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On some subgroup chains related to Kneser's theorem

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Résumé. Un résultat récent de Balandraud démontre que pour toute partie S d'un groupe abélien G, il existe un sous-groupe H non-trivial tel que l'inégalité $|TS| \leq |T| + |S| - 2$ n'a lieu que si $H \subset Stab(TS)$. On remarque que le théorème de Kneser n'implique que l'inégalité $\{1\} \neq Stab(TS)$.

Ce renforcement du théorème de Kneser se déduit des propriétés plaisantes d'un certain ensemble partiellement ordonné étudié par Balandraud. Nous considérons un ensemble partiellement ordonné analogue pour les groupes non forcément abéliens et à l'aide d'outils classiques de théorie additive des nombres, généralisons certains des résultats suscités. En particulier nous obtenons des démonstrations courtes des résultats de Balandraud dans le cas abélien.

Abstract. A recent result of Balandraud shows that for every subset S of an abelian group G there exists a non trivial subgroup H such that $|TS| \leq |T| + |S| - 2$ holds only if $H \subset Stab(TS)$. Notice that Kneser's Theorem only gives $\{1\} \neq Stab(TS)$.

This strong form of Kneser's theorem follows from some nice properties of a certain poset investigated by Balandraud. We consider an analogous poset for nonabelian groups and, by using classical tools from Additive Number Theory, extend some of the above results. In particular we obtain short proofs of Balandraud's results in the abelian case.

1. Introduction

In order to avoid switching from multiplicative to additive notation, all groups will be written multiplicatively.

Kneser's addition theorem [5] states that if S, T are finite subsets of an abelian group G then $|ST| \leq |S| + |T| - 2$ holds only if ST is periodic (i.e, there is a non trivial subgroup H such that $HST = ST$.) Kneser's Theorem is a fundamental tool in Additive number Theory. Proofs of this result may be found in [4, 5, 6, 7, 8, 9, 11].

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In all previously known proofs of Kneser's Theorem, the subgroup H depends crucially on both sets S and T. With the goal of breaking this double dependence in S and T, Balandraud investigated in recent work [1, 2] the properties of a combinatorial poset that we now present.

Let S be a finite subset containing 1 of a group G and let $X \subset G$. Following Lee [7], we introduce the following notion (where the reference to S is implicit): let

$$\widetilde{X} = \{x \in G : xS = XS\}.$$

A subset X such that $\widetilde{X} = X$ is called a *full* subset. Equivalently, a subset X is a full subset if for all $z \notin X$ we have $zS \not\subset XS$.

This notion is rediscovered by Balandraud in [1, 2] where full subsets are named "cells", and also by Grynkiewicz [3] where the term "nonextendible subset" is used. The following lemma is straightforward:

Lemma 1.1 ([1, 2, 7]). *Let G be a group and $1 \in S \subset G$ be a finite subset. Then $X \subset \widetilde{X}$ and $\widetilde{X}S = XS$.*

We shall use a slightly modified version of Balandraud's terminology:

A cell (or a full subset) X is called a *u-cell* if $|XS| - |X| = u$. A u-cell with minimal cardinality is called a *u-kernel* (of S).

Throughout the paper, by a cell we always mean a cell of S.

Balandraud showed that, for a finite set S in an abelian group G, in the poset of j–cells containing the unity ordered by inclusion with $1 \le j \le |S| - 2$, the set of kernels forms a chain of subgroups. Moreover, if there exists a u–cell, then there is a unique u–kernel containing the unit element which is contained in all u–cells containing the unit element.

One of the consequences of this work is a new proof and the following strengthening of Kneser's Theorem:

Theorem 1.1 (Balandraud). *For any non-empty finite subset S of an abelian group G, there exists a finite subgroup H of G such that for any finite subset T of G one of the following conditions hold :*

- $|TS| \ge |T| + |S| - 1$
- $HTS = TS$ *and* $|TS| \le |HS| + |HT| - |H|$

As far as the authors are aware this is a surprising and strong formulation that was not observed before and does not follow straightforwardly from the classical forms of Kneser's Theorem.

The purpose of the present note is to give a short proof that, in the poset of j–cells *that are subgroups* ordered by inclusion with $0 \le j \le |S| - 1$, the set of kernels forms a chain of subgroups. Moreover, each u-kernel of this poset is unique and contained in all u–cells of this poset. The proof works for general, not necessarily abelian groups.

