András BIRÓ

Characterizations of groups generated by Kronecker sets

<http://jtnb.cedram.org/item?id=JTNB_2007__19_3_567_0>
Characterizations of groups generated by 
Kronecker sets

par András BIRÓ

Résumé. Ces dernières années, depuis l'article [B-D-S], nous avons étudié la possibilité de caractériser les sous-groupes dénombrables du tore \( T = \mathbb{R}/\mathbb{Z} \) par des sous-ensembles de \( \mathbb{Z} \). Nous considérons ici de nouveaux types de sous-groupes: soit \( K \subseteq T \) un ensemble de Kronecker (un ensemble compact sur lequel toute fonction continue \( f : K \to T \) peut être approchée uniformément par des caractères de \( T \)) et \( G \) le groupe engendré par \( K \). Nous prouvons (théorème 1) que \( G \) peut être caractérisé par un sous-ensemble de \( \mathbb{Z}^2 \) (au lieu d’un sous-ensemble de \( \mathbb{Z} \)). Si \( K \) est fini, le théorème 1 implique notre résultat antérieur de [B-S]. Nous montrons également (théorème 2) que si \( K \) est dénombrable alors \( G \) ne peut pas être caractérisé par un sous-ensemble de \( \mathbb{Z} \) (ou une suite d’entiers) au sens de [B-D-S].

Abstract. In recent years, starting with the paper [B-D-S], we have investigated the possibility of characterizing countable subgroups of the torus \( T = \mathbb{R}/\mathbb{Z} \) by subsets of \( \mathbb{Z} \). Here we consider new types of subgroups: let \( K \subseteq T \) be a Kronecker set (a compact set on which every continuous function \( f : K \to T \) can be uniformly approximated by characters of \( T \)), and \( G \) the group generated by \( K \). We prove (Theorem 1) that \( G \) can be characterized by a subset of \( \mathbb{Z}^2 \) (instead of a subset of \( \mathbb{Z} \)). If \( K \) is finite, Theorem 1 implies our earlier result in [B-S]. We also prove (Theorem 2) that if \( K \) is uncountable, then \( G \) cannot be characterized by a subset of \( \mathbb{Z} \) (or an integer sequence) in the sense of [B-D-S].

1. Introduction

Let \( T = \mathbb{R}/\mathbb{Z} \), where \( \mathbb{R} \) denotes the additive group of the real numbers, \( \mathbb{Z} \) is its subgroup consisting of the integers. If \( x \in \mathbb{R} \), then \( ||x|| \) denotes its distance to the nearest integer; this function is constant on cosets by \( \mathbb{Z} \), so it is well-defined on \( T \). A set \( K \subseteq T \) is called a Kronecker set if it

Manuscrit reçu le 13 mai 2005.
Research partially supported by the Hungarian National Foundation for Scientific Research (OTKA) Grants No. T032236, T 042750, T043623 and T049693.
is nonempty, compact, and for every continuous function $f : K \to T$ and $\delta > 0$ there is an $n \in \mathbb{Z}$ such that
\[ \max_{\alpha \in K} \| f(\alpha) - n\alpha \| < \delta. \]
If $K \subseteq T$ is a finite set, it is a Kronecker set if and only if its elements are independent over $\mathbb{Z}$ (this is essentially Kronecker’s classical theorem on simultaneous diophantine approximation). There are many uncountable Kronecker sets, see e.g. [L-P], Ch. 1.

In [B-D-S] and in [B-S], we proved for a subgroup $G \subseteq T$ generated by a finite Kronecker set that $G$ can be characterized by a subset of the integers in certain ways. In fact we dealt with any countable subgroup of $T$ in [B-D-S], and the result of [B-S] was generalized also for any countable subgroup in [B]. For further generalizations and strengthenings of these results, see [Bi1], [Bi2], [D-M-T], [D-K], [B-S-W].

In the present paper, we prove such a characterization of a group generated by a general Kronecker set by a subset of $\mathbb{Z}^2$ (instead of a subset of $\mathbb{Z}$). We also show, on the contrary, that using a subset of $\mathbb{Z}$, the characterization is impossible, if $K$ is uncountable. More precisely, we prove the following results.

Throughout the paper, let $K$ be a fixed Kronecker set, $G$ the subgroup of $T$ generated by $K$, and let $\epsilon > 0$ be a fixed number. Write
\[ l(x) = \frac{-1}{\log_2 x} \text{ for } 0 < x < 1/2, \]
and extend it to every $x \geq 0$ by $l(0) = 0$, and $l(x) = 1$ for $x \geq 1/2$.

**Theorem 1.** There is an infinite subset $A \subseteq \mathbb{Z}^2$ such that for every $\alpha \in G$ we have
\[ \sum_{n = (n_1, n_2) \in A} l^{1+\epsilon} \left( \min (\|n_1\alpha\|, \|n_2\alpha\|) \right) < \infty, \quad (1.1) \]
and if $\beta \in T$ satisfies
\[ \min (\|n_1\beta\|, \|n_2\beta\|) < \frac{1}{10} \quad (1.2) \]
for all but finitely many $n = (n_1, n_2) \in A$, then $\beta \in G$. Moreover, $A$ has the additional property that if $\alpha_1, \alpha_2, \ldots, \alpha_t \in G$ are finitely many given elements, then there is a function $f : A \to \mathbb{Z}$ such that $f(n) = n_1$ or $f(n) = n_2$ for every $n = (n_1, n_2) \in A$, and for every $1 \leq i \leq t$ we have
\[ \sum_{n \in A} l^{1+\epsilon} (\|f(n)\alpha_i\|) < \infty. \quad (1.3) \]

If $K$ is finite, the theorem of [B-S] follows at once from Theorem 1, since we can take all elements of $K$ as $\alpha_1, \alpha_2, \ldots, \alpha_t$ (see also Lemma 2 (i) in Section 3). Note that the statement of the Theorem in [B-S] contains a misprint: $\lim \inf$ should be replaced by $\lim \sup$ there.
Theorem 2. If $K$ is uncountable, and $A \subseteq \mathbb{Z}$ is an infinite subset, then

$$G \neq \left\{ \beta \in T : \lim_{n \in A} \| n\beta \| = 0 \right\}.$$ 

This is in fact an easy corollary of a result of Aaronson and Nadkarni, but since the proof of that result is very sketchy in [A-N], we present its proof (see Section 4, Prop. 1.).

