Gabriele NEBE et Kristina SCHINDELAR

S-extremal strongly modular lattices

<http://jtnb.cedram.org/item?id=JTNB_2007__19_3_683_0>
S-extremal strongly modular lattices

par Gabriele NEBE et Kristina SCHINDELAR

Résumé. Un réseau fortement modulaire est dit s-économique, s'il maximise le minimum du réseau et son ombre simultanément. La dimension des réseaux s-économiques dont le minimum est pair peut être bornée par la théorie des formes modulaires. En particulier de tels réseaux sont extrémaux.

Abstract. S-extremal strongly modular lattices maximize the minimum of the lattice and its shadow simultaneously. They are a direct generalization of the s-extremal unimodular lattices defined in [6]. If the minimum of the lattice is even, then the dimension of an s-extremal lattices can be bounded by the theory of modular forms. This shows that such lattices are also extremal and that there are only finitely many s-extremal strongly modular lattices of even minimum.

1. Introduction.

Strongly modular lattices have been defined in [11] to generalize the notion of unimodular lattices. For square-free \( N \in \mathbb{N} \) a lattice \( L \subseteq (\mathbb{R}^n, (.,.)) \) in Euclidean space is called strongly \( N \)-modular, if \( L \) is integral, i.e. contained in its dual lattice

\[
L^* = \{x \in \mathbb{R}^n \mid (x, \ell) \in \mathbb{Z} \forall \ell \in L\}
\]

and isometric to its rescaled partial dual lattices \( \sqrt{d}(L^* \cap \frac{1}{d}L) \) for all \( d | N \). The simplest strongly modular lattice is

\[
C_N := \perp_{d|N} \sqrt{d}\mathbb{Z}
\]

of dimension \( \sigma_0(N) \), the number of divisors of \( N \). For

\[
N \in \mathcal{L} = \{1, 2, 3, 5, 6, 7, 11, 14, 15, 23\}
\]

which is the set of square-free numbers such that \( \sigma_1(N) = \sum_{d|N} d \) divides 24, Theorems 1 and 2 in [13] bound the minimum \( \min(L) := \min\{(\ell, \ell) \mid \)
of a strongly $N$-modular lattice that is rational equivalent to $C_N^k$ by
\begin{equation}
\min(L) \leq 2 + 2\left\lfloor \frac{k}{s(N)} \right\rfloor, \text{ where } s(N) = \frac{24}{\sigma_1(N)}.
\end{equation}
For $N \in \{1, 3, 5, 7, 11\}$ there is one exception to this bound: $k = s(N) - 1$ and $L = S^{(N)}$ of minimum 3 (see [13, Table 1]). Lattices achieving this bound are called extremal.
For an odd strongly $N$-modular lattice $L$ let
\[ S(L) = L_0^* \setminus L^* \]
denote the shadow of $L$, where $L_0 = \{ \ell \in L \mid (\ell, \ell) \in 2\mathbb{Z} \}$ is the even sublattice of $L$. For even strongly $N$-modular lattices $L$ let $S(L) := L^*$. Then the shadow-minimum of an $N$-modular lattice is defined as
\[ \text{smin}(L) := \min\{N(x, x) \mid x \in S(L)\}. \]
In particular smin$(L) = 0$ for even lattices $L$. In this paper we show that for all $N \in \mathcal{L}$ and for all strongly $N$-modular lattices $L$ that are rational equivalent to $C_N^k$
\[ 2\min(L) + \text{smin}(L) \leq k\frac{\sigma_1(N)}{2} + 2 \quad \text{if } N \text{ is odd and} \]
\[ \min(L) + \text{smin}(L) \leq k\frac{\sigma_1(N/2)}{2} + 1 \quad \text{if } N \text{ is even} \]
with the exceptions $L = S^{(N)}$, $k = s(N) - 1$ ($N \neq 23, 15$ odd) where the bound has to be increased by 2 and $L = O^{(N)}$, $k = s(N)$ and $N$ even, where the bound has to be increased by 1 (see [13, Table 1] for the definition of the lattices $S^{(N)}$, $O^{(N)}$ and also $E^{(N)}$). Lattices achieving this bound are called $s$-extremal. The theory of modular forms allows us to bound the dimension $\sigma_0(N)k$ of an $s$-extremal lattice of even minimum $\mu$ by
\[ 2k < \mu s(N). \]
In particular $s$-extremal lattices of even minimum are automatically extremal and hence by [12] there are only finitely many strongly $N$-modular $s$-extremal lattices of even minimum. This is also proven in Section 3, where explicit bounds on the dimension of such $s$-extremal lattices and some classifications are obtained. It would be interesting to have a similar bound for odd minimum $\mu \geq 3$. Of course for $\mu = 1$, the lattices $C_N^k$ are $s$-extremal strongly $N$-modular lattices of minimum 1 for arbitrary $k \in \mathbb{N}$ (see [9]), but already for $\mu = 3$ there are only finitely many $s$-extremal unimodular lattices of minimum 3 (see [10]). The $s$-extremal strongly $N$-modular lattices of minimum $\mu = 2$ are classified in [9] and some $s$-extremal lattices of minimum 3 are constructed in [15]. For all calculations we used the computer algebra system MAGMA [2].
2. S-extremal lattices.

For a subset $S \subset \mathbb{R}^n$, which is a finite union of cosets of an integral lattice we put its theta series

$$\Theta_S(z) := \sum_{v \in S} q^{(v,v)}, \quad q = \exp(\pi i z).$$

The theta series of strongly $N$-modular lattices are modular forms for a certain discrete subgroup $\Gamma_N$ of $SL_2(\mathbb{R})$ (see [13]). Fix $N \in \mathcal{L}$ and put

$$g_1^{(N)}(z) := \Theta_{CN}(z) = \prod_{d \mid N} \Theta_{\mathbb{Z}}(dz) = \prod_{d \mid N} \prod_{j=1}^{\infty} (1 - q^{2dj})(1 + q^{d(2j-1)})^2$$

(see [4, Section 4.4]). Let $\eta$ be the Dedekind eta-function

$$\eta(z) := q^{1/24} \prod_{j=1}^{\infty} (1 - q^{2j})$$

and put $\eta^{(N)}(z) := \prod_{d \mid N} \eta(dz)$.

If $N$ is odd define

$$g_2^{(N)}(z) := \left( \frac{\eta^{(N)}(z/2)\eta^{(N)}(2z)}{\eta^{(N)}(z)^2} \right)^{s(N)}$$

and if $N$ is even then

$$g_2^{(N)}(z) := \left( \frac{\eta^{(N/2)}(z/2)\eta^{(N/2)}(4z)}{\eta^{(N/2)}(z)\eta^{(N/2)}(2z)} \right)^{s(N)}.$$

The meromorphic function $g_2^{(N)}$ generates the field of modular functions of $\Gamma_N$. It is a power series in $q$ starting with

$$g_2^{(N)}(z) = q - s(N)q^2 + \ldots.$$

Using the product expansion of the $\eta$-function we find that

$$q^{-1}g_2^{(N)}(z) = \prod_{d \mid N} \prod_{j=1}^{\infty} (1 + q^{d(2j-1)})^{-s(N)}.$$

For even $N$ one has to note that

$$q^{-1}g_2^{(N)}(z) = \prod_{d \mid N, d \neq N/2} \prod_{j=1}^{\infty} \left( \frac{1 + q^{4dj}}{1 + q^{dj}} \right)^{s(N)}$$

$$= \prod_{d \mid N, d \neq N/2} \prod_{j=1}^{\infty} (1 + q^{2d(2j-1)})^{-s(N)}(1 + q^{d(2j-1)})^{-s(N)}.$$. 
By [13, Theorem 9, Corollary 3] the theta series of a strongly $N$-modular lattice $L$ that is rational equivalent to $C_N^k$ is of the form

\begin{equation}
\Theta_L(z) = g_1^{(N)}(z)^k \sum_{i=0}^{b} c_i g_2^{(N)}(z)^i
\end{equation}

for $c_i \in \mathbb{R}$ and some explicit $b$ depending on $k$ and $N$. The theta series of the rescaled shadow $S := \sqrt{N}S(L)$ of $L$ is

\begin{equation}
\Theta_S(z) = s_1^{(N)}(z)^k \sum_{i=0}^{b} c_i s_2^{(N)}(z)^i
\end{equation}

where $s_1^{(N)}$ and $s_2^{(N)}$ are the corresponding “shadows” of $g_1^{(N)}$ and $g_2^{(N)}$ as defined in [13] (see also [9]).

