho)$ for $j = 1, 2$ and for any $\rho \in L$. Since $S/R$ is finite, $S$ is the unique valuation ring of $L$ containing $R$. Thus each valuation ring $T$ of $E$ containing $R$ must also contain $S$. (There may be several such $T$ if $R$ is not complete.) Fix one of these valuation rings $T$ of $E$, and let $v_T$ be the corresponding valuation on $E$. Then any valuation $v'$ on $E$ with $v'(\mu) = v_T(\mu)$ for all $\mu \in K$ necessarily satisfies $v'(\rho) = v_T(\rho)$ for all $\rho \in L$. In particular, for each $g \in G$, the valuation $v_E \circ g$ on $E$ must have the same restriction to $L$ as $v_E$. Thus, for each $\eta \in N$, we have $v_E(\sigma_\eta(\rho)) = v_E(\rho)$ for all $\rho \in L$. 


For \( j = 1 \) or \( 2 \), let
\[
e_j = \sum_{\eta \in \mathcal{N}} \lambda_\eta \eta \quad \text{with} \quad \lambda_\eta \in k'.
\]
Then, as \( e_j \in H \), we have
\[
e_j(\rho) = \sum_{\eta \in \mathcal{N}} \lambda_\eta \sigma_\eta(\rho)
\]
by (2.1). But \( \lambda_\eta \) is algebraic over \( \mathbb{F}_p \), so either \( \lambda_\eta = 0 \) or \( v_E(\lambda_\eta) = 0 \). We then have
\[
v_E(e_j \rho) \geq \min_{\eta \in \mathcal{N}} (v_E(\lambda_\eta) + v_E(\sigma_\eta(\rho))) \geq 0 + v_E(\rho).
\]
As \( \rho \), \( e_j \rho \in L \), it follows that \( v_L(e_j \rho) \geq v_L(\rho) \) as required. \( \Box \)

We can now prove case (i) of Theorem 2(b).

**Corollary 3.2.** Let \( S/R \) be as in Theorem 2, and suppose that \([ L : K ]\) is not a power of \( p \). Then, for any \( b \in \mathbb{Z} \), there exists some \( \rho \in L \) with \( v_L(\rho) = b \) such that \( \rho \) is not a normal basis generator for \( L/K \) with respect to \( H \).

**Proof.** Take any \( \rho' \in L \) with \( v_L(\rho') = b \). With \( e_1, e_2 \in H \) as in Lemma 3.1, we have
\[
\rho' = e_1 \rho' + e_2 \rho', \quad v_L(e_1 \rho') \geq b, \quad v_L(e_2 \rho') \geq b.
\]
Both inequalities cannot be strict since \( v_L(\rho') = b \), so without loss of generality we have \( v_L(e_1 \rho') = b \). Set \( \rho = e_1 \rho' \). Then \( v_L(\rho) = b \) but \( \rho \) cannot be a normal basis generator with respect to \( H \), since \( e_2 \rho = (e_2 e_1) \rho' = 0 \). \( \Box \)

4. **Proof of Theorem 2**

For this section, the hypotheses of Theorem 2 are in force. In particular, \( S/R \) is a finite extension of discrete valuation rings of characteristic \( p > 0 \), and the corresponding extension of fields of fractions \( L/K \) is separable of degree \( n \). Also, \( L/K \) is \( H \)-Galois for some \( K \)-Hopf algebra \( H \).

By Corollary 3.2, we may assume that \( n = [ L : K ] \) is a power of \( p \). Let \( e \) be the ramification index of \( S/R \), let \( w = v_L(\mathcal{D}_{S/R}) \), and let \( \pi \) and \( \Pi \) be uniformisers for \( R \) and \( S \) respectively. By definition of the different, we have
\[
\text{Tr}_{L/K}(\Pi^{-w}S) \subseteq R, \quad \text{Tr}_{L/K}(\Pi^{-w-1}S) \not\subseteq R,
\]
and therefore
\[
\text{Tr}_{L/K}(\Pi^{e-w}S) \subseteq \pi R, \quad \text{Tr}_{L/K}(\Pi^{e-w-1}S) = R.
\]
Hence there is some \( x_1 \in L \) with \( v_L(x_1) = e - w - 1 \) and \( \text{Tr}_{L/K}(x_1) = 1 \). For 
\[ 2 \leq i \leq e, \text{ pick } x_i' \in L \text{ with } v_L(x_i') = e - w - i, \text{ and set } x_i = x_i' - \text{Tr}_{L/K}(x_i')x_1. \]
Since \( \text{Tr}_{L/K}(x_i') \in R \) and \( v_L(x_i') < v_L(x_1) \), we have
\[ (4.1) \quad v_L(x_i) = e - w - i \text{ for } 1 \leq i \leq e, \]
and clearly
\[ (4.2) \quad \text{Tr}_{L/K}(x_i) = \begin{cases} 1 & \text{if } i = 1; \\ 0 & \text{otherwise.} \end{cases} \]

We first consider the totally ramified case \( e = n \). Then \( x_1, \ldots, x_n \) is a \( K \)-basis for \( L \), since the \( v_L(x_i) \) represent all residue classes modulo \( n \).