We give the proof of Theorem 1 in Section 2. We mention that the basic idea is the same as in [Bi2]. Some lemmas needed in the proof of Theorem 1 are presented in Section 3. We remark that Lemma 4 is very important in the proof, and it provides the main reason why we need an $\epsilon > 0$ in the theorem. The proof of Theorem 2 is given in Section 4. Section 5 contains a few comments and open questions.

2. Proof of Theorem 1

We will use Lemmas 2, 3 and 4, these lemmas are stated and proved in Section 3, so see that section if we refer to one of these lemmas.

If $x \in \mathbb{R}$, we also write $x$ for the coset of $x$ modulo $\mathbb{Z}$, so we consider $x$ as an element of $T$. The fractional part function $\{x\}$ is well-defined on $T$.

Let $T^{(2)}$ be the subgroup of $T$ defined by

$$T^{(2)} = \left\{ \frac{a}{2^N} : N \geq 0, 1 \leq a \leq 2^N \right\}.$$ 

For $N \geq 0$ and $1 \leq a \leq 2^N$ let

$$K_{N,a} = \left\{ \alpha \in K : \frac{a - 1}{2^N} < \{\alpha\} < \frac{a}{2^N} \right\}.$$ 

Since $K$ is a Kronecker set, we can easily see that $K \cap T^{(2)} = \emptyset$, and so every $K_{N,a}$ is an open-closed subset of $K$, and

$$K = \bigcup_{a=1}^{2^N} K_{N,a}$$

(disjoint union). Let $F$ be the set of functions $f : K \to T^{(2)}$ which are constant on each small set of one of these subdivisions, i.e.

$$F = \left\{ f : K \to T^{(2)} : |f(K_{N,a})| \leq 1 \text{ for some } N \geq 0 \text{ and for every } 1 \leq a \leq 2^N \right\},$$

where $|f(K_{N,a})|$ denotes the cardinality of the set $f(K_{N,a})$, and we write $\leq 1$ because it may happen that some set $K_{N,a}$ is empty. Observe that $F$ is countable. Every element of $F$ is a continuous function on $K$, and $F$ is a group under pointwise addition. For a pair $(N,a)$ with $N \geq 0$ and $1 \leq a \leq 2^N$ let $F_{N,a} \subseteq F$ be the subgroup

$$F_{N,a} = \{ f \in F : f(\alpha) = 0 \text{ for } \alpha \in K \setminus K_{N,a}, |f(K_{N,a})| \leq 1 \}.$$
For any $N \geq 0$ let $g_N \in F$ be defined by

$$g_N(\alpha) = \frac{a}{2^N}$$

for every $\alpha \in K_{N,a}$ and for every $1 \leq a \leq 2^N$, and let $f_{N,a,r} \in F_{N,a}$ be defined by ($N \geq 0$, $1 \leq a \leq 2^N$, $r \geq 1$ are fixed):

$$f_{N,a,r}(\alpha) = \begin{cases} 2^{-r}, & \text{if } \alpha \in K_{N,a} \\ 0, & \text{if } \alpha \in K \setminus K_{N,a}. \end{cases}$$

Clearly

$$\max_{\alpha \in K} \|g_N(\alpha) - \alpha\| \leq 2^{-N} \text{ for every } N \geq 0. \quad (2.1)$$

Remark that the functions $g_N$ are not necessarily distinct, but if $N \geq 0$ is fixed, then

$$|\{\nu \geq 0 : g_\nu = g_N\}| < \infty, \quad (2.2)$$

since otherwise (2.1), applied for the elements $\nu$ of this set, would give $g_N(\alpha) = \alpha$ for every $\alpha \in K$, which is impossible by $K \cap T^{(2)} = \emptyset$.

For every $f \in F$ take a number $C(f) > 0$, and for every $N \geq 0$ a number $R(N) > 0$, we assume the following inequalities:

$$\sum_{f \in F} C(f)^{-\epsilon} < \infty, \quad \sum_{N=0}^{\infty} R(N)^{-\epsilon} < \infty, \quad (2.3)$$

and (it is possible by (2.2)):

$$C(g_N) > N \text{ for every } N \geq 0. \quad (2.4)$$

For every $f \in F$ and for every integer $j \geq 1$ we take an integer $m_j(f)$ such that

$$\max_{\alpha \in K} \|f(\alpha) - m_j(f)\alpha\| < 2^{-j-2^jC(f)}, \quad (2.5)$$

which is possible, since $K$ is a Kronecker set. Moreover, we can assume that if $j, j^* \geq 1$, $f, f^* \in F$, then

$$m_j(f^*) \neq m_j(f) \text{ if } (j, f) \neq (j^*, f^*). \quad (2.6)$$

Indeed, there are countably many pairs $(j, f)$, and for a fixed pair $(j, f)$ there are infinitely many possibilities for $m_j(f)$ in (2.5), so we can define recursively the integers $m_j(f)$ to satisfy (2.5) and (2.6).