If $N$ is odd, then

\[
s_1^{(N)} = 2^{\sigma_0(N)} q^{\sigma_1(N)/4}(1 + q^2 + \ldots)
\]

and

\[
s_2^{(N)} = 2^{-s(N)\sigma_0(N)/2} (-q^{-2} + s(N) + \ldots).
\]

If $N$ is even, then

\[
s_1^{(N)} = 2^{\sigma_0(N)/2} q^{\sigma_1(N)/2}(1 + 2q + \ldots),
\]

\[
s_2^{(N)} = 2^{-s(N)\sigma_0(N)/2} (-q^{-1} + s(N) + \ldots).
\]

**Theorem 2.1.** Let $N \in \mathcal{L}$ be odd and let $L$ be a strongly $N$-modular lattice in the genus of $C_N^k$. Let $\sigma := \min(L)$ and let $\mu := \min(L)$. Then

\[
\sigma + 2\mu \leq k \frac{\sigma_1(N)}{4} + 2
\]

unless $k = s(N) - 1$ and $\mu = 3$. In the latter case the lattice $S^{(N)}$ is the only exception (with $\min(S^{(N)}) = 3$ and $\min(S^{(N)}) = 4 - \sigma_1(N)/4$).

**Proof.** The proof is a straightforward generalization of the one given in [6]. We always assume that $L \neq S^{(N)}$ and put $g_1 := g_1^{(N)}$ and $g_2 := g_2^{(N)}$. Let $m := \mu - 1$ and assume that $\sigma + 2\mu \geq k \frac{\sigma_1(N)}{4} + 2$. Then from the expansion of

\[
\Theta_S = \sum_{j=\sigma}^{\infty} b_j q^j = s_1^{(N)}(z)^k \sum_{i=0}^{b} c_i s_2^{(N)}(z)^i
\]

in formula (2.2) above we see that $c_i = 0$ for $i > m$ and (2.1) determines the remaining coefficients $c_0 = 1$, $c_1, \ldots, c_m$ uniquely from the fact that

\[
\Theta_L = 1 + \sum_{j=\mu}^{\infty} a_j q^j \equiv 1 \pmod{q^{m+1}}.
\]
The number of vectors of norm \( k \frac{\sigma_1(N)}{4} + 2 - 2 \mu \) in \( S = \sqrt{N}S(L) \) is

\[
c_m(-1)^m 2^{-m \sigma_0(N)} s(N)/2 + k \sigma_0(N)
\]

and nonzero, iff \( c_m \neq 0 \). The expansion of \( g_1^{-k} \) in a power series in \( g_2 \) is given by

\[
g_1^{-k} = \sum_{i=0}^{m} c_i g_2^i - a_{m+1} q^{m+1} g_1^{-k} + * q^{m+2} + \ldots = \sum_{i=0}^{\infty} \tilde{c}_i g_2^i
\]

with \( \tilde{c}_i = c_i \ (i = 0, \ldots, m) \) and \( \tilde{c}_{m+1} = -a_{m+1} \). Hence Bürmann-Lagrange (see for instance [16]) yields that

\[
c_m = \frac{1}{m!} \frac{\partial^{m-1}}{\partial q^{m-1}} \left( \frac{\partial}{\partial q} (g_1^{-k})(q g_2^{-1})^m \right)_{q=0} = \frac{-k}{m} \left( \text{coeff. of } q^{m-1} \text{ in } (g_1'/g_1)/f_1 \right)
\]

with \( f_1 = (q^{-1} g_2)^m g_1^k \). Using the product expansion of \( g_1 \) and \( g_2 \) above we get

\[
f_1 = \prod_{d \mid N} \prod_{j=1}^{\infty} (1 - q^{2dj})^k (1 + q^{d(2j-1)})^{2k-s(N)m}.
\]

Since

\[
g_1'/g_1 = \sum_{d \mid N} \frac{\partial}{\partial q} \frac{\theta_3(dz)}{\theta_3(dz)}
\]

is alternating as a sum of alternating power series, the series \( P := g_1'/g_1/f_1 \) is alternating, if \( 2k - s(N)m \geq 0 \). In this case all coefficients of \( P \) are nonzero, since all even powers of \( q \) occur in \((1 - q^2)^{-1}\) and \( g_1'/g_1 \) has a non-zero coefficient at \( q^1 \). Otherwise write

\[
P = g_1' \prod_{d \mid N} \prod_{j=1}^{\infty} \frac{(1 + q^{d(2j-1)})^{s(N)m-2k-2}}{(1 - q^{2dj})^{k+1}}.
\]

If \( 2k - s(N)m < -2 \) then \( P \) is a positive power series in which all \( q \)-powers occur. Hence \( c_m < 0 \) in this case. If the minimum \( \mu \) is odd then this implies that \( b_\sigma < 0 \) and hence the nonexistence of an \( s \)-extremal lattice of odd minimum for \( s(N)m - 2 > 2k \). Assume now that \( 2k - s(N)m = -2 \), i.e. \( k = s(N)m/2 - 1 \). By the bound in [13] one has

\[
m + 1 \leq 2 \left[ \frac{k}{s(N)} \right] + 2 = 2 \left[ \frac{m}{2} - \frac{1}{s(N)} \right] + 2.
\]

This is only possible if \( m \) is odd. Since \( g_1' \) has a non-zero constant term, \( P \) contains all even powers of \( q \) in particular the coefficient of \( q^{m-1} \) is positive. The last case is \( 2k - s(N)m = -1 \). Then clearly \( m \) and \( s(N) \) are
odd and \( P = GH^{(m-1)/2} \) where
\[
G = g' \prod_{d | N} \prod_{j=1}^{\infty} (1 + q^{d(2j-1)})^{-1}(1 - q^{2dj})^{-(s(N)+1)/2}
\]
and
\[
H = \prod_{d | N} \prod_{j=1}^{\infty} (1 - q^{2dj})^{-s(N)}.
\]
If \( m \) is odd then the coefficient of \( P \) at \( q^{m-1} \) is
\[
\int_{c+1+iy_0}^{c-1+iy_0} e^{-(m-1)\pi i z} G(e^{\pi i z}) H(e^{\pi i z})^{(m-1)/2} dz
\]
which may be estimated by the saddle point method as illustrated in [8, Lemma 1]. In particular this coefficient grows like a constant times
\[
\frac{e^{(m-1)/2}}{m^{1/2}}
\]
where \( c = F(y_0), F(y) = e^{2\pi y} H(e^{-2\pi y}) \) and \( y_0 \) is the first positive zero of \( F' \). Since \( c > 0 \) and also \( F''(y_0) > 0 \) and the coefficient of \( P \) at \( q^{m-1} \) is positive for the first few values of \( m \) (we checked 10000 values), this proves that \( b_\sigma > 0 \) also in this case. \( \square \)

To treat the even \( N \in \mathcal{L} \), we need two easy (probably well known) observations:

**Lemma 2.1.** Let
\[
f(q) := \prod_{j=1}^{\infty} (1 + q^{2j-1})(1 + q^{2(2j-1)}).
\]
Then the \( q \)-series expansion of \( 1/f \) is alternating with non zero coefficients at \( q^a \) for \( a \neq 2 \).