Let \( \rho \in L \) with \( v_L(\rho) \equiv -w - 1 \pmod{n} \). We may write
\[ \rho = \sum_{i=1}^{n} a_i x_i \]
with the \( a_i \in K \). Then \( v_L(\rho) = \min_i \{nv_K(a_i) + (n - w - i) \} \). The hypothesis
on \( \rho \) means that the minimum must occur at \( i = 1 \). In particular, \( a_1 \neq 0 \). Then, by (4.2), we have
\[ \text{Tr}_{L/K}(\rho) = \sum_{i=1}^{n} a_i \text{Tr}_{L/K}(x_i) = a_1 \neq 0, \]
and by Lemma 2.3, \( \rho \) is a normal basis generator for \( L/K \) with respect to \( H \). This completes the proof of Theorem 2(a).

Next let \( b \in \mathbb{Z} \) with \( b \not\equiv -1 - w \pmod{n} \). Then \( b = n(s + 1) - w - i \) with 
\[ 2 \leq i \leq n \text{ and } s \in \mathbb{Z}. \]
Set \( \rho = \pi^s x_i, \) so \( v_L(\rho) = b \) by (4.1). But \( \text{Tr}_{L/K}(\rho) = 0 \) by (4.2), so that \( \rho \) cannot be a normal basis generator by Corollary 2.2.
This completes the proof of Theorem 2 for totally ramified extensions.

Finally, suppose that \( S/R \) is not totally ramified. Given \( b \in \mathbb{Z} \), write 
\( b = e(s + 1) - w - i \) with \( 1 \leq i \leq e \) and \( s \in \mathbb{Z} \). If \( i \neq 1 \) then \( \rho = \pi^s x_i \)
satisfies \( v_L(\rho) = b \) and \( \text{Tr}_{L/K}(\rho) = 0 \), so as before \( \rho \) cannot be a normal
basis generator. It remains to consider the case \( i = 1 \). Let \( l, k \) be the residue
fields of \( S, R \) respectively. Then \( l/k \) has degree \( f > 1 \) with \( ef = n \). (Note,
however, that \( l/k \) need not be separable.) Pick \( \omega \in l \) with \( \omega \not\in k \), let \( \Omega \in S \)
be any element whose image in \( l \) is \( \omega \), and set
\[ \rho = \pi^s (\Omega - \text{Tr}_{L/K}(x_1 \Omega)) x_1. \]
Then \( \text{Tr}_{L/K}(x_1 \Omega) \in \text{Tr}_{L/K}(D_{S/R}^{-1}) \subseteq R \). Since \( \omega \) and \( 1 \) are elements of \( l \)
which are linearly independent over \( k \), it follows that \( v_L(\Omega - \text{Tr}_{L/K}(x_1 \Omega)) = v_L(\Omega) = 0 \), and hence \( v_L(\rho) = es + v_L(x_1) = b \). But once more we have
\( \text{Tr}_{L/K}(\rho) = 0 \), so that \( \rho \) cannot be a normal basis generator for \( L/K \) with
respect to \( H \). This concludes the proof of Theorem 2.
5. An example

We end with an example of a family of extensions \( L/K \) which are \( H \)-Galois for a suitable Hopf algebra \( H \), but which are not Galois. Theorem 2 will give an integer certificate for normal basis generators in \( L/K \), although Theorem 1 is not applicable.

Fix a prime number \( p \), and let \( K = \mathbb{F}_p((T)) \) be the field of formal Laurent series over the finite field \( \mathbb{F}_p \) of \( p \) elements. Then \( K \) is complete with respect to the discrete valuation \( v_K \) such that \( v_K(T) = 1 \), and the valuation ring is \( R = \mathbb{F}_p[[T]] \). Take any integer \( f \geq 2 \), and set \( q = p^f \). Let \( b > 0 \) be an integer which is not divisible by \( p \), and let \( \alpha \in K \) be any element with \( v_K(\alpha) = -b \). The field we consider is \( L = K(\theta) \), where \( \theta \) is a root of the polynomial \( g(X) = X^q - X - \alpha \in K[X] \).

