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par Omar KIHEL et Florian LUCA

Résumé. Dans cet article, nous étudions quelques variations sur l’équation diophantienne de Brocard-Ramanujan.

Abstract. In this paper, we discuss variations on the Brocard-Ramanujan Diophantine equation.

1. Introduction

Brocard (see [4, 5]), and independently Ramanujan (see [15, 16]), posed the problem of finding all integral solutions to the diophantine equation

\[ n! + 1 = x^2. \]  

Although it is unlikely that equation (1) has any solution with \( n > 7 \), the fact that it has only finitely many solutions has only been conditionally proved by Overholt (see [13]). He showed that the weak form of Szpiro’s conjecture implies that equation (1) has only finitely many solutions. The weak form of Szpiro’s conjecture is a special case of the \( ABC \) conjecture and asserts that there exists a constant \( s \) such that if \( A, B, \) and \( C \) are positive integers satisfying \( A + B = C \) with \( \gcd(A, B) = 1 \), then

\[ C \leq N(ABC)^s, \]

where \( N(k) \) is the product of all primes dividing \( k \) taken without repetition. Berend and Osgood [1] showed that if \( P \in \mathbb{Z}[X] \) is a polynomial of degree \( \geq 2 \), then the density of the set of positive integers \( n \) for which there exists an integer \( x \) satisfying the more general diophantine equation

\[ n! = P(x) \]

is zero. Erdős and Obláth [7] and Pollack and Shapiro [14] showed that if \( P(x) = x^d \pm 1 \) and \( d \geq 3 \), then equation (2) has no solution with \( n > 1 \). Generalizing Overholt’s result, the second author [11] showed that the full \( ABC \) conjecture implies that equation (2) has only finitely many solutions. A wealth of information about this equation can be found in the recent paper [2].

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In this paper, we discuss some variations on the above diophantine equations. We look at the following diophantine equations:

\[(3) \quad x^p \pm y^p = \prod_{k \times n \atop k = 1}^n k,\]

and

\[(4) \quad P(x) = \prod_{k \times n \atop k = 1}^n k,\]

where \(P \in \mathbb{Q}[x]\) is a polynomial of degree \(\geq 2\). Here and in what follows, we write \(k \times n\) to mean that \(k\) does not divide \(n\).

2. New results

In what follows, we use the Vinogradov symbols \(\gg, \ll\) and \(\asymp\) as well as the Landau symbols \(O\) and \(o\) with their regular meanings. Recall that \(A \ll B, B \gg A\) and \(A = O(B)\) are all equivalent and mean that \(|A| \leq c|B|\) holds with some positive constant \(c\).

**Theorem 1.** The Diophantine equation

\[x^p \pm y^p = \prod_{k \times n \atop k = 1}^n k\]

admits only finitely many integer solutions \((x, y, p, n)\) with \(p \geq 3\) a prime number and \(\gcd(x, y) = 1\).

**Proof.** There is no loss of generality to consider only the ‘+’ sign and to assume that \(|x| > |y|\). Since the right hand side is positive, we get that \(x\) is positive. Note that \(\gcd(x, y) = 1\) implies that no prime \(q \leq n\) coprime to \(n\) divides either \(x\) or \(y\). Now either \(x \leq n\), or \(x \geq n + 1\). In the first case,

\[(n - 1)!^{1/2} \leq x^p + |y|^p \leq 2n^p,
\]

therefore by Stirling’s formula

\[
\frac{(n - 1)}{2} (1 + o(1)) \log n \leq p \log n + \log 2.
\]

For large \(n\), the above inequality implies that \(p \geq n/3\). Note however that

\[x^p + y^p = (x + y)(x^p + y^p)/(x + y),\]

and, by Fermat’s Little Theorem, it follows easily that \((x^p + y^p)/(x + y) = \delta m\), where \(\delta \in \{1, p\}\), and every prime factor of \(m\) is \(1 \pmod{p}\). Since every prime factor of \(m\) is \(\leq n \leq 3p\), it follows that

\[
\frac{x^p + y^p}{x + y} \leq p(p + 1)(2p + 1).
\]
However,
\[
\frac{x^p + y^p}{x + y} \geq x^{p-2} \geq 2^{p-2}.
\]
Indeed, the above inequality holds for positive \(y\) because
\[
x^p + y^p > x^p \geq 2x^{p-1} = (2x)x^{p-2} \geq (x + y)x^{p-2},
\]
and for negative \(y\) because
\[
x^p + y^p = x^{p-1} + x^{p-2}(-y) + \cdots + (-y)^{p-1} > x^{p-1} > x^{p-2}.
\]
We thus get the inequality
\[
2^{p-2} \leq p(p + 1)(2p + 1),
\]
which shows that \(p\) is bounded, and since \(p \geq n/3\), we get that \(n\) is bounded as well in this case.