**Proof.**
\[
1/f = \prod_{j=1}^{\infty} (1 + q^{2j-1} + q^{2(2j-1)} + q^{3(2j-1)})^{-1} = \prod_{j=1}^{\infty} \sum_{\ell=0}^{\infty} q^{4\ell(2j-1)} - q^{(4\ell+1)(2j-1)}
\]
is alternating as a product of alternating series. The coefficient of \( q^a \) is non-zero, if and only if \( a \) is a sum of numbers of the form \( 4\ell(2j-1) \) and \( (4\ell + 1)(2j-1) \) with distinct \( \ell \). One obtains 0 and 1 with \( \ell = 0 \) and \( j = 1 \) and \( 3 = 1(2 \cdot 2 - 1) \) and \( 6 = 1 + 5 \). Since one may add arbitrary multiples of 4, this shows that the coefficients are all non-zero except for the case that \( a = 2 \). \( \square \)
Lemma 2.2. Let \( g_1 := g_1^{(N)} \) for even \( N \) such that \( N/2 \) is odd and denote by \( g_1' \) the derivative of \( g_1 \) with respect to \( q \). Then \( \frac{g_1'}{g_1} \) is an alternating series with non-zero coefficients for all \( q^a \) with \( a \neq 1 \mod 4 \). The coefficients for \( q^a \) with \( a \equiv 1 \mod 4 \) are zero.

Proof. Using the product expansion

\[
g_1 = \prod_{d|N} \prod_{j=1}^{\infty} (1 - q^{2jd})(1 + q^{2j-1}d^2)
\]

we calculate

\[
g_1' / g_1 = \sum_{d|N} \sum_{j=1}^{\infty} \frac{2(2j-1)dq^{d(2j-1)-1}}{1 - q^{d(2j-1)}} - \frac{2djq^{2dj-1}}{1 - q^{2dj}} - \frac{4djq^{adj-1}}{1 - q^{4dj}}
\]

\[
+ \frac{2(4j-2)dq^{d(4j-2)-1}}{1 - q^{d(4j-2)}}
\]

\[
= \sum_{d|N} \sum_{j=1}^{\infty} \frac{(4j-2)dq^{d(2j-1)d-1}}{1 + q^{d(2j-1)d}} - \frac{8djq^{adj-1}}{1 - q^{4dj}}
\]

\[
+ \frac{(4j-2)d(q^{d(4j-2)d-1} - 3q^{d(8j-4)d-1})}{1 - q^{d(8j-4)d}}
\]

\[
= \sum_{d|N} \sum_{j=1}^{\infty} \sum_{\ell=1}^{\infty} -8jdq^{4j\ell\ell-1} - 3(4j-2)dq^{d(8j-4)d\ell-1}
\]

\[
+ (4j-2)dq^{d(2j-1)d(4\ell-2)-1} - (-1)^\ell(4j-2)dq^{d(2j-1)d\ell-1}
\]

Hence the coefficient of \( q^a \) is positive if \( a \) is even and negative if \( a \equiv -1 \mod 4 \). The only cancellation that occurs is for \( a \equiv 1 \mod 4 \). In this case the coefficient of \( q^a \) is zero. \( \square \)

Theorem 2.2. Let \( N \in \mathcal{L} \) be even and let \( L \) be a strongly \( N \)-modular lattice in the genus of \( C^k_N \). Let \( \sigma := \text{smin}(L) \) and let \( \mu := \text{min}(L) \). Then

\[
\sigma + \mu \leq k \frac{\sigma_1(N/2)}{2} + 1
\]

unless \( k = s(N) \) and \( \mu = 3 \) where this bound has to be increased by 1. In these cases \( L \) is the unique lattice \( L = O^{(N)} \) (from [13, Table 1]) of minimum 3 described in [9, Theorem 3].

Proof. As in the proof of Theorem 2.1 let \( g_1 := g_1^{(N)} \) and \( g_2 := g_2^{(N)} \), \( m := \mu - 1 \) and assume that \( \sigma + \mu \geq k \frac{\sigma_1(N/2)}{2} + 1 \). Again all coefficients \( c_i \) in (2.2) and (2.1) are uniquely determined by the conditions that \( \text{smin}(L) \geq k \frac{\sigma_1(N/2)}{4} - m \) and \( \Theta_L \equiv 1 \mod q^{m+1} \). The number of vectors of norm
\[
k^{\sigma_1(N/2)} - m \text{ in } S = \sqrt{N}S(L) \text{ is } c_m(-1)^m 2^{\sigma_0(N)k/2-ms(N)}. \text{ As in the proof of Theorem 2.1 the formula of Bürmann-Lagrange yields that}
\]
\[
c_m = \frac{-k}{m} \text{ (coeff. of } q^{m-1} \text{ in } (g_1'/g_1)/f_1)\]

with \(f_1\) as in the proof of Theorem 2.1. We have
\[
f_1 = \prod_{d \mid N} f(dq)^{2k-s(N)m} \prod_{j=1}^{\infty} (1-q^{2dj})^k(1-q^{4dj})^k
\]

where \(f\) is as in Lemma 2.1. If \(2k - s(N)m > 0\) then \(1/f_1\) is alternating by Lemma 2.1 and \(g_1'/g_1\) is alternating (with a non-zero coefficient at \(q^3\)) by Lemma 2.2 and we can argue as in the proof of Theorem 2.1. Since \(k > 0\) all even coefficients occur in the product
\[
\prod_{j=1}^{\infty} (1-q^{2j})^{-k}
\]
hence all coefficients in \((g_1'/g_1)/f_1\) are non-zero. If \(2k - s(N)m = 0\) similarly the only zero coefficient in \((g_1'/g_1)/f_1\) is at \(q^1\) yielding the exception stated in the Theorem. Now assume that \(2k - s(N)m < 0\) and write
\[
P = (g_1'/g_1)/f_1 = g_1' \prod_{d \mid N} f(dq)^{s(N)m-2k-2} \prod_{j=1}^{\infty} ((1-q^{2dj})(1-q^{4dj}))^{k+1}.
\]

If \(2k - s(N)m < -2\) then \(P\) is a positive power series in which all \(q\)-powers occur and hence \(c_m < 0\). If the minimum \(\mu\) is odd then this implies that \(b_\sigma < 0\) and hence the nonexistence of an \(s\)-extremal lattice of odd minimum for \(s(N)m - 2 > 2k\). Assume now that \(2k - s(N)m = -2\), i.e. \(k = s(N)m/2 - 1\). Then again \(m\) is odd and since \(g_1'\) has a non-zero constant term \(P\) contains all even powers of \(q\). In particular the coefficient of \(q^{m-1}\) is positive. The last case is \(2k - s(N)m = -1\) and dealt with as in the proof of Theorem 2.1.

From the proof of Theorem 2.1 and 2.2 we obtain the following bound on the minimum of an \(s\)-extremal lattice which is sometimes a slight improvement of the bound (1.1).