Let $j(N,a,r) \geq 1$ be integers for every triple $(N,a,r) \in V$, where

$$V = \left\{ (N,a,r) : N \geq 0, 1 \leq a \leq 2^N, r > R(N) \right\},$$

satisfying that if $(N^*,a^*,r^*) \in V$ is another such triple, then

$$j(N,a,r) \neq j(N^*,a^*,r^*), \text{ if } (N,a,r) \neq (N^*,a^*,r^*). \quad (2.7)$$
We easily see from (2.6) and (2.7) that for \((N, a, r)(N^*, a^*, r^*) \in V\) we have
\[
m_{j(N, a, r)}(f_{N, a, r}) \neq m_{j(N^*, a^*, r^*)}(f_{N^*, a^*, r^*}), \quad \text{if} \quad (N, a, r) \neq (N^*, a^*, r^*). \tag{2.8}
\]
Define
\[
H_1 = \left\{ m_{j(N, a, r)}(f_{N, a, r}) : (N, a, r) \in V \right\}. \tag{2.9}
\]
We claim that
\[
\sum_{n \in H_1} t^{1+\varepsilon}(\|n\alpha\|) < \infty \tag{2.10}
\]
for every \(\alpha \in K\). Indeed, let \(\alpha \in K\) be fixed. We have
\[
\|m_{j(N, a, r)}(f_{N, a, r})\alpha\| \leq \|f_{N, a, r}(\alpha)\| + 2^{-1-2j(N, a, r)}C(f_{N, a, r}) \tag{2.11}
\]
by (2.5). Now, on the one hand,
\[
\sum_{a=1}^{2^N} t^{1+\varepsilon}(\|f_{N, a, r}(\alpha)\|) = t^{1+\varepsilon}(2^{-r}), \quad \sum_{N=0}^{\infty} \sum_{r > R(N)} t^{1+\varepsilon}(2^{-r}) < \infty \tag{2.12}
\]
by (2.3); on the other hand, using (2.7) and (2.3), we get
\[
\sum_{(N, a, r) \in V} t^{1+\varepsilon}\left(2^{-1-2j(N, a, r)}C(f_{N, a, r})\right) \leq \sum_{f \in F} \sum_{j \geq 1} (C(f)2^j)^{-(1+\varepsilon)} < \infty. \tag{2.13}
\]
In view of Lemma 2 (i), (2.11)-(2.13), and the definition of \(H_1\) in (2.9), we get (2.10).

If \(s\) is a nonnegative integer, the following set is a compact subset of \(T\):
\[
K_s = \left\{ \alpha = \sum_{i=1}^{t} k_i\alpha_i : \ t \geq 1, \ \alpha_1, \alpha_2, \ldots, \alpha_t \in K, \ k_1, k_2, \ldots, k_t \in \mathbb{Z}, \ \sum_{i=1}^{t} |k_i| \leq s \right\}.
\]

**Lemma 1.** There is a subset \(H\) of the integers such that \(H_1 \subseteq H\) and on the one hand we have
\[
\sum_{n \in H} t^{1+\varepsilon}(\|n\alpha\|) < \infty \tag{2.14}
\]
for every \(\alpha \in K\); on the other hand, if \(\beta \in T\) has the property that
\[
\|n\beta\| < \frac{1}{10} \tag{2.15}
\]
for all but finitely many \(n \in H\), then there is a group homomorphism \(\phi_\beta = \phi : F \to T\) which satisfies the following properties:

(i) for all but finitely many pairs \((f, j)\) with \(f \in F, \ j \geq 1\) we have
\[
\|\phi(f) - m_j(f)\beta\| < 2^{-C(f)-j}; \tag{2.16}
\]
(ii) for every \((N,a)\) pair with \(N \geq 0\), \(1 \leq a \leq 2^N\), if \(K_{N,a} \neq \emptyset\), there is a unique integer \(k_{N,a}\) for which
\[
\phi(f) = k_{N,a}f(\alpha)
\] (2.17)
for every \(f \in F_{N,a}\), where \(\alpha \in K_{N,a}\) is arbitrary; if \(K_{N,a} = \emptyset\), we put \(k_{N,a} = 0\), and then for large \(N\) we have
\[
\max_{1 \leq a \leq 2^N} |k_{N,a}| \leq 2^{R(N)};
\] (2.18)

(iii) if \(N\) is large enough, then writing \(s = \sum_{a=1}^{2^N} |k_{N,a}|\), there is an \(\alpha \in K_s\) such that
\[
\|\alpha - \beta\| \leq \frac{1}{N} + s2^{-N}.
\] (2.19)