We now assume that \(x \geq n + 1\). If \(y > 0\), then \(n^n > n! \geq x^p \geq (n + 1)^p\), and if \(y < 0\), then
\[
n^n > (x + y) \left( x^{p-1} + x^{p-2}(-y) + \cdots + (-y)^{p-1} \right) > x^{p-1} \geq (n + 1)^{p-1}.
\]
In both cases, we get \(p \leq n\). We write again
\[
x^p + y^p = (x + y) \left( \frac{x^p + y^p}{x + y} \right),
\]
and we use the fact that \((x^p + y^p)/(x + y) = \delta m\), where \(\delta \in \{1, p\}\), and every prime factor of \(m\) is 1 (mod \(p\)). If \(y > 0\), we get
\[
m = \frac{x^p + y^p}{\delta(x + y)} > \frac{x^p}{2xp} = \frac{x^{p-1}}{2p},
\]
while if \(y < 0\), then
\[
m = \frac{1}{\delta} \left( \frac{x^p + y^p}{x + y} \right) = \frac{1}{\delta} \left( x^{p-1} + x^{p-2}(-y) + \cdots + (-y)^{p-1} \right) > \frac{x^{p-1}}{2p}.
\]
Thus, we always have
\[
m \geq \frac{x^{p-1}}{2p} > \frac{(2xp)(p-1)/p}{4p} > \frac{\left( \prod_{k=1}^{n} k \right)^{(p-1)/p}}{4p},
\]
where we used the fact that \(\prod_{k=1}^{n} k = x^p + y^p < 2x^p\). Let \(M\) be the largest divisor of \(\prod_{k=1}^{n} k\) build up only from primes of the form \(q \equiv 1 \pmod{p}\).

Then \(m \mid M\), and so
\[
\log M > \frac{p - 1}{p} \log \left( \prod_{k=1}^{n} k \right) - \log(4p) > \frac{p - 1}{2p} (n - 1) \log \left( \frac{n - 1}{e} \right) - \log(4p).
\]
In the above inequality, we used Stirling’s formula as well as the fact that
\( \prod_{k=1}^{n} k > (n - 1)!^{1/2} \). It is clear that for all \( q \leq n \), the order of \( q \) in \( n! \) is
\[
\left\lfloor \frac{n}{q} \right\rfloor + \left\lfloor \frac{n}{q^2} \right\rfloor + \cdots < n \sum_{i \geq 1} \frac{1}{q^i} = \frac{n}{q - 1}.
\]
Thus,
\[
\log M < n \sum_{q \equiv 1 \pmod{p}} \frac{\log q}{q - 1}.
\]
Comparing the above inequalities, we get
\[
\frac{p - 1}{2p} \log(n - 1) - \frac{p - 1}{p} - \frac{\log 4p}{n - 1} < \frac{n}{n - 1} \sum_{q \equiv 1 \pmod{p}} \frac{\log q}{q - 1},
\]
which together with the fact that \( p \leq n \) leads to
\[
\frac{p - 1}{2p} \log(n - 1) \leq \frac{n}{n - 1} \sum_{q \equiv 1 \pmod{p}} \frac{\log q}{q - 1} + O(1).
\]
Writing \( q = 1 + pt \) for some \( t \leq n/p \) and using the trivial inequality
\[
\sum_{q \equiv 1 \pmod{p}} \frac{\log q}{q - 1} \leq \frac{\log n}{p} \sum_{t \leq n/p} \frac{1}{t} \ll \frac{\log^2 n}{p},
\]
we get
\[
\frac{1}{3} \log(n - 1) \leq \frac{p - 1}{2p} \log(n - 1) \ll \frac{\log^2 n}{p} + O(1),
\]
therefore \( p \ll \log n \). Using the Montgomery-Vaughan Theorem concerning primes in arithmetic progressions (see [12]) as well as partial summation, we deduce that
\[
\sum_{q \equiv 1 \pmod{p}} \frac{\log q}{q - 1} \ll \frac{\log n}{p},
\]
and therefore get
\[
\frac{1}{3} \log(n - 1) \leq \frac{p - 1}{2p} \log(n - 1) \ll \frac{\log n}{p} + O(1),
\]
which leads to \( p \ll 1 \). Since now \( p \) may be assumed fixed, we may apply Dirichlet’s theorem on primes in arithmetical progressions, to get that
\[
\sum_{q \equiv 1 \pmod{p}} \frac{\log q}{q - 1} = \frac{\log(n - 1)}{p - 1} + O(1),
\]
and now we are led to
\[ \frac{p - 1}{2p} \log(n - 1) \leq \frac{\log(n - 1)}{p - 1} + O(1), \]
which tells us that
\[ \left( \frac{p - 1}{2p} - \frac{1}{p - 1} \right) = O \left( \frac{1}{\log(n - 1)} \right), \]
which admits only finitely many solutions \((p, n)\) with \(p \geq 5\) because
\[ \frac{p - 1}{2p} - \frac{1}{p - 1} = \frac{p^2 - 4p + 1}{2p(p - 1)} \geq \frac{6}{40} \]
for \(p \geq 5\). Finally, since \(n^n > x^{p-1}\) and \(n\) and \(p\) are bounded, we get that \(x\) is also bounded. The statement with \(p = 3\) follows from the same arguments by strengthening the inequality
\[ \prod_{k \times n}^{n} k > (n - 1)!^{1/2}, \]
to say
\[ \prod_{k \times n}^{n} k > n^{2/3} \]
for \(n\) sufficiently large. To see that this last inequality holds, note that
\[ \prod_{k \times n}^{n} k \geq \frac{n!}{n^{\tau(n)}}, \]
where \(\tau(n)\) is the number of divisors of \(n\). Thus, it suffices to show that the inequality
\[ n^{\tau(n)} < n^{1/3} \]
holds for large \(n\), and this inequality is implied by
\[ \tau(n) \log n < \frac{n}{3} \log(n/e). \]
Since \(\log(n/e) > (\log n)/2\) if \(n\) is large, it follows that it is enough that the inequality
\[ \tau(n) < \frac{n}{6} \]
holds for large \(n\), and this last inequality is obvious. \(\Box\)