**Corollary 2.1.** Let \(L\) be an \(s\)-extremal strongly \(N\)-modular lattice in the genus of \(C_N^k\) with odd minimum \(\mu := \min(L)\). Then
\[
\mu < \frac{2k+2}{s(N)} + 1.
\]
3. S-extremal lattices of even minimum.

In this section we use the methods of [8] to show that there are only finitely many s-extremal lattices of even minimum. The first result generalizes the bound on the dimension of an s-extremal lattice of even minimum that is obtained in [6] for unimodular lattices. In particular such s-extremal lattices are automatically extremal. Now [12, Theorem 5.2] shows that there are only finitely many extremal strongly $N$-modular lattices which also implies that there are only finitely many such s-extremal lattices with even minimum. To get a good upper bound on the maximal dimension of an s-extremal strongly $N$-modular lattice, we show that the second (resp. third) coefficient in the shadow theta series becomes eventually negative.

**Theorem 3.1.** Let $N \in \mathcal{L}$ and let $L$ be an s-extremal strongly $N$-modular lattice in the genus of $C^k_N$. Assume that $\mu := \min(L)$ is even. Then

$$s(N)(\mu - 2) \leq 2k < \mu s(N).$$

**Proof.** The lower bound follows from (1.1). As in the proof of Theorem 2.1 we obtain the number $a_\mu$ of minimal vectors of $L$ as

$$a_\mu = \frac{k}{\mu - 1} \text{ (coeff. of } q^{\mu-1} \text{ in } (g_1'/g_1)/f_2)$$

with

$$f_2 = (q^{-1} g_2)^\mu g_1^k.$$

If $N$ is odd, then

$$f_2 = \prod_{d|N} \prod_{j=1}^\infty (1 - q^{2dj})^k(1 + q^{d(2j-1)})^{2k-s(N)\mu}$$

and for even $N$ we obtain

$$f_2 = \prod_{d|\frac{N}{2}} f(dq)^{2k-s(N)\mu} \prod_{j=1}^\infty (1 - q^{2dj})^k(1 + q^{4dj})^k$$

where $f$ is as in Lemma 2.1. If $2k - s(N)\mu \geq 0$ then in both cases $(g_1'/g_1)/f_2$ is an alternating series and since $\mu - 1$ is odd the coefficient of $q^{\mu-1}$ in this series is negative. Therefore $a_\mu$ is negative which is a contradiction. □

We now proceed as in [8] and express the first coefficients of the shadow theta series of an s-extremal $N$-modular lattice.

**Lemma 3.1.** Let $N \in \mathcal{L}$, $s_1 := s_1^{(N)}$ and $s_2 := s_2^{(N)}$. Then $s_1^k \sum_{i=0}^m c_i s_2^i$ starts with $(-1)^m 2^{\sigma_0(N)(k-\text{ms}(N)/2)}q^{k\sigma_1(N)/4-2m}$ times

$$c_m - (2^{s(N)\sigma_0(N)/2} c_{m-1} + (s(N)m - k)c_m)q^2.$$
if $N$ is odd, and with $(-1)^m2^{\sigma_0(N)/2}q^{m\sigma_0(N)/4}\sigma_0(N)/q^{k\sigma_1(N/2)/2-m}$ times
\[
c_m - (2^{s(N)\sigma_0(N)/4}c_{m-1} + (s(N)m - 2k)c_m)q
+ (2^{s(N)\sigma_0(N)/2}c_{m-2} + 2^{s(N)\sigma_0(N)/4}(s(N)(m - 1) - 2k)c_{m-1}
+ (s(N)^2\frac{m(m-1)}{2}) - 2kms(N) + 2k(k - 1) + 2^{s(N)\sigma_0(N)/4}\frac{m(s(N)+1)}{4})c_m)q^2
\]
if $N$ is even.

Proof. If $N$ is odd then
\[
s_1 = 2^{\sigma_0(N)/2}q^{\sigma_1(N)/4}(1 + q^2) + \ldots
s_2 = 2^{-s(N)\sigma_0(N)/2}(-q^{-2} + s(N)) + \ldots
\]
and for even $N$
\[
s_1 = 2^{\sigma_0(N)/2}q^{\sigma_1(N/2)/2}(1 + 2q + 0q^2) + \ldots
s_2 = 2^{-s(N)\sigma_0(N)/4}(-q^{-1} + s(N)) - \frac{s(N)+1}{4}q + \ldots
\]
Explicit calculations prove the lemma. \qed

We now want to use [8, Lemma 1] to show that the coefficients $c_m$ and $c_{m-1}$ determined in the proof of Theorem 2.1 for the theta series of an $s$-extremal lattice satisfy $(-1)^jc_j > 0$ and $c_m/c_{m-1}$ is bounded.

If $L$ is an $s$-extremal lattice of even minimum $\mu = m + 1$ in the genus of $C^k_N$, then Theorem 3.1 yields that
\[k = \frac{s(N)}{2}(m - 1) + b\] for some $0 \leq b < s(N)$.

Let
\[
\psi := \psi^{(N)} := \prod_{j=1}^{\infty} \prod_{d|N}(1 - q^{2jd})\quad \text{and}\quad \varphi := \varphi^{(N)} := \prod_{j=1}^{\infty} \prod_{d|N}(1 + q^{(2j-1)d}).
\]

Then
\[
c_{m-\ell} = \frac{k}{m-\ell} \text{ coeff. of } q^{m-\ell-1} \text{ in } g^{\prime}_1\psi^{-k-1}\varphi^s(N)(m-\ell-1) - 2(k+1)
= \frac{k}{m-\ell} \text{ coeff. of } q^{m-\ell-1} \text{ in } G^{(b)}_\ell H^{m-\ell-1}
\]
where
\[
G^{(b)}_\ell = g^{(b)}_1\psi^{-b-1-\ell}s(N)/2\varphi^{-2b-2+(1-\ell)s(N)} = G^{(0)}_\ell(\psi^{-1}\varphi^{-2})^b
\]
and
\[
H = \psi^{-s(N)/2} = 1 + \frac{s(N)}{2}q^2 + \ldots
\]
In particular the first two coefficients of $H$ are positive and the remaining coefficients are nonnegative. Since also odd powers of $q$ arise in $G^{(b)}_\ell$ the coefficient $\beta_{m-\ell-1}$ of $q^{m-\ell-1}$ in $G^{(b)}_\ell H^{m-\ell-1}$ is by Cauchy’s formula
\[
\beta_{m-\ell-1} = \frac{1}{2} \int_{-1+iy}^{1+iy} e^{-\pi i(m-\ell-1)z} G^{(b)}_\ell(e^{\pi iz})H^{m-\ell-1}(e^{\pi iz})dz
\]
for arbitrary $y > 0$.