**Proof.** Define
\[
H_2 = \{2^r (m_{j+1}(f) - m_j(f)) : f \in F, j \geq 1, 0 \leq r \leq j - 1 + C(f)\}.
\]
Let us choose for every triple \(f_1, f_2, f_3 \in F\) with \(f_3 = f_1 + f_2\) an infinite subset \(J_{f_1,f_2,f_3}\) of the positive integers such that (the first summation below is over every such triple from \(F\))
\[
\Sigma := \sum_{f_3=f_1+f_2} \sum_{j \in J_{f_1,f_2,f_3}} \left(2^j \min(C(f_1), C(f_2), C(f_3))\right)^{-\epsilon} < \infty. \tag{2.20}
\]
Since \(C(f) > 0\) for every \(f \in F\), \(\epsilon > 0\) and \(F\) is countable, this is obviously possible. Then define (we mean again that \(f_1, f_2, f_3\) run over every such triple from \(F\))
\[
H_3 = \left\{2^r (m_j(f_1) + m_j(f_2) - m_j(f_3)) : f_3 = f_1 + f_2, j \in J_{f_1,f_2,f_3}, 0 \leq r \leq j - 2\right\},
\]
\[
H_4 = \{2^r (m_1(g_N) - 1) : N \geq 1, 0 \leq r \leq \log_2 N\}.
\]
Let \(H = \bigcup_{i=1}^4 H_i\). We first prove (2.14). If \(f \in F, j \geq 1\) and \(\alpha \in K\), then
\[
\|(m_{j+1}(f) - m_j(f))\| \alpha \| \leq 2^{-(j+C(f)-1)-(2^j-1)\epsilon} < \infty.
\] (2.21)
by (2.5), therefore, using also Lemma 2 (ii) and (2.3), we obtain
\[
\sum_{n \in H_2} \max_{\alpha \in K} l^{1+\epsilon}(\|n\alpha\|) \leq m \sum_{f \in F} \sum_{j \geq 1} C(f)^{-\epsilon}(2^j - 1)^{-\epsilon} < \infty. \tag{2.22}
\]
If \(\alpha \in K, f_1, f_2, f_3 \in F, f_3 = f_1 + f_2\) and \(j \in J_{f_1,f_2,f_3}\), then by (2.5) we get
\[
\|(m_j(f_1) + m_j(f_2) - m_j(f_3))\| \leq 2^{-(j-2)}2^{-2^j \min(C(f_1), C(f_2), C(f_3))}.
\] (2.23)
and so by Lemma 2 (ii) and (2.20) we get
\[
\sum_{n \in H_3} \max_{\alpha \in K} l^{1+\epsilon}(\|n\alpha\|) \leq m \Sigma < \infty. \tag{2.24}
\]
If \( N \geq 1 \) and \( \alpha \in K \), then
\[
\|(m_1(g_N) - 1)\alpha\| \leq \|m_1(g_N)\alpha - g_N(\alpha)\| + \|g_N(\alpha) - \alpha\| \leq 2^{1-N} \tag{2.25}
\]
by (2.1), (2.4) and (2.5), so by the definition of \( H_4 \), we obtain
\[
\sum_{n \in H_4} \max_{\alpha \in K} l^{1+\epsilon}(\|n\alpha\|) \leq \sum_{N=1}^{\infty} (1 + \log_2 N) l^{1+\epsilon} \left(2^{1-N-\log_2 N}\right) < \infty. \tag{2.26}
\]
The relations (2.10), (2.22), (2.24) and (2.26) prove (2.14).

Now, assume that for a \( \beta \in T \) we have an \( n_0 > 0 \) such that (2.15) is true if \( n \in H \) and \( |n| > n_0 \). Since \( K \) is a Kronecker set, so \( \|n\alpha\| > 0 \) for \( 0 \neq n \in \mathbb{Z}, \alpha \in K \). Therefore, we see from (2.21) (and (2.3)) that
\[
0 < |m_{j+1}(f) - m_j(f)| \leq n_0
\]
can hold only for finitely many pairs \( f \in F, j \geq 1 \); we see from (2.23) that if \( f_1, f_2, f_3 \in F \) are given with \( f_3 = f_1 + f_2 \), then
\[
0 < |m_j(f_1) + m_j(f_2) - m_j(f_3)| \leq n_0
\]
can hold only for finitely many \( j \geq 1 \); and from (2.25) that
\[
0 < |m_1(g_N) - 1| \leq n_0
\]
can hold only for finitely many \( N \). Then, by Lemma 3, we obtain the following inequalities (using \( H_2 \subseteq H, H_3 \subseteq H, H_4 \subseteq H \), respectively):
\[
\|(m_{j+1}(f) - m_j(f))\beta\| < \frac{1/10}{2^{-j+C(f)}} \tag{2.27}
\]
for all but finitely many pairs \( f \in F, j \geq 1 \);
\[
\|(m_j(f_1) + m_j(f_2) - m_j(f_3))\beta\| < \frac{1/10}{2^{-j}} \tag{2.28}
\]
for every triple \( f_1, f_2, f_3 \in F \) with \( f_3 = f_1 + f_2 \) and for large enough \( j \in J_{f_1,f_2,f_3} \);
\[
\|(m_1(g_N) - 1)\beta\| < \frac{1/10}{N/2} \tag{2.29}
\]
for large enough \( N \).

Then from (2.27), for all but finitely many pairs \( f \in F, j_1 \geq 1 \) we have
\[
\|(m_{j_2}(f) - m_{j_1}(f))\beta\| < \frac{2/5}{2^C(f)} \sum_{j=j_1}^{j_2-1} 2^{-j} \tag{2.30}
\]
for every \( j_2 > j_1 \). This implies that \( m_j(f)\beta \) is a Cauchy sequence for every \( f \in F \), so
\[
\phi(f) := \lim_{j \to \infty} m_j(f)\beta \tag{2.31}
\]
exists, (2.16) is satisfied for all but finitely many pairs \( f \in F, j \geq 1 \) by (2.30), and since every \( J_{f_1,f_2,f_3} \) is an infinite set, \( \phi : F \to T \) is a group.
homomorphism by \((2.28)\) and \((2.31)\). We also see that for large \(N\), by \((2.16)\), \((2.4)\) and \((2.29)\), we have
\[
\| \phi(g_N) - \beta \| \leq \frac{1}{N}. \tag{2.32}
\]

If \((N,a)\) is a fixed pair with \(N \geq 0\), \(1 \leq a \leq 2^N\) and \(K_{N,a} \neq \emptyset\), then
\[
\| \phi(f_{N,a,r}) \| \leq \left\| \phi(f_{N,a,r}) - m_{j(N,a,r)}(f_{N,a,r}) \beta \right\| + \left\| m_{j(N,a,r)}(f_{N,a,r}) \beta \right\|,
\]
and so
\[
\limsup_{r \to \infty} \| \phi(f_{N,a,r}) \| \leq \frac{1}{10}
\]
by \((2.16)\), \((2.7)\), using also the assumption on \(\beta\), \((2.8)\) and \(H_1 \subseteq H\). Then \((2.17)\) follows from Lemma 4, because \(F_{N,a}\) is obviously isomorphic to \(T^{(2)}\).