Before stating and proving Theorem 2, we restate the \(ABC\) conjecture mentioned already in the introduction. The \(ABC\) conjecture asserts that for any \(\varepsilon > 0\) there exists a constant \(C(\varepsilon)\) depending only on \(\varepsilon\), such that
if $A$, $B$ and $C$ are three nonzero coprime integers satisfying $A + B = C$, then
\[
\max(|A|, |B|, |C|) < C(\varepsilon)N(ABC)^{1+\varepsilon}.
\]

**Theorem 2.** Let $P \in \mathbb{Q}[x]$ be a polynomial of degree $\geq 2$. Then the $ABC$ conjecture implies that the equation
\[
P(x) = \prod_{k=1}^{n} k
\]
has only finitely many solutions $(x, n)$, where $x$ is a rational number and $n$ is a positive integer.

**Proof.** We write the equation as
\[
a_0 x^d + a_1 x^{d-1} + \cdots + a_d = a_{d+1} \prod_{k=1}^{n} k,
\]
where $a_0, \ldots, a_{d+1}$ are integers with $a_0 a_{d+1} \neq 0$. Here, one may choose $a_{d+1}$ to be the least common denominator of all the coefficients of $P(x)$. Multiplying both sides of the above equation by $a_0^{d-1}$ and letting $y = a_0 x$, it follows that we arrive at the equation
\[
y^d + b_1 y^{d-1} + \cdots + b_d = b_{d+1} \prod_{k=1}^{n} k,
\]
where $b_i = a_i a_0^{i-1}$ for $i = 1, \ldots, d$, and $b_{d+1} = a_{d+1} a_0^{d-1}$. Note now that $y$ is an integer because for every fixed positive integer $n$ the above equation shows that the rational number $y$ is a root of a monic polynomial with integer coefficients; hence, an algebraic integer, and therefore a rational integer. With the substitution $z = y + b_1/d$, we may rewrite the above equation as
\[
z^d + c_2 z^{d-2} + \cdots + c_d = c_{d+1} \prod_{k=1}^{n} k,
\]
where $c_i$ are rational numbers whose denominator divides $d^d$. Finally, we multiply the above equation by $d^d$ and use the substitution $t = dz$ to arrive at
\[
t^d + e_2 t^{d-2} + \cdots + e_d = e_{d+1} \prod_{k=1}^{n} k,
\]
where $t$ and $e_i$ are integers for $i = 2, \ldots, d + 1$. Let $j \leq d$ be the largest index in $\{2, \ldots, d\}$ such that $e_j \neq 0$. If this index does not exist, then the
above equation is
\[ t^d = e_{d+1} \prod_{k \times n}^{n} k. \]

By the Prime Number Theorem, for large \( n \), the interval \( (n/2, n) \) contains \( \approx n/(2 \log n) \) prime numbers \( p \) and none of those divides \( n \). Since \( d > 1 \) and \( t \) is an integer, it follows that every such prime number divides \( e_{d+1} \).