Put $F(y) := e^{\pi y} H(e^{-\pi y})$ and let $y_0$ be the first positive zero of $F'$. Then we check that $d_1 := F(y_0) > 0$ and $d_2 := F''(y_0)/F(y_0) > 0$. Now $H$ has two saddle points in $[-1 + i y_0, 1 + i y_0]$ namely at $\pm 1 + i y_0$ and $i y_0$. By the saddle point method (see [1, (5.7.2)]) we obtain

$$\beta_{m-\ell-1} \sim d_1^{m-\ell-1} (G_{\ell}^{(b)}(e^{-\pi y_0}) + (-1)^{m-\ell-1} G_{\ell}^{(b)}(-e^{-\pi y_0})),$$

$$\times (2\pi(m - \ell - 1)d_2)^{-1/2}$$

as $m$ tends to infinity. In particular

$$c_m \sim d_1 \frac{G_{\ell}^{(b)}(e^{-\pi y_0}) + (-1)^{m-1} G_{\ell}^{(b)}(-e^{-\pi y_0})}{G_{\ell}^{(b)}(e^{-\pi y_0}) + (-1)^m G_{\ell}^{(b)}(-e^{-\pi y_0})} c_{m-1}.$$  

**Lemma 3.2.** For $N \in \mathcal{L}$ and $b \in \{0, \ldots, s(N)-1\}$ let $k := \frac{s(N)}{2}(m-1)+b = js(N)+b$, $G_{\ell}^{(b)}$, $d_1, d_2, y_0$ be as above where $m = 2j+1$ is odd. Then $c_{2j+1}/c_{2j}$ tends to

$$Q(N, b) := \frac{d_1 G_{\ell}^{(b)}(e^{-\pi y_0}) + (-1)^{m-1} G_{\ell}^{(b)}(-e^{-\pi y_0})}{G_{\ell}^{(b)}(e^{-\pi y_0}) + (-1)^m G_{\ell}^{(b)}(-e^{-\pi y_0})} \in \mathbb{R}_{<0}$$

if $j$ goes to infinity.

By Lemma 3.1 the second coefficient $b_{\sigma+2}$ in the shadow theta series of a putative $s$-extremal strongly $N$-modular lattice of even minimum $\mu = m+1$ in the genus of $C_N^k$ ($k = \frac{s(N)}{2}(m-1)+b$ as above) is a positive multiple of

$$2^{s(N)} \frac{s_0(N)}{2} c_{m-1} + (s(N)m - k)c_m$$

$$\sim (2^{s(N)} \frac{s_0(N)}{2} + Q(N, b) \frac{s(N)(m + 1) - 2b}{2})c_{m-1}$$

when $m$ tends to infinity. In particular this coefficient is expected to be negative if

$$\mu = m + 1 > B(N, b) := \frac{2}{s(N)} \left( b + \frac{2^{s(N)}s_0(N)/2}{-Q(N, b)} \right).$$

Since all these are asymptotic values, the actual value $\mu_-(N, b)$ of the first even minimum $\mu$ where $b_{\sigma+2}$ becomes negative may be different. In all cases, the second coefficient of the relevant shadow theta series seems to remain negative for even minimum $\mu \geq \mu_-(N, b)$.

For odd $N \in \mathcal{L}$ the values of $B(N, b)$ and $\mu_-(N, b)$ are given in the following tables:
\[
\begin{array}{cccccccccc}
N = 1 & b = 0 & b = 1 & b = 2 & b = 3 & b = 4 & b = 5 & b = 6 & b = 7 & b = 8 \\
Q(1,b) & -380 & -113 & -43.8 & -18.4 & -8 & -3.53 & -1.57 & -0.71 & -0.33 \\
B(1,b) & 0.9 & 3.1 & 7.96 & 18.8 & 43 & 97.1 & 217.4 & 480.4 & 1036.6 \\
\mu_-(1,b) & 6 & 6 & 12 & 20 & 44 & 96 & 216 & 478 & 1032 \\
k_-(1,b) & 48 & 49 & 122 & 219 & 508 & 1133 & 2574 & 5719 & 12368 \\
\end{array}
\]

\[
\begin{array}{cccccccccc}
N = 1 & b = 9 & b = 10 & b = 11 & b = 12 & b = 13 & b = 14 & b = 15 \\
Q(1,b) & -0.16 & -0.08 & -0.05 & -0.04 & -0.03 & -0.027 & -0.026 \\
B(1,b) & 2131.3 & 4012.4 & 6597.4 & 9240.4 & 11239.4 & 12433.6 & 13049.1 \\
\end{array}
\]

\[
\begin{array}{cccccccccc}
N = 1 & b = 16 & b = 17 & b = 18 & b = 19 & b = 20 & b = 21 & b = 22 & b = 23 \\
Q(1,b) & -0.026 & -0.025 & -0.025 & -0.025 & -0.025 & -0.025 & -0.025 & -0.025 \\
B(1,b) & 13342 & 13477 & 13538 & 13565 & 13577 & 13582 & 13585 & 13586 \\
\end{array}
\]

\[
\begin{array}{cccccccccc}
N = 3 & b = 0 & b = 1 & b = 2 & b = 3 & b = 4 & b = 5 \\
Q(3,b) & -15.6 & -2 & -0.45 & -0.2 & -0.16 & -0.15 \\
B(3,b) & 1.36 & 11 & 47.6 & 107.13 & 137.07 & 144.34 \\
\mu_-(3,b) & 6 & 12 & 44 & 100 & 126 & 130 \\
k_-(3,b) & 12 & 31 & 128 & 297 & 376 & 389 \\
\end{array}
\]

\[
\begin{array}{cccccccccc}
N = 5 & b = 0 & b = 1 & b = 2 & b = 3 & N = 7 & b = 0 & b = 1 & b = 2 \\
Q(5,b) & -5 & -0.73 & -0.31 & -0.25 & Q(7,b) & -2.88 & -0.51 & -0.32 \\
B(5,b) & 1.6 & 11 & 27 & 33.5 & B(7,b) & 1.85 & 11 & 17.8 \\
\mu_-(5,b) & 6 & 12 & 22 & 24 & \mu_-(7,b) & 6 & 10 & 12 \\
k_-(5,b) & 8 & 21 & 42 & 47 & k_-(7,b) & 6 & 13 & 17 \\
\end{array}
\]

\[
\begin{array}{cccccc}
N = 11 & b = 0 & b = 1 & N = 15 & b = 0 & N = 23 & b = 0 \\
Q(11,b) & -1.72 & -0.45 & Q(15,b) & -2.03 & Q(23,b) & -1.08 \\
B(11,b) & 2.33 & 9.8 & B(15,b) & 3.93 & B(23,b) & 3.69 \\
\mu_-(11,b) & 6 & 6 & \mu_-(15,b) & 6 & \mu_-(23,b) & 6 \\
k_-(11,b) & 4 & 5 & k_-(15,b) & 2 & k_-(23,b) & 2 \\
\end{array}
\]
For even \( N \in \mathcal{L} \) the situation is slightly different. Again \( k = b + \frac{s(N)}{2} (m - 1) \) for some \( 0 \leq b < s(N) \). From Lemma 3.1 the second coefficient \( b_{\sigma+1} \) in the s-extremal shadow theta series is a nonzero multiple of \( 2^{s(N)} \sigma_0(N)/4 c_{m-1} + (s(N) - 2b) c_m \) and in particular its sign is asymptotically independent of \( m \). Therefore we need to consider the third coefficient \( b_{\sigma+2} \), which is by Lemma 3.1 for odd \( m \) a positive multiple of

\[
-a^2 c_{m-2} + a(2k - s(m - 1)) c_{m-1} + (2km - s^2 m(m-1)/2 - 2k(k-1) - am s + 4) c_m
\]

where for short \( a := 2^{s\sigma_0(N)/4} \) and \( s := s(N) \). For \( k = \frac{s(N)}{2} (m - 1) + b \) this becomes

\[
-a^2 c_{m-2} + 2abc_{m-1} + (m(2b - s - 1) - a \frac{s + 4}{4} + s + 2) c_m + \frac{2s + s^2}{2} c_m.
\]

Since the quotients \( c_{m-1}/c_{m-2} \) and \( c_m/c_{m-2} \) are bounded, there is an explicit asymptotic bound \( B(N, b) \) for \( \mu = m + 1 \) after which this coefficient should become negative. Again, the true values \( \mu_-(N, b) \) differ and the results are displayed in the following table.