We now prove \((2.18)\). Assume that \(N\) is large and
\[
|k_{N,a}| > 2^{R(N)} \tag{2.33}
\]
for some \(1 \leq a \leq 2^N\). Take an integer \(r\) such that
\[
2|k_{N,a}| \leq 2^r \leq 4|k_{N,a}|. \tag{2.34}
\]
Then \(r > R(N)\), so \(m_{j(N,a,r)}(f_{N,a,r}) \in H_1 \subseteq H\), and so for large \(N\) we have (see \((2.8)\)) that
\[
\left\| m_{j(N,a,r)}(f_{N,a,r}) \beta \right\| < \frac{1}{10}. \tag{2.35}
\]
But \((2.34)\) and \((2.17)\) imply
\[
\| \phi(f_{N,a,r}) \| \geq \frac{1}{4},
\]
which contradicts \((2.35)\) for large \(N\) by \((2.16)\) and \((2.7)\). Therefore \((2.33)\) cannot be true for large \(N\), so \((2.18)\) is proved. To prove \((2.19)\), if \(N \geq 0\), \(1 \leq a \leq 2^N\) are arbitrary and \(k_{N,a} \neq 0\), which implies \(K_{N,a} \neq \emptyset\) by definition, we take an \(\alpha_{N,a} \in K_{N,a}\), and then, by the definition of \(g_N\) and by the already proved properties of \(\phi\), we have
\[
\| \phi(g_N) - \sum_{1 \leq a \leq 2^N, k_{N,a} \neq 0} k_{N,a} \alpha_{N,a} \| \leq 2^{-N} \sum_{a=1}^{2^N} |k_{N,a}|,
\]
and together with \((2.32)\), this proves \((2.19)\). \hfill \Box

**Proof of Theorem 1.** For every \(N \geq 0\) we take some integer \(j(N) \geq 1\) such that the sequence \(j(N)\) is strictly increasing and
\[
\sum_{N=0}^{\infty} 2^{N-1} (R(N) + 2) 2^{1+\epsilon} \left(2^{-j(N)}\right) < \infty. \tag{2.36}
\]
Let
\[
U = \left\{ (N,a) : N \geq 0, 1 \leq a \leq 2^{N-1}, K_{N,2a-1} \neq \emptyset, K_{N,2a} \neq \emptyset \right\},
\]
define \( A^* \subseteq \mathbb{Z}^2 \) as
\[
A^* = \left\{ (m_{j(N)}(f_{N,2a-1,r_1}), m_{j(N)}(f_{N,2a,r_2})) : (N,a) \in U, \right.
\]
\[1 \leq r_1, r_2 \leq R(N) + 2\]
and let \( A = A^* \cup \{(n,n) : n \in H\} \). Note that if \((N,a), (N^*,a^*) \in U\), and \(1 \leq r_1 \leq R(N) + 2, 1 \leq r_1^* \leq R(N^*) + 2\), then
\[
m_{j(N)}(f_{N,2a-1,r_1}) \neq m_{j(N^*)}(f_{N^*,2a^*-1,r_1^*}), \quad \text{if} \quad (N,a) \neq (N^*,a^*). \quad (2.37)
\]
Indeed, this follows from the fact that \( j \) is strictly increasing (so one-to-one), using (2.6) and the definition of \( U \).

Assume that \( \beta \in T \) satisfies (1.2) for all but finitely many \( n = (n_1, n_2) \in A \). Then (2.15) is true for all but finitely many \( n \in H \), we can apply Lemma 1. If \( N \) is large, and we assume that \( k_{N,2a-1} \neq 0 \) and \( k_{N,2a} \neq 0 \) for some \( 1 \leq a \leq 2^{N-1} \) (this implies \( (N,a) \in U \) by the definitions), then by (2.18) we can take a pair \( 1 \leq r_1, r_2 \leq R(N) + 2 \) such that
\[
2 |k_{N,2a-1}| \leq 2^{r_1} \leq 4 |k_{N,2a-1}|, \quad 2 |k_{N,2a}| \leq 2^{r_2} \leq 4 |k_{N,2a}|.
\]
Then by (2.17), we have
\[
\|\phi(f_{N,2a-1,r_1})\| \geq \frac{1}{4}, \quad \|\phi(f_{N,2a,r_2})\| \geq \frac{1}{4},
\]
and, in view of (2.16), \( j(N) \to \infty \), the definition of \( A \), (2.37) and the property of \( \beta \), this is a contradiction for large \( N \). Therefore, if \( N \) is large, then \( k_{N,2a-1}k_{N,2a} = 0 \) for every \( 1 \leq a \leq 2^{N-1} \), and since clearly \( k_{N,2a-1} + k_{N,2a} = k_{N-1,a} \), this easily implies that \( \sum_{a=1}^{2^N} |k_{N,a}| \) is constant for large \( N \). In view of (2.19) and the compactness of the sets \( K_s \), this proves that \( \beta \in G \).