From this statement Kneser's theorem allows one to deduce Balandraud's results for the abelian case, and in particular Theorem 1.1. Kneser's Theorem [5, 6] has several equivalent forms. We use the following one; see e.g [4, 9]:

Theorem 1.2 (Kneser). *Let G be an abelian group and $X, Y \subset G$ be finite subsets such that $|XY| \leq |X| + |Y| - 2$. Then*

$$|XY| = |HX| + |HY| - |H|,$$

where $H = stab(XY) = \{x : xXY = XY\}$.

Our main tool is the following Theorem of Olson [10, Theorem 2]. We give an equivalent formulation here where we use left–cosets instead of right–cosets.

Theorem 1.3 (Olson [10]). *Let X, Y be finite subsets of a group G, and let H and K be subgroups such that $HX = X$, $KY = Y$ and $KX \neq X$, $HY \neq Y$. Then*

$$|X \setminus Y| + |Y \setminus X| \geq |H| + |K| - 2|H \cap K|.$$

In particular either $|X \setminus Y| \geq |H| - |H \cap K|$ or $|Y \setminus X| \geq |K| - |H \cap K|$.

We shall use the following lemma.

Lemma 1.2 ([1, 2]). *Let G be a group and $1 \in S \subset G$ be a finite subset. Then the intersection of two cells M_1, M_2 of S is a cell of S.*

Proof. Let $x \notin M_1 \cap M_2$. There is i with $x \notin M_i$. Then $xS \not\subset M_iS$. Hence $xS \not\subset (M_1 \cap M_2)S$. □

We can now state our main result, namely Theorem 2.1 below.

2. An application of Olson's Theorem

Balandraud [1, 2] proved that, in the abelian case, the set of kernels containing the unit element and ordered by inclusion is a chain of subgroups. In the non abelian case we can prove only that the set of kernels that are *subgroups* forms a chain. The abelian case can then be easily recovered, since Kneser's Theorem implies (as we shall see below) that a kernel containing the unit element is a subgroup.

Theorem 2.1. *Let S be a finite subset containing 1 of a group G. Let M be a u–kernel of S which is a subgroup. Let N be a subgroup which is a v–cell and suppose $u, v \leq |S| - 1$.*

(i) *If either N is a v–kernel or $u = v$ then $M \subset N$ or $N \subset M$.*
(ii) *If N is a v–kernel and $v \leq u$ then $M \subset N$.*

Proof. Suppose that $M \not\subset N$ and $N \not\subset M$. Note that, since M is a cell, if $NMS = MS$ then $NM = M$, thus $N \subset M$ against our assumption. Hence we may assume $NMS \neq MS$ and similarly $MNS \neq NS$. By Theorem 1.3 we have one of the two following cases.

Case 1: $|MS| - |(MS) \cap (NS)| = |(MS) \setminus (NS)| \geq |M| - |M \cap N|$. It follows that $|(M\cap N)S| - |M\cap N| \leq |(MS)\cap(NS)| - |M\cap N| \leq |MS| - |M|$. On the other hand we have $u = |MS| - |M| < |S| \leq |(M \cap N)S|$. Since $|MS| - |M|$ is a multiple of $|M \cap N|$ we have

$$u = |MS| - |M| = |(M \cap N)S| - |M \cap N|.$$

By Lemma 1.2, $M \cap N$ is a cell. Since M is a u–kernel, we have $M \cap N = M$, a contradiction.

Case 2: $|NS| - |(NS) \cap (MS)| = |(NS) \setminus (MS)| \geq |N| - |N \cap M|$. It follows that $|(N\cap M)S| - |N\cap M| \leq |(NS)\cap(MS)| - |N\cap M| \leq |NS| - |N|$. On the other hand we have $|NS| - |N| < |S| \leq |(N\cap M)S|$. Since $|NS| - |N|$ is a multiple of $|N \cap M|$ we have

$$|NS| - |N| = |(N \cap M)S| - |N \cap M|. \tag{1}$$

Assume first $u = v$. Then $u = |MS| - |M| = |NS| - |N| = |(N \cap M)S| - |N \cap M|$. Since M is a u–kernel, we have $M \cap N = M$, a contradiction.

Assume that N is a v–kernel. Then (1) implies $N \cap M = N$, a contradiction. This proves (i).

