Systems of quadratic diophantine inequalities

par WOLFGANG MÜLLER

1. Introduction

Let $Q_1, \ldots, Q_r$ be quadratic forms in $s$ variables with real coefficients. We ask whether the system of quadratic inequalities

\[ |Q_1(x)| < \epsilon, \ldots, |Q_r(x)| < \epsilon \]

has a nonzero integer solution for every $\epsilon > 0$. If some $Q_i$ is rational and $\epsilon$ is small enough then for $x \in \mathbb{Z}^s$ the inequality $|Q_i(x)| < \epsilon$ is equivalent to the equation $Q_i(x) = 0$. Hence if all forms are rational then for sufficiently small $\epsilon$ the system (1.1) reduces to a system of equations. In this case W. SCHMIDT [10] proved the following result. Recall that the real pencil generated by the forms $Q_1, \ldots, Q_r$ is defined as the set of all forms

\[ Q_\alpha = \sum_{i=1}^{r} \alpha_i Q_i \]

where $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{R}^r$, $\alpha \neq 0$. The rational and complex pencil are defined similarly. Suppose that $Q_1, \ldots, Q_r$ are rational quadratic forms. Then the system $Q_1(x) = 0, \ldots, Q_r(x) = 0$ has a nonzero integer solution provided that

\[ A \text{ a real quadratic form is called rational if its coefficients are up to a common real factor rational. It is called irrational if it is not rational.} \]
(i) the given forms have a common nonsingular real solution, and either
(ii) each form in the complex pencil has rank > 4r^2 + 4r, or
(iii) each form in the rational pencil has rank > 4r^3 + 4r^2.

Recently, R. DIETMANN [7] relaxed the conditions (iiia) and (iib). He
replaced them by the weaker conditions

(iiia') each form in the complex pencil has rank > 2r^2 - 3r, or
(iiib') each form in the rational pencil has rank > 2r^3 if r is even and rank
> 2r^3 + 2r if r is odd.

If r = 2 the existence of a nonsingular real solution of Q_1(x) = 0
and Q_2(x) = 0 follows if one assumes that every form in the real pencil
is indefinite (cf. SWINNERTON-DYER [11] and COOK [6]). As noted by
W. SCHMIDT [10] this is false for r > 2.

We want to consider systems of inequalities (1.1) without hidden equa-
lities. A natural condition is to assume that all forms in the real pencil are
irrational. Note that if Q_\alpha is rational and \epsilon is small enough, then (1.1) and
x \in \mathbb{Z}^s imply Q_\alpha(x) = 0. We prove

**Theorem 1.1.** Let Q_1, \ldots, Q_r be quadratic forms with real coefficients.
Then for every \epsilon > 0 the system (1.1) has a nonzero integer solution pro-
vided that

(i) the system Q_1(x) = 0, \ldots, Q_r(x) = 0 has a nonsingular real solution,
(ii) each form in the real pencil is irrational and has rank > 8r.

In the case r = 1 much more is known. G.A. MARGULIS [9] proved that
for an irrational nondegenerate form Q in s \geq 3 variables the set \{Q(x) | x \in \mathbb{Z}^s\} is dense in \mathbb{R} (Oppenheim conjecture). In the case r > 1 all known
results assume that the forms Q_i are diagonal\(^2\). For more information on
these results see E.D. FREEMAN [8] and J. BRUDERN, R.J. COOK [4].

In 1999 V. BENTKUS and F. GÖTZE [2] gave a completely different proof
of the Oppenheim conjecture for s > 8. We use a multidimensional variant
of their method to count weighted solutions of the system (1.1). To do this
we introduce for an integer parameter N \geq 1 the weighted exponential sum

\begin{equation}
S_N(\alpha) = \sum_{x \in \mathbb{Z}^s} w_N(x)e(Q_\alpha(x)) \quad (\alpha \in \mathbb{R}^r) .
\end{equation}

Here Q_\alpha is defined by (1.2), e(x) = \exp(2\pi ix) as usual , and

\begin{equation}
w_N(x) = \sum_{n_1+n_2+n_3+n_4=x} p_N(n_1)p_N(n_2)p_N(n_3)p_N(n_4)
\end{equation}

\(^2\)Note added in proof: Recently, A. GORODNIK studied systems of nondiagonal forms. In
his paper On an Oppenheim-type conjecture for systems of quadratic forms, Israel J. Math.
149 (2004), 125–144, he gives conditions (different from ours) that guarantee the existence of a
nonzero integer solution of (1.1). His Conjecture 13 is partially answered by our Theorem 1.1.
denotes the fourfold convolution of \( p_N \), the density of the discrete uniform probability distribution on \([-N, N] \cap \mathbb{Z}^s \). Since \( w_N \) is a probability density on \( \mathbb{Z}^s \) one trivially obtains \( |S_N(\alpha)| \leq 1 \). The key point in the analysis of BENTKUS and GÖTZE is an estimate of \( S_N(\alpha + \varepsilon)S_N(\alpha - \varepsilon) \) in terms of \( \varepsilon \) alone. Lemma 2.2 gives a generalization of their estimate to the case \( r > 1 \). It is proved via the double large sieve inequality. It shows that for \( N^{-2} < |\varepsilon| < 1 \) the exponential sums \( S_N(\alpha - \varepsilon) \) and \( S_N(\alpha + \varepsilon) \) cannot be simultaneously large. This information is almost sufficient to integrate \( 18N(a) \) within the required precision. As a second ingredient we use for \( 0 < T_0 \leq 1 \leq T_1 \) the uniform bound

\[
\lim_{N \to \infty} \sup_{T_0 \leq |\alpha| \leq T_1} |S_N(\alpha)| = 0.
\]

Note that (1.5) is false if the real pencil contains a rational form. The proof of (1.5) follows closely BENTKUS and GÖTZE [2] and uses methods from the geometry of numbers.

2. The double large sieve bound

The following formulation of the double large sieve inequality is due to BENTKUS and GÖTZE [2]. For a vector \( T = (T_1, \ldots, T_s) \) with positive real coordinates write \( T^{-1} = (T_1^{-1}, \ldots, T_s^{-1}) \) and set

\[
B(T) = \{(x_1, \ldots, x_s) \in \mathbb{R}^s \mid |x_j| \leq T_j \text{ for } 1 \leq j \leq s\}.
\]

**Lemma 2.1 (Double large sieve).** Let \( \mu, \nu \) denote measures on \( \mathbb{R}^s \) and let \( S, T \) be \( s \)-dimensional vectors with positive coordinates. Write

\[
J = \int_{B(S)} \left( \int_{B(T)} g(x)h(y)e(\langle x, y \rangle) \, d\mu(x) \right) \, d\nu(y),
\]

where \( \langle ., . \rangle \) denotes the standard scalar product in \( \mathbb{R}^s \) and \( g, h : \mathbb{R}^s \to \mathbb{C} \) are measurable functions. Then

\[
|J|^2 \ll A(2S^{-1}, g, \mu)A(2T^{-1}, h, \nu) \prod_{j=1}^s (1 + S_j T_j),
\]

where

\[
A(S, g, \mu) = \int \left( \int_{y \in x + B(S)} |g(y)| \, d\mu(y) \right) |g(x)| \, d\mu(x).
\]
The implicit constant is an absolute one. In particular, if $|g(x)| \leq 1$ and $|h(x)| \leq 1$ and $\mu, \nu$ are probability measures, then

$$(2.3) \qquad |J|^2 \ll \sup_{x \in \mathbb{R}^s} \mu(