Now, let \( \alpha_1, \alpha_2, \ldots, \alpha_t \) be given distinct elements of \( K \). Then it is clear that if \( N \) is large enough \( (N \geq N_0) \), then for any \( 1 \leq a \leq 2^{N-1} \) we can take a \( \delta(N,a) \in \{0,1\} \) such that
\[
\alpha_1, \alpha_2, \ldots, \alpha_t \notin K_{N,2a-\delta(N,a)},
\]
i.e.
\[
f_{N,2a-\delta(N,a),r} (\alpha_i) = 0
\]
for every \( r \geq 1, 1 \leq i \leq t \). Then, defining \( \delta(N,a) \in \{0,1\} \) arbitrarily for \( 0 \leq N < N_0, 1 \leq a \leq 2^{N-1} \), by (2.5) and (2.36) we have
\[
\sum_{N=0}^{\infty} 2^{N-1} \sum_{1 \leq r_1, r_2 \leq R(N)+2} l^{1+\epsilon} \left( \|m_{j(N)}(f_{N,2a-\delta(N,a),r_2-\delta(N,a)}\alpha_i)\| \right) < \infty
\]
for \( 1 \leq i \leq t \). This, together with (2.14), means that defining \( f \) on \( A^* \) by
\[
f \left( (m_{j(N)}(f_{N,2a-1,r_1}), m_{j(N)}(f_{N,2a,r_2})) \right) = m_{j(N)}(f_{N,2a-\delta(N,a),r_2-\delta(N,a)}),
\]
(the definition is correct by (2.37)), and extending $f$ to $A$ by $f((n, n)) = n$ for $n \in H$, we have (1.3) for every $1 \leq i \leq t$. We proved the existence of such an $f$ for $\alpha_1, \alpha_2, \ldots, \alpha_t \in K$, but since $K$ generates $G$, such an $f$ exists also for $\alpha_1, \alpha_2, \ldots, \alpha_t \in G$, in view of Lemma 2 (i). Then (1.1) follows easily, so the theorem is proved. 

3. Some lemmas

Lemma 2. (i) There is a constant $M > 0$ such that if $x, y \geq 0$, then
\[ l^{1+\epsilon}(x + y) \leq M(l^{1+\epsilon}(x) + l^{1+\epsilon}(y)). \]
(ii) There is an $m > 0$ constant such that for any $a > 0$ we have
\[ \sum_{r=0}^{\infty} l^{1+\epsilon}(2^{-r-a}) \leq ma^{-\epsilon}. \]

Proof. For statement (i) we may obviously assume that $0 < x, y < 1/4$. Then
\[ x + y \leq 2 \max(x, y) \leq \sqrt{\max(x, y)}, \]
and so
\[ l^{1+\epsilon}(x + y) \leq l^{1+\epsilon}\left(\sqrt{\max(x, y)}\right) = \left(-\log_2\left(\sqrt{\max(x, y)}\right)\right)^{-(1+\epsilon)} = 2^{1+\epsilon}l^{1+\epsilon}(\max(x, y)), \]
which proves (i). Statement (ii) is trivial from the definitions.

Lemma 3. If $\omega \in T$, $k \geq 1$ is an integer, and
\[ \|\omega\|, \|2\omega\|, \|4\omega\|, \ldots, \|2^k \omega\| \leq \delta < \frac{1}{10}, \]
then $\|\omega\| \leq \frac{\delta}{2^k}$.

Proof. This is easy, and proved as Lemma 3 of [B-S].

Lemma 4. If $\phi : T^{(2)} \to T$ is a group homomorphism and
\[ \limsup_{r \to \infty} \left\| \phi\left(\frac{1}{2^r}\right) \right\| < \frac{1}{4}, \]
then there is a unique integer $k$ such that $\phi(\alpha) = k\alpha$ for every $\alpha \in T^{(2)}$.

Proof. The uniqueness is obvious, we prove the existence. It is well-known that the Pontriagin dual of the discrete group $T^{(2)}$ is the additive group $\mathbb{Z}_2$ of 2-adic integers. Hence there is a 0-1 sequence $b_r$ ($r \geq 0$) such that
\[ \phi(\alpha) = \left(\sum_{r=0}^{\infty} b_r 2^r\right)\alpha \]
for every $\alpha \in T^{(2)}$, hence
\[ \phi \left( \frac{1}{2^r} \right) = \frac{b_0}{2^r} + \frac{b_1}{2^{r-1}} + \cdots + \frac{b_{r-1}}{2}, \quad (3.3) \]
for every $r \geq 1$. We see from (3.3) that if $b_{r-1} = 1$, $b_{r-2} = 0$, then
\[ \frac{1}{2} \leq \left\{ \phi \left( \frac{1}{2^r} \right) \right\} \leq \frac{3}{4}, \]
which is impossible for large enough $r$, in view of (3.1). Consequently, the sequence $b_r$ is constant for large enough $r$. If this constant is 0, i.e. $b_r = 0$ for $r \geq r_0$, then using (3.2), we get the lemma at once. If the constant is 1, so $b_r = 1$ for $r \geq r_0$, then, since
\[ \sum_{r=0}^{\infty} 2^r = -1 \]
in $\mathbb{Z}_2$, one obtains the lemma from (3.2) with
\[ k = -1 - \left( \sum_{i=0}^{r_0-1} 2^i \right). \]
\[ \square \]

4. Proof of Theorem 2

If $G$ is a group and $d$ is a metric on $G$, we say that $(G, d)$ is a Polish group, if $d$ is a complete metric, and $G$ with this metric is a separable topological group.

The following proposition essentially appears on p. 541. of [A-N], but since they give only a brief indication of the proof, we think that it is worth to include a proof here.

Proposition 1. Assume that $K$ is an uncountable compact subset of $T$, and $K$ is independent over $\mathbb{Z}$. Let $G \leq T$ be the subgroup generated by $K$. Let $d$ be a metric defined on $G$ such that $(G, d)$ is a Polish group. Then the injection map
\[ i : (G, d) \to T, \quad i(g) = g \text{ for every } g \in G \]
is not continuous (we take on $T$ its usual topology, inherited from $\mathbb{R}$).