In particular, \( n/2 < e_{d+1} \), which shows that \( n \) is bounded.

Assume now that \( j \) exists and rewrite the equation as
\[ (5) \quad t^j + (e_2 t^{j-2} + \cdots + e_j) = e_{d+1} \prod_{k \times n}^{n} k. \]

Since for large \( t \) we have that \( |t^d + e_2 t^{d-2} + \cdots + e_d| \gg |t|^d \), it follows that
\[ |t|^d \gg |t^d + e_2 t^{d-2} + \cdots + e_d| \gg \prod_{k \times n}^{n} k \gg (n-1)!^{1/2}, \]
therefore, by taking logarithms and invoking Stirling’s formula, we get
\[ (6) \quad |t| \geq \exp \left( \frac{1}{2d} (1 + o(1)) n \log n + O(1) \right). \]

We now set \( A = t^j, \ B = (e_2 t^{j-2} + \cdots + e_j), \ C = e_{d+1} \prod_{k \times n}^{n} k, \) and we apply the \( ABC \) conjecture to equation (5). We note that our \( A, B, C \) are not necessarily coprime, but their greatest common divisor is \( O(1) \). Indeed, let \( D_1 = \gcd(t, e_j) \). Clearly, \( D_1 \leq |e_j| \), and
\[ \gcd(A, B) = \gcd(t^j, e_2 t^{j-2} + \cdots + e_j) \mid (\gcd(t, e_2 t^{j-2} + \cdots + e_j))^j \mid D_1^d. \]

Thus, we may apply the \( ABC \)-conjecture and get that
\[ |t|^j \ll \left( |t||B| \prod_{p \leq n} p \right)^{1+\varepsilon} \ll |t|^{(j-1)(1+\varepsilon)} \cdot 4^{(1+\varepsilon)n}. \]

Choosing \( \varepsilon = 1/j \), we get that \( (j-1)(1+\varepsilon) = j - 1/j \), and that the inequality
\[ |t|^{1/j} \ll 4^{1+\varepsilon}n < 4^{2n} \]
holds. This last inequality leads to
\[ (7) \quad |t| \leq \exp \left( 2jn \log 4 + O(1) \right) \leq \exp(2dn \log 4 + O(1)). \]

Comparing (6) and (7), we get
\[ 2dn \log 4 - \frac{1}{2d} (1 + o(1)) n \log n = O(1), \]
which certainly implies that \( n \) is bounded. \( \Box \)

Dabrowski (see [6]), showed that if \( P(x) = x^2 - A \), where \( A \) is an integer which is not a perfect square, then equation (2) has only finitely many solutions. We consider the diophantine equation

\[
(8) \quad x^2 - A = \prod_{k=1}^{n} k,
\]

and prove the following result.

**Theorem 3.** If the integer \( A \) is not a perfect square, and \( n \) and \( x \) are positive integers satisfying equation (8), then either

\[
 n \leq p \text{ or } n = 2p,
\]

where \( p \) is the smallest prime such that \( (\frac{A}{p}) = -1 \). Here, \( (\frac{\cdot}{p}) \) stands for the Legendre symbol. In particular, equation (8) has only finitely many positive integer solutions.

Before proving Theorem 3, we need the following Lemma.

**Lemma 4.** Every prime \( p \leq n \) divides \( \prod_{k=1}^{n} k \), except when \( n = p, 2p \), cases in which \( p \) is the only prime \( \leq n \) which does not divide \( \prod_{k=1}^{n} k \).

**Proof.** Suppose that \( p \leq n \) and that \( n \neq p, 2p \). If \( p \in (n/2, n) \), then \( p \) does not divide \( n \), therefore it divides \( \prod_{k=1}^{n} k \). We now assume that \( p < \frac{n}{2} \).

Hence, there exists a positive integer \( i \) such that

\[
 \frac{n}{2} < 2^i p < n.
\]

(i) If \( \frac{n}{2} < 2^i p < n \), then \( 2^i p \) does not divide \( n \), and so it divides \( \prod_{k=1}^{n} k \).

(ii) If \( \frac{n}{2} = 2^i p \), then \( 3 \cdot 2^{i-1} p < n \), and does not divide \( n \), therefore it divides \( \prod_{k=1}^{n} k \).

\( \Box \)

**Proof of Theorem 3.** Since \( A \) is not a perfect square, there exists a prime \( p \) such that \( (\frac{A}{p}) = -1 \). Then \( p \) does not divide \( \prod_{k=1}^{n} k \). Lemma 4 now shows that either \( n \leq p \) or \( n = 2p \).