<table>
<thead>
<tr>
<th>( N = 2 )</th>
<th>( b = 0 )</th>
<th>( b = 1 )</th>
<th>( b = 2 )</th>
<th>( b = 3 )</th>
<th>( b = 4 )</th>
<th>( b = 5 )</th>
<th>( b = 6 )</th>
<th>( b = 7 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( B(2, b) )</td>
<td>-4.9</td>
<td>10</td>
<td>52.5</td>
<td>170.1</td>
<td>382.6</td>
<td>575.9</td>
<td>677.7</td>
<td>725.7</td>
</tr>
<tr>
<td>( \mu_-(2, b) )</td>
<td>10</td>
<td>22</td>
<td>34</td>
<td>166</td>
<td>374</td>
<td>564</td>
<td>666</td>
<td>716</td>
</tr>
<tr>
<td>( k_-(2, b) )</td>
<td>56</td>
<td>81</td>
<td>210</td>
<td>659</td>
<td>1492</td>
<td>2253</td>
<td>2662</td>
<td>2863</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( N = 6 )</th>
<th>( b = 0 )</th>
<th>( b = 1 )</th>
<th>( N = 14 )</th>
<th>( b = 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( B(6, b) )</td>
<td>1</td>
<td>33.58</td>
<td>( B(14, b) )</td>
<td>2</td>
</tr>
<tr>
<td>( \mu_-(6, b) )</td>
<td>10</td>
<td>28</td>
<td>( \mu_-(14, b) )</td>
<td>10</td>
</tr>
<tr>
<td>( k_-(6, b) )</td>
<td>8</td>
<td>27</td>
<td>( k_-(14, b) )</td>
<td>4</td>
</tr>
</tbody>
</table>

### 3.1. Explicit classifications.

In this section we classify the s-extremal strongly \( N \)-modular lattices \( L_N(\mu, k) \) rational equivalent to \( C_N^k \) for certain \( N \) and even minimum \( \mu \). For \( N \in \{11, 14, 15, 23\} \) a complete classification is obtained. For convenience we denote the uniquely determined modular form that should be the theta series of \( L_N(\mu, k) \) by \( \theta_N(\mu, k) \) and its shadow by \( \sigma_N(\mu, k) \).

Important examples are the unique extremal even strongly \( N \)-modular lattices \( E^{(N)} \) of minimum 4 and with \( k = s(N) \) from [13, Table 1]. For odd
3.1 suggests to write \( k = \frac{s(N)(\mu - 2)}{2} + b \) for some \( 0 \leq b \leq s(N) - 1 \) and we will organize the classification according to the possible \( b \). Note that for every \( b \) the maximal minimum \( \mu \) is bounded by \( \mu(N, b) \) above.

If \( N = 14, 15 \) or \( 23 \), then \( s(N) = 1 \) and hence Theorem 3.1 implies that \( k = \frac{\mu - 2}{2} \). For \( N = 15, 23 \) the only possibility is \( k = 1 \) and \( \mu = 4 \) and \( L_N(4, 1) = E'(N) \). The second coefficient of \( \sigma_{14}(4, 1) \) and \( \sigma_{14}(8, 3) \) is negative, hence the only \( s \)-extremal strongly 14-modular lattice with even minimum is \( L_{14}(6, 2) \) of minimum 6. The series \( \sigma_{14}(6, 2) \) starts with \( 8q^3 + 8q^5 + 16q^6 + \ldots \). Therefore the even neighbour of \( L_{14}(6, 2) \) in the sense of \([13, \text{Theorem 8}]\) is the unique even extremal strongly 14-modular lattice of dimension 8 (see \([14, \text{p. 160}]\)). Constructing all odd 2-neighbours of this lattice, it turns out that there is a unique such lattice \( L_{14}(6, 2) \). Note that \( L_{14}(6, 2) \) is an odd extremal strongly modular lattice in a jump dimension and hence the first counterexample to conjecture (3) in the Remark after \([13, \text{Theorem 2}]\).

For \( N = 11 \) and \( b = 0 \) the only possibility is \( \mu = 4 \) and \( k = 2 = s(N) \) whence \( L_{11}(4, 2) = E^{(11)} \). If \( b = 1 \) then either \( \mu = 2 \) and \( L_{11}(2, 1) = \left( \begin{array}{c} 21 \\ 16 \end{array} \right) \) or \( \mu = 4 \). An explicit enumeration of the genus of \( G_{11}^3 \) with the Kneser neighbouring method \([7]\) shows that there is a unique lattice \( L_{11}(4, 3) \).

Now let \( N = 7 \). For \( b = 0 \) again the only possibility is \( k = s(N) \) and \( L_7(4, 3) = E^{(7)} \). For \( b = 1 \) and \( b = 2 \) one obtains unique lattices \( L_7(2, 1) \) (with Grammatrix \( \left( \begin{array}{c} 21 \\ 14 \end{array} \right) \) ) \( L_7(4, 4) \) and \( L_7(4, 5) \). There is no contradiction for the existence of lattices \( L_7(6, 7) \), \( L_7(6, 8) \), \( L_7(8, 10) \), \( L_7(8, 11) \), though a complete classification of the relevant genera seems to be difficult. For the lattice \( L_7(6, 8) \) we tried the following: Both even neighbours of such a lattice are extremal even 7-modular lattices. Starting from the extremal 7-modular lattice constructed from the structure over \( \mathbb{Z}[\sqrt{2}] \) of the Barnes-Wall lattice as described in \([14]\), we calculated the part of the Kneser 2-neighbouring graph consisting only of even lattices of minimum 6 and therewith found 126 such even lattices 120 of which are 7-modular. None of the edges between such lattices gave rise to an \( s \)-extremal lattice. The lattice \( L_7(10, 14) \) does not exist because \( \theta_7(10, 14) \) has a negative coefficient at \( q^{13} \).

Now let \( N := 6 \). For \( k = \mu - 2 \) the second coefficient in the shadow theta series is negative, hence there are no lattices \( L_6(\mu, \mu - 2) \) of even minimum \( \mu \). For \( k = \mu - 1 < 27 \) the modular forms \( \theta_6(\mu, \mu - 1) \) and \( \sigma_6(\mu, \mu - 1) \) seem to have nonnegative integral coefficients. The lattice \( L_6(2, 1) \) is unique and already given in \([9]\). For \( \mu = 4 \) the even neighbour of any lattice \( L_6(4, 3) \) (as defined in \([13, \text{Theorem 8}]\) is one of the five even extremal strongly
6-modular lattices given in [14]. Constructing all odd 2-neighbours of these lattices we find a unique lattice \( L_6(4,3) \) as displayed below.

For \( N = 5 \) the lattice \( L_5(4,4) = E^{(5)} \) is is the only s-extremal lattice of even minimum \( \mu \) for \( k = 2(\mu - 2) \), because \( \mu_-(5,0) = 6 \). For \( k = 2(\mu - 2) + 1 \) the shadow series \( \sigma_5(2,1) \), \( \sigma_5(4,5) \) and \( \sigma_5(6,9) \) have non-integral respectively odd coefficients so the only lattices that might exist here are \( L_5(8,13) \) and \( L_5(10,17) \). The s-extremal lattice \( L_5(2,2) = \left( \frac{21}{13} \right) \perp \left( \frac{21}{13} \right) \) is unique. The theta series \( \theta_5(2,3) \) starts with \( 1 + 20q^3 + \ldots \), hence \( L_5(2,3) = S^{(5)} \) has minimum 3. The genus of \( C_5^6 \) contains 1161 isometry classes, 3 of which represent s-extremal lattices of minimum 4 and whose Grammatrices \( L_5(4,6)_{a,b,c} \) are displayed below. For \( k = 7 \) a complete classification of the genus of \( C_5^6 \) seems to be out of range. A search for lattices in this genus that have minimum 4 constructs the example \( L_5(4,7)_a \) displayed below of which we do not know whether it is unique. For the remaining even minima \( \mu < \mu_-(5, b) \) we do not find a contradiction against the existence of such s-extremal lattices.