Proof. Let $Q$ be a countable dense subgroup in $(G, d)$ (such a subgroup clearly exists, since $(G, d)$ is separable). Consider $Q$ with the discrete topology (discrete metric). Then $(Q, G)$ is a Polish (polonais) transformation group in the sense of [E], moreover, it clearly satisfies Condition C on p. 41. of [E]. Since $Q$ is not locally closed in $G$ by our conditions, condition (5) of Theorem 2.6 of [E] is not satisfied. Hence (9) of that theorem is also false, therefore there is a Borel measure $\mu$ on $G$ with $\mu(G) = 1$ such that

(i) each $Q$-invariant measurable subset of $G$ has measure 0 or 1;
(ii) each point of $G$ has measure 0.
Indeed, $\mu(G) = 1$ can be assumed, since $\mu$ is nontrivial and finite by [E], (i) follows since $\mu$ is ergodic in the sense of [E], and (ii) is true by (i), because $\mu$ is not concentrated in a $Q$-orbit.

The measure $\mu$ then has the following additional property, which is a strengthening of (ii):

(iii) if $F \subseteq G$ is a closed subset (in the $d$-topology) and $\mu(F) > 0$, then there is an $A \subseteq F$ with $0 < \mu(A) < \mu(F)$.

It follows by another application of Theorem 2.6 of [E]. Indeed, let $\{0\}$ be the trivial group, then $\{0\}, F$ is a polonais transformation group satisfying Condition C on p.41. of [E], (5) of Theorem 2.6 is true, hence (8) of Theorem 2.6, using (ii), gives (iii).

Now, we are able to prove the proposition. Assume that $i : (G, d) \to T$ is continuous, and we will get a contradiction. For $t \geq 1, n_1, n_2, \ldots, n_t \in \mathbb{Z}$ set

$$E(n_1, n_2, \ldots, n_t) = \{n_1x_1 + n_2x_2 + \ldots + n_tx_t : x_1, x_2, \ldots, x_t \in K\}.$$ 

Every $E(n_1, n_2, \ldots, n_t)$ is a closed set in $(G, d)$, since it is closed in $T$ and $i$ is continuous. Since $G = \bigcup_{t \geq 1, n_1, n_2, \ldots, n_t \in \mathbb{Z}} E(n_1, n_2, \ldots, n_t),$ hence $\mu(E(n_1, n_2, \ldots, n_t)) > 0$ for some values of the parameters.

Let $g \in G, t \geq 1, n_1, n_2, \ldots, n_t \in \mathbb{Z}$ be minimal with the property that

$$\mu(g + E(n_1, n_2, \ldots, n_t)) > 0,$$

in the sense that

$$\mu(h + E(m_1, m_2, \ldots, m_r)) = 0 \quad (4.1)$$

for every $h \in G, r \geq 1, m_1, m_2, \ldots, m_r \in \mathbb{Z}$ with

$$|m_1| + |m_2| + \ldots + |m_r| + |r| < |n_1| + |n_2| + \ldots + |n_t| + |t|. \quad (4.2)$$

By (iii), writing $F = g + E(n_1, n_2, \ldots, n_t)$, there is an $A \subseteq F$ with $0 < \mu(A) < \mu(F)$. Then $\mu\left(\bigcup_{q \in Q}(q + A)\right) > 0$, hence $\mu\left(\bigcup_{q \in Q}(q + A)\right) = 1$ by (i). We prove that

$$\mu\left(\left(\bigcup_{q \in Q}(q + A)\right) \cap (F \setminus A)\right) = 0.$$

This will give a contradiction, because $\mu(F \setminus A) > 0$. Since $Q$ is countable, it is enough to prove that $\mu((q + A) \cap F) = 0$ for every $0 \neq q \in Q$, which follows, if we prove

$$\mu((q + F) \cap F) = 0 \quad (4.3)$$

for every $0 \neq q \in Q$. 

Assume that \( q + f_1 = f_2, f_1 = g + e_1, f_2 = g + e_2 \), where \( f_1, f_2 \in F, e_1, e_2 \in E(n_1, n_2, \ldots, n_t) \). For \( i = 1, 2 \) let
\[
e_i = n_1 x_{i1} + n_2 x_{i2} + \ldots + n_t x_{it}
\]
with \( x_{ij} \in K \) for \( i = 1, 2, 1 \leq j \leq t \). Let
\[
q = \nu_1 x_{01} + \nu_2 x_{02} + \ldots + \nu_s x_{0s}
\]
with \( s \geq 1 \), and \( \nu_i \in \mathbb{Z}, x_{0l} \in K \) for \( 1 \leq l \leq s \). Since \( q + e_1 = e_2, q \neq 0 \), and \( K \) is independent over \( \mathbb{Z} \), there are integers \( 1 \leq i \leq 2, 1 \leq j \leq t \) and \( 1 \leq l \leq s \) such that \( x_{ij} = x_{0l} \). Therefore, if
\[
e_i \in E \text{ for some } 1 \leq i \leq 2.
\]
Hence \( f_2 \in (g + E) \bigcup (g + q + E) \).