In the general case in which \( A \) is any integer in equation (8), we have the conditional result given in the following theorem.

**Theorem 5.** If the weak form of Hall’s conjecture is true, then equation (8) has only finitely many solutions.
The weak form of Hall’s conjecture is a special case of the $ABC$-conjecture and asserts that for every $\varepsilon > 0$, there exists a constant $C(\varepsilon)$ depending only on $\varepsilon > 0$, such that if $x, y, \text{ and } k$ are nonzero integers satisfying $x^2 = y^3 + k$, then 

$$\max(|x^2|, |y^3|) \leq C(\varepsilon)|k|^{6+\varepsilon}.$$ 

**Proof of Theorem 5.** We assume that $n \geq 3$. Let $d$ and $y$ be the two integers with $d$ cubefree such that $\prod_{k=1}^{n} k = dy^3$. Then, from Chebyshev’s bound we obtain that

$$d \leq \left( \prod_{p < n} p \right)^2 < 4^{2(n-1)}.$$ 

Equation (8) gives

$$dy^3 + A = x^2,$$ 

so

$$Y^3 + d^2 A = X^2,$$ 

where $X = dx$ and $Y = dy$. Taking $\varepsilon = 1$ in the weak form of Hall’s conjecture we get that

$$d^2 \prod_{k=1}^{n} k = d^3 y^3 = Y^3 \leq C(1)|d^2 A|^7.$$ 

Since

$$(n-1)!^{1/2} \leq \prod_{k=1}^{n} k,$$ 

it follows, from Stirling’s formula, that

$$(4(n-1)^{(n-1)}e^{-(n-1)})^{1/2} \leq (n-1)!^{1/2} \leq \prod_{k=1}^{n} k.$$ 

Hence,

$$(4(n-1)^{(n-1)}e^{-(n-1)})^{1/2} \leq \prod_{k=1}^{n} k \ll |d^2|^{6}|A|^7 \ll 4^{24(n-1)}|A|^7.$$ 

Thus,

$$\left( \frac{n-1}{4^{48}} \right)^{n-1} \ll |A|^7.$$ 

This proves that $n$ is bounded. \qed
We considered equation (8) with $A = 1$, namely

\begin{equation}
\prod_{k=1}^{n} k + 1 = y^2,
\end{equation}

and did some computations. Except from the obvious solutions $n = 4$ and $n = 5$, we didn’t find any other solution for equation (9) up to $n = 10^5$.

Finally, we look at yet another variant of the Brocard-Ramanujan diophantine equation, namely

\begin{equation}
1 + \prod_{\substack{k \leq n \\ \gcd(k,n) = 1}} k = y^2.
\end{equation}

**Theorem 6.** Suppose that there exist integers $n > 4$ and $y$ satisfying equation (10). Then either $n$ is equal to $p^\alpha$ or $2p^\alpha$ for some prime $p$ and positive integer $\alpha$, or all odd primes dividing $n$ are $\pm 1$ (mod 8).

**Proof.** We know from Gauss generalization to Wilson’s Theorem (see [17]) that

\[
\prod_{\substack{k \leq n \\ \gcd(k,n) = 1}} k = \begin{cases}
-1 & \text{if } n = 4, p^\alpha, 2p^\alpha, \\
1 & \text{otherwise}.
\end{cases}
\]

Thus, if $n \neq 4, p^\alpha, 2p^\alpha$, then

\[
\prod_{\substack{k \leq n \\ \gcd(k,n) = 1}} k + 1 \equiv 2 \pmod{n}.
\]

This implies that $y^2 \equiv 2 \pmod{n}$. In particular, $y^2 \equiv 2 \pmod{q}$ holds for all odd prime factors $q$ of $n$. Hence, $(\frac{2}{q}) = 1$, leading to the conclusion that $q \equiv \pm 1 \pmod{8}$.

We remark that results of Landau (see pages 668–669 in [10]), together with the Prime Number Theorem, imply that if $x$ is any positive real number, then the number of positive integers $n \leq x$ such that $n = p^\alpha, 2p^\alpha$, or $n$ is free of prime factors $\equiv \pm 3 \pmod{8}$ is $\ll x/\sqrt{\log x}$. In particular, the set of $n$ for which equation (10) can have a positive integer solution $y$ is of asymptotic density zero, which is an analogue of the result of Berend and Osgood from [1] for the particular polynomial $P(X) = X^2 - 1$ and our variant of the Brocard-Ramanujan equation.

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