For \( N = 3 \) and \( b = 0 \) again \( E^{(3)} = L_3(4,6) \) is the unique s-extremal lattice. For \( k = 3(\mu - 2) + 1 \), the theta series \( \theta_3(8, 19) \) and \( \theta_3(10, 25) \) as well as their shadows seem to have integral non-negative coefficients, whereas \( \sigma_3(4,7) \) and \( \sigma_3(6,13) \) have non-integral coefficients. The remaining theta-series and their shadows again seem to have integral non-negative coefficients. The lattices of minimum 2 are already classified in [9]. In all cases \( L_3(2, b) \) \( 2 \leq b \leq 5 \) is unique but \( L_3(2,5) = S^{(3)} \) has minimum 3.

Now let \( N := 2 \). For \( b = 0 \) and \( b = 1 \) the second coefficient in \( \sigma_2(\mu, 4(\mu - 2) + b) \) is always negative, proving the non-existence of such s-extremal lattices. The lattices of minimum 2 are already classified in [9]. There is a unique lattice \( L_2(2,2) \cong D_4 \), no lattice \( L_2(2,3) \) since the first coefficient of \( \sigma_2(2,3) \) is 3, unique lattices \( L_2(2,b) \) for \( b = 4 \), 5 and 7 and two such lattices \( L_2(2,6) \).

For \( N = 1 \) we also refer to the paper [6] for the known classifications. Again for \( b = 0 \), the Leech lattice \( L_1(4,24) = E^{(1)} \) is the unique s-extremal lattice. For \( \mu = 2 \), these lattices are already classified in [5]. The possibilities for \( b = k \) are 8, 12, 14 \( \leq b \leq 22 \). For \( \mu = 4 \), the possibilities are either \( b = 0 \) and \( k = 24 \) or \( 8 \leq b \leq 23 \) whence \( 32 \leq k \leq 47 \) since the other shadow series have non-integral coefficients. The lattices \( L_1(4,32) \) are classified in [3]. For \( \mu = 6 \) no such lattices are known. The first possible dimension is 56, since the other shadow series have non-integral coefficients.

Since for odd \( N \) the value \( \mu_-(N,0) = 6 \) and the s-extremal lattices of minimum 4 with \( k = s(N) \) are even and hence isometric to \( E^{(N)} \) we obtain the following theorem.
Gabriele Nebe, Kristina Schindelar

**Theorem 3.2.** Let $L$ be an extremal and $s$-extremal lattice rational equivalent to $C_N^k$ for some $N \in \mathcal{L}$ such that $k$ is a multiple of $s(N)$. Then $\mu := \min(L)$ is even and $k = s(N)(\mu - 2)/2$ and either $\mu = 4$, $N$ is odd and $L = E(N)$ or $\mu = 6$, $N = 14$ and $L = L_{14}(6, 2)$.

For $N \in \{11, 14, 15, 23\}$ the complete classification of $s$-extremal strongly $N$-modular lattices in the genus of $C_N^k$ is as follows:

<table>
<thead>
<tr>
<th>$N$</th>
<th>23</th>
<th>15</th>
<th>14</th>
<th>11</th>
<th>11</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>min</td>
<td>4</td>
<td>4</td>
<td>6</td>
<td>2</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>$k$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>lattice</td>
<td>$E^{(23)}$</td>
<td>$E^{(15)}$</td>
<td>$E^{(14)}$</td>
<td>$L_{11}(2, 1)$</td>
<td>$E^{(11)}$</td>
<td>$L_{11}(4, 3)$</td>
</tr>
</tbody>
</table>

For the remaining $N \in \mathcal{L}$, the results are summarized in the following tables. The last line, labelled with # displays the number of lattices, where we display $-$ if there is no such lattice, ? if we do not know such a lattice, + if there is a lattice, but the lattices are not classified. We always write $k = \ell s(N) + b$ with $0 \leq b \leq s(N) - 1$ such that $\mu = \min(L) = 2\ell + 2$ by Theorem 3.1 and $\dim(L) = k\sigma_0(N)$.

**For $N = 7$, $s(N) = 3$, $k = \ell s(N) + b$**

<table>
<thead>
<tr>
<th>$b$</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ell$</td>
<td>$\geq 2$</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>min</td>
<td>$\geq 4$</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>#</td>
<td>1</td>
<td>-</td>
<td>1</td>
</tr>
</tbody>
</table>

**For $N = 6$, $s(N) = 2$, $k = \ell s(N) + b$**

<table>
<thead>
<tr>
<th>$b$</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ell$</td>
<td>$\geq 1$</td>
<td>0</td>
</tr>
<tr>
<td>min</td>
<td>$\geq 4$</td>
<td>2</td>
</tr>
<tr>
<td>#</td>
<td>-</td>
<td>1</td>
</tr>
</tbody>
</table>

**For $N = 5$, $s(N) = 4$, $k = \ell s(N) + b$**

<table>
<thead>
<tr>
<th>$b$</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ell$</td>
<td>$\geq 2$</td>
<td>0</td>
</tr>
<tr>
<td>min</td>
<td>$\geq 4$</td>
<td>2</td>
</tr>
<tr>
<td>#</td>
<td>1</td>
<td>-</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$b$</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ell$</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>min</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>#</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>
Grammatrices of the new s-extremal lattices:

$$L_{14}(6,2) = \begin{pmatrix} 6 & 3 & 0 & 2 & 3 & 3 & 1 & 2 \\ 3 & 6 & 3 & 2 & 3 & 3 & 3 & 2 \\ 0 & 3 & 6 & 0 & 3 & 2 & 2 & 3 \\ 2 & 2 & 0 & 6 & 2 & 1 & 1 & 3 \\ -3 & 3 & 3 & 3 & 2 & 6 & 3 & 3 & 3 \\ 3 & 3 & 2 & 1 & 3 & 7 & 4 & 2 \\ -1 & 3 & 2 & 1 & 3 & 4 & 7 & 1 \\ -2 & 2 & 3 & 3 & 3 & 3 & 2 & 1 & 7 \end{pmatrix}, \quad L_{11}(4,3) = \begin{pmatrix} 4 & 0 & 0 & 2 & 2 & 1 \\ 0 & 4 & 0 & 2 & 2 & 1 \\ 0 & 0 & 4 & 2 & 1 & 2 \\ 0 & 0 & 4 & 2 & 1 & 2 \\ 0 & 0 & 4 & 2 & 1 & 2 \\ 0 & 0 & 4 & 2 & 1 & 2 \\ 0 & 0 & 4 & 2 & 1 & 2 \end{pmatrix}$$