Since \( \mu(g + E) = \mu(g + q + E) = 0 \) by (4.1), (4.2), so (4.3) is true, and the proposition is proved. \( \square \)

Proof of Theorem 2. Assume that
\[
G = \bigg\{ \beta \in T : \lim_{n \in A} \|n \beta\| = 0 \bigg\}
\]
for some infinite \( A \subseteq \mathbb{Z} \). For \( x, y \in G \) let
\[
d(x, y) = \|x - y\| + \max_{n \in A} \|n (x - y)\|. \tag{4.4}
\]
It is clear that \( d \) is a metric on \( G \), and \((G, d)\) is a topological group. We show that \( d \) is complete. Let \( \beta_j \in G, j \geq 1 \) be a Cauchy sequence with respect to \( d \). Then \( \beta_j \) is a Cauchy sequence also in \( T \) by (4.4), so there is a \( \beta \in T \) such that \( \|\beta_j - \beta\| \to 0 \) as \( j \to \infty \). Now, for \( n \in A, j_1, j_2 \geq 1 \) we have
\[
\|n (\beta_{j_1} - \beta)\| \leq \|n (\beta_{j_1} - \beta_{j_2})\| + \|n (\beta_{j_2} - \beta)\|. \tag{4.5}
\]
Letting \( j_2 \to \infty \) for fixed \( n \) and \( j_1 \) we get
\[
\|n \beta\| \leq \|n \beta_{j_1}\| + \limsup_{j_2 \to \infty} d(\beta_{j_1}, \beta_{j_2}),
\]
and \( \beta_{j_1} \in G \) gives
\[
\limsup_{n \in A} \|n \beta\| \leq \limsup_{j_2 \to \infty} d(\beta_{j_1}, \beta_{j_2})
\]
for every \( j_1 \geq 1 \), which proves \( \beta \in G \). Let \( \epsilon > 0 \), then we can take \( j_2, N \geq 1 \) so that
\[
\| n (\beta_{j_2} - \beta) \| + \sup_{j_1 \geq j_2} d(\beta_{j_1}, \beta_{j_2}) < \epsilon
\]
for every \( n \in A, |n| \geq N \). Hence for \( j_1 \geq j_2, n \in A, |n| \geq N \) we have
\[
\| n (\beta_{j_1} - \beta) \| < \epsilon \text{ by (4.5).}
\]
Since for any fixed \( |n| < N \) we know that \( \| n (\beta_{j_1} - \beta) \| \to 0 \) as \( j_1 \to \infty \), this proves \( d(\beta_{j_1}, \beta) \to 0 \), so \( d \) is complete.

Let \( X \) be a countable dense subset in \( T \), and for \( N, l \geq 1 \) integers, \( x \in X \) let
\[
U_{N,l,x} = \left\{ \beta \in G : \| \beta - x \| + \max_{n \in A, |n| \leq N} n(\beta - x) + \max_{n \in A, |n| > N} \| n\beta \| < \frac{1}{l} \right\}.
\]
It is easy to check that if we take an element from each nonempty \( U_{N,l,x} \), then we get a countable dense subset of \( (G, d) \). So the conditions of Proposition 1 are satisfied, hence \( i : (G, d) \to T \) is not continuous. But this contradicts (4.4), so the theorem is proved. \( \square \)

5. Some remarks and problems

If \( K \) is finite, it follows from [Bi2], Theorem 1 (ii) that Theorem 1 of the present paper would be false for \( \epsilon = 0 \). But we cannot decide the following

**Problem 1.** Let \( K \) be uncountable. Is Theorem 1 true with \( \epsilon = 0 \)?

The following proposition is a consequence of [V], p.140, Theorem 2' (the quoted theorem of Varopoulos is stronger than this statement):

**Proposition 2.** Let \( L \subseteq T \) be a compact set with \( L \cap G = \emptyset \), then there is an infinite subset \( A \subseteq \mathbb{Z} \) such that
\[
G = \left\{ \beta \in G \cup L : \lim_{n \in A} \| n\beta \| = 0 \right\}.
\]

Compare Proposition 2 with our Theorem 2. We do not know whether Proposition 2 can be strengthened in the following way:

**Problem 2.** Let \( L \subseteq T \) be a compact set with \( L \cap G = \emptyset \). Is there an infinite subset \( A \subseteq \mathbb{Z} \) such that
\[
G = \left\{ \beta \in G \cup L : \lim_{n \in A} \| n\beta \| = 0 \right\},
\]
and
\[
\sum_{n \in A} \| n\alpha \| < \infty
\]
for every \( \alpha \in G \)?

We state without proof our following partial result in this direction.
Theorem 3. Let $L \subseteq T$ be a compact set with $L \cap G = \emptyset$, and let $v$ be a strictly increasing continuous function on the interval $[0,1/2]$ with $v(0) = 0$. Then there is an infinite subset $A \subseteq \mathbb{Z}$ such that we have

$$\sum_{n \in A} l^{1+\epsilon} (\|n\alpha\|) < \infty$$

for every $\alpha \in G$, but

$$\sum_{n \in A} v (\|n\beta\|) = \infty$$

for every $\beta \in L$.

Remark that this theorem implies at once the result mentioned on p.40. of [H-M-P], namely that $G$ is a saturated subgroup of $T$ (for the definition of a saturated subgroup, see [H-M-P] or [N], Ch. 14). We note that the above-mentioned Theorem 2' on [V], p.140, also implies that $G$ is saturated.

Finally, we mention that Theorem 2 and Proposition 2 together show that if $K$ is uncountable, then $G$ is a g-closed but not basic g-closed subgroup of $T$ in the terminology of [D-M-T]. This answers the question of D. Dikranjan (oral communication) about the existence of such subgroups of $T$.

References


[Bi2] A. Biró, Characterizing sets for subgroups of compact groups II.: the general case. Preprint, 2004


András Biró
A. Rényi Institute of Mathematics
Hungarian Academy of Sciences
1053 Budapest, Reáltanoda u. 13-15., Hungary
E-mail: biroand@renyi.hu