To see that \( L \) is not Galois over \( K \), consider the unramified extension \( F = \mathbb{F}_q K \) of \( K \) (where \( \mathbb{F}_q \) is the field of \( q \) elements), and let \( E = LF \). Then \( E \) is the splitting field of \( g \) over \( K \), and the roots of \( g \) in \( E \) are \( \{\theta + \omega \mid \omega \in \mathbb{F}_q\} \). Thus \( E \) is the Galois closure of \( L/K \), and it follows in particular that \( L/K \) is not Galois. We are therefore in the situation of §2, with \( G = \text{Gal}(E/K) \) of order \( fq \), and with \( G' = \text{Gal}(E/L) \cong \text{Gal}(F/K) \cong \text{Gal}(\mathbb{F}_q/\mathbb{F}_p) \) cyclic of order \( f \). Moreover, \( G' \) has a normal complement \( N = \text{Gal}(E/F) \cong \mathbb{F}_q \) in \( G \). Thus \( G \cong N \rtimes G' \) (and, since \( \mathbb{F}_q/\mathbb{F}_p \) has a normal basis, it is easy to see that any generator of \( G' \) acts on \( N \) with minimal polynomial \( X^f - 1 \)). In the terminology of [5, §4], \( L/K \) is an almost classically Galois extension. It therefore admits at least one Hopf-Galois structure, namely that corresponding to the group \( N \).

Now \( E/F \) is totally ramified of degree \( q \), and the ramification filtration of \( \text{Gal}(E/F) \) has only one break, occurring at \( b \). Hence, by Hilbert’s formula [6, IV, Prop. 4], \( v_E(D_{E/F}) = (b+1)(q-1) \). As \( E/L \) and \( F/K \) are unramified, it follows that \( L/K \) is totally ramified, and, using the transitivity of the different [6, III, Prop. 8], that \( v_L(D_{L/K}) = (b+1)(q-1) \). Thus Theorem 2(a) applies with \( w \equiv -1 - b \pmod{q} \). Hence any \( \rho \in L \) with \( v_L(\rho) \equiv b \pmod{q} \) is a normal basis generator with respect to any Hopf-Galois structure on \( L/K \).

Following a suggestion of the referee, we specialise this example further. Let us take \( b = q - 1 \) and \( \alpha = T^{1-q} \). Then \( v_L(\theta) = 1 - q \). We obtain a uniformising parameter for \( S \) by setting \( \eta = T \theta \). Then \( \eta \) is a root of the Eisenstein polynomial \( X^q - T^{q-1}X - T \), so \( D_{L/K} \) is generated by \( T^{q-1} \) and \( w \equiv 0 \pmod{q} \). Hence any element \( \rho \) of \( L \) with \( v_L(\rho) \equiv -1 \pmod{q} \) is a normal basis generator with respect to any Hopf-Galois structure on \( L/K \). This can easily be verified directly for \( \rho = \eta^{q-1} \) and the Hopf-Galois structure corresponding to \( N \) as above. Indeed, let \( \sigma_w \) be the element of \( N = \text{Gal}(E/F) \) corresponding to \( \omega \in \mathbb{F}_q \), so \( \sigma_w(\eta) = \eta + \omega T \). We first claim that \( \eta^{q-1} \) is a normal basis generator for the Galois extension \( E/F \),
or equivalently, that $F[N] \cdot \eta^{q-1} = E$. We have

$$\sigma_{\omega}(\eta^{q-1}) = (\eta + \omega T)^{q-1} = \sum_{i=0}^{q-1} \eta^{q-1-i}(-\omega T)^{i},$$

so the claim follows from the non-vanishing of the Vandermonde matrix $((-\omega)^i)_{\omega \in \mathbb{F}_q, 0 \leq i < q}$. Since the $F[N]$-module $E$ is free on the generator $\eta^{q-1}$, and $H = F[N]^G$ is a $K$-subalgebra of $F[N]$, it follows that $H \cdot \eta^{q-1}$ has dimension $\dim_K(H) = q = [L : K]$ over $K$. But $\eta \in L$ and $H \cdot L = L$, so we must have $H \cdot \eta^{q-1} = L$. Thus $\eta^{q-1}$ is a normal basis generator for $L/K$ over $H$, as required.

**Remark (Galois extensions).** If we apply the preceding construction starting with $\mathbb{F}_q((T))$ rather than $\mathbb{F}_p((T))$ (that is, we just consider the extension $E/F$ above) then we obtain a Galois (indeed, abelian) extension of degree $q$ for which we have given a direct verification that $\eta^{q-1}$ is a normal basis generator. This provides an explicit example of the situation considered in [9]

**Remark (Global examples).** We can easily adapt the above arguments to the case where $K$ is not complete. Let $K$ be a function field of dimension 1 with field of constants $\mathbb{F}_p$, and choose any valuation $v_K$ on $K$ which corresponds to a place of $K$ with residue field $\mathbb{F}_p$. With $q$, $b$ and $\alpha$ as above, let $L = K(\theta)$ where $\theta^q - \theta = \alpha$. Then the extension $L/K$ has degree $q$ and is a totally ramified at $v_K$. As before, $L/K$ is not Galois but does admit at least one Hopf-Galois structure, and Theorem 2(a) shows that any $\rho \in L$ with $v_L(\rho) \equiv b \pmod{q}$ is a normal basis generator for $L/K$ with respect to any Hopf-Galois structure on $L/K$.

**References**


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