$$L_{7}(4,4) = \begin{pmatrix} 4 & 0 & 0 & 2 & 2 & 2 & 2 & 1 \\ 0 & 4 & 0 & 2 & 2 & 1 & 2 & 2 \\ 0 & 0 & 4 & 2 & 1 & 2 & 1 & 2 \\ 0 & 0 & 4 & 2 & 1 & 2 & 1 & 2 \\ 0 & 0 & 4 & 2 & 1 & 2 & 1 & 2 \\ 2 & 2 & 2 & 0 & 5 & 2 & 1 & 0 \\ 2 & 2 & 2 & 0 & 5 & 2 & 1 & 0 \\ 1 & 1 & 2 & 1 & 5 & 1 & 5 & 1 \\ -2 & 2 & 3 & 3 & 0 & 1 & 5 & 3 & 1 \\ -1 & 2 & 2 & 2 & 2 & 1 & 1 & 5 & 1 \\ -1 & 2 & 2 & 2 & 2 & 1 & 1 & 5 & 1 \\ -1 & 2 & 2 & 2 & 2 & 1 & 1 & 5 & 1 \\ -1 & 2 & 2 & 2 & 2 & 1 & 1 & 5 & 1 \\ -1 & 2 & 2 & 2 & 2 & 1 & 1 & 5 & 1 \end{pmatrix}, \quad L_{7}(4,5) = \begin{pmatrix} 4 & 0 & 0 & 2 & 1 & 1 & 1 & 2 & 1 & 1 \\ 0 & 4 & 0 & 2 & 1 & 2 & 1 & 2 & 2 \\ 0 & 0 & 4 & 2 & 1 & 2 & 1 & 2 & 2 \\ 0 & 0 & 4 & 2 & 1 & 2 & 1 & 2 & 2 \\ 0 & 0 & 4 & 2 & 1 & 2 & 1 & 2 & 2 \\ 1 & 1 & 2 & 0 & 5 & 2 & 1 & 0 & 2 & 1 \\ 1 & 1 & 2 & 0 & 5 & 2 & 1 & 0 & 2 & 1 \\ 1 & 1 & 2 & 0 & 5 & 2 & 1 & 0 & 2 & 1 \\ 1 & 1 & 2 & 0 & 5 & 2 & 1 & 0 & 2 & 1 \\ 1 & 1 & 2 & 0 & 5 & 2 & 1 & 0 & 2 & 1 \\ 2 & 2 & 2 & 3 & 0 & 0 & 1 & 5 & 3 & 1 \\ 2 & 2 & 2 & 3 & 0 & 0 & 1 & 5 & 3 & 1 \\ 1 & 2 & 2 & 2 & 2 & 1 & 1 & 3 & 6 & 2 \\ 1 & 2 & 2 & 2 & 2 & 1 & 1 & 3 & 6 & 2 \\ -1 & 2 & 2 & 2 & 2 & 1 & 1 & 3 & 6 & 2 \\ -1 & 2 & 2 & 2 & 2 & 1 & 1 & 3 & 6 & 2 \end{pmatrix}$$
\[
L_b(4,3) = \begin{pmatrix}
4 & 1 & 2 & 1 & 0 & 1 & 1 & 1 & 1 & -2 & 2 & 0 & 0 & 1 \\
1 & 4 & 2 & 0 & 1 & 1 & 1 & 0 & 2 & 0 & 1 \\
-2 & 4 & 1 & 1 & -1 & 1 & 0 & 0 & 0 & 2 & 2 & 1 & 1 \\
0 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 2 & 2 & -4 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
0 & 2 & 1 & 1 & 1 & -1 & 2 & -5 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 1 & 1 & 0 & 0 & 0 & 1 & 5 & 0 & 1 & 1 \\
-1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 0 & 1 & 2 & 1 & 0 \\
\end{pmatrix}, \quad L_5(4,6) = \begin{pmatrix}
4 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 2 \\
1 & 4 & 1 & 0 & 1 & 2 & 2 & 2 & 1 & 1 & 1 \\
0 & 2 & 0 & 4 & 0 & 4 & 1 & 2 & 2 & 2 & 2 \\
2 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 2 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
-1 & 1 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 \\
0 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
2 & 2 & 2 & 1 & 0 & 0 & 0 & 0 & 2 & 2 & 1 \\
\end{pmatrix}
\]

\[
L_5(4,6)_b = \begin{pmatrix}
4 & 1 & 1 & 0 & 2 & 0 & 1 & 0 & 1 & -1 & 2 \\
1 & 4 & 1 & 2 & 1 & 1 & 1 & 0 & 1 & 0 & 2 \\
1 & 1 & 0 & 1 & 2 & 2 & 2 & 2 & 1 & 1 & 1 \\
0 & 2 & 0 & 4 & 0 & 1 & 1 & 2 & 0 & 0 & 2 \\
2 & 1 & 1 & 0 & 4 & 0 & 1 & 2 & 1 & 1 & 1 \\
0 & 1 & 2 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
-1 & 1 & 2 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 0 \\
0 & 1 & 2 & 2 & 1 & 2 & 1 & 2 & 5 & 1 & 2 \\
0 & 1 & 1 & 0 & 1 & 2 & 1 & 1 & 1 & 5 & 2 \\
-1 & 1 & 2 & 1 & 1 & 1 & 3 & 2 & 5 & 5 & 1 \\
2 & 2 & 2 & 2 & 1 & 0 & 0 & 0 & 0 & 2 & 2 \\
\end{pmatrix}, \quad L_5(4,6)_c = \begin{pmatrix}
4 & 2 & 0 & 0 & 1 & 2 & 0 & 2 & 0 & 2 & 2 & 0 \\
2 & 4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 4 & 0 & 2 & 2 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 4 & 0 & 2 & 2 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 4 & 0 & 2 & 2 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 4 & 0 & 2 & 2 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 4 & 0 & 2 & 2 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 4 & 0 & 2 & 2 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 4 & 0 & 2 & 2 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 4 & 0 & 2 & 2 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 4 & 0 & 2 & 2 & 2 & 0 & 0 \\
\end{pmatrix}
\]

\[
L_5(4,7)_a = \begin{pmatrix}
4 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 \\
0 & 4 & 0 & 1 & 2 & 0 & 2 & 1 & 1 & 0 & 0 & 1 \\
0 & 4 & 0 & 0 & 2 & 0 & 2 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 2 & 0 & 0 & 1 & 1 & 2 & 1 & 0 & 2 \\
0 & 0 & 2 & 2 & 5 & 2 & 2 & 0 & 2 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 5 & 1 & 1 & 2 & 2 & 0 & 0 \\
0 & 0 & 2 & 2 & 1 & 2 & 1 & 2 & 5 & 1 & 2 & 2 \\
-2 & 2 & 0 & 1 & 1 & 0 & 1 & 2 & 5 & 5 & 1 & 1 \\
0 & 0 & 1 & 2 & 0 & 1 & 1 & 5 & 2 & 2 & 1 & 1 \\
-2 & 0 & 1 & 1 & 2 & 2 & 2 & 1 & 2 & 5 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 2 & 2 & 5 & 0 & 0 & 0 \\
-2 & 0 & 0 & 0 & 2 & 0 & 1 & 2 & 1 & 0 & 0 & 5 \\
0 & 0 & 2 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 2 & 5 \\
\end{pmatrix}
\]
S-extremal strongly modular lattices


Gabriele Nebe
Lehrstuhl D für Mathematik
RWTH Aachen
52056 Aachen, Germany
E-mail: nebe@math.rwth-aachen.de
URL: http://www.math.rwth-aachen.de/~Gabriele.Nebe/

Kristina Schindelar
Lehrstuhl D für Mathematik
RWTH Aachen
52056 Aachen, Germany
E-mail: schindelar@math.rwth-aachen.de