Assume now that $v \leq u$. Suppose $M \not\subset N$. By (i) we have $N \subset M$, which implies in particular that $|MS| - |M|$ is a multiple of N. Therefore, from $u = |MS| - |M| < |S| \leq |NS|$ we have $u = |MS| - |M| \leq |NS| - |N| = v$ which gives $u = v$. But then $M \not\subset N$ and $N \subset M$ imply $|N| < |M|$, and since N is now a u–cell, this contradicts M being a u–kernel. □

We can now deduce Balandraud's description for kernels and cells :

Corollary 2.1 (Balandraud [1, 2])**.** *Let G be an abelian group and $S \subset G$ be a finite subset with $1 \in S$. Let M be a u–kernel of S containing 1 with $1 \leq u \leq |S| - 2$. Then,*

(i) *M is a subgroup.*
(ii) *Each u-cell is M–periodic.*
(iii) *Each v–kernel containing 1 with $u < v \leq |S| - 2$ is a proper subgroup of M.*

Proof. Let X be a u-cell with $u \leq |S| - 2$. By Kneser's Theorem, the inequality $|XS| - |X| = u \leq |S| - 2$ implies

$$u = |XS| - |HX| = |HS| - |H|, \tag{2}$$

where H is the stabilizer of XS. Since X is a cell and $HXS = XS$, we have $X = HX$. Note that, since G is abelian, $(\{y\} \cup H)S = HS$ implies $y \in Stab(HS) \subset Stab(XS)$, so that $y \in H$. This observation and (2) imply that H is an u–cell. In particular, by taking $X = M$, the period K of MS is a u–cell. Since $KMS = MS$ and M is a u–cell, we have $K \subset KM \subset M$. Since M is a u–kernel we have $M = K$. This proves (i).

Now let H be the stabilizer of XS, where X is a u–cell. As shown in the preceding paragraph H is also a u–cell. By Theorem 2.1 we have $M \subset H$ and thus $MH = H$. Since X is a cell and $HXS = XS$, we have $X = HX = MHX$. Hence $X \subset MX \subset MHX = X$ implies $X = MX$. This proves (ii).

Finally, by (i), a v–kernel N is a subgroup. By Theorem 2.1 we have $N \subset M$. □

From Corollary 2.1, one can deduce Theorem 1.1.

Proof of Theorem 1.1. We may assume without loss of generality that $1 \in S$.

Case 1: There is no m–cell for any $1 \leq m \leq |S| - 2$.

- either we have $|TS| \geq |S| + |T| - 1$ for any non-empty finite T, in which case the theorem clearly holds with $H = \{1\}$.
- or there exists some non-empty finite T such that $|TS| \leq |S| + |T| - 2$. Without loss of generality, we may also suppose $1 \in T$. Put $X = \widetilde{T}$. By Lemma 1.1 we have $XS = TS$. Now X is an m–cell with $m = |XS| - |X| \leq |TS| - |T| \leq |T| - 2$. But since no such cell exists for $1 \leq m \leq |S| - 2$, we have that $X = T$. Then T itself must be a cell (a 0-cell) i.e. $|TS| = |T|$. We therefore have $HT = TH = T = TS = HTS$ where H is the (necessarily finite) subgroup generated by S. We have just proved that the theorem holds in this case with $H = \langle S \rangle$.

Case 2: There exists an m–cell with $1 \leq m \leq |S| - 2$. We may therefore consider the largest integer $u \leq |S| - 2$ for which S admits a u–cell. Let H be the u–kernel containing 1. Note that $u \leq |S| - 2$ implies that H is different from $\{1\}$. Now let T be any finite non-empty subset such that $|TS| - |T| \leq |S| - 2$. We shall prove that $HTS = TS$.

Put $X = \widetilde{T}$. By Lemma 1.1, $XS = TS$. Note that we then have $|XS| - |X| \leq |TS| - |T| \leq |S| - 2$, so that X is a v–cell for some $v \leq u$. By Corollary 2.1 (ii) we have $TS = XS = MXS = MTS$ where M is the v-kernel containing 1. By part (i) of Corollary 2.1, H is a subgroup of M so that $TS = XS = HTS$ as well.

Finally, $|ST| \leq |HS| + |HT| - |H|$ follows from $|ST|$ being a multiple of $|H|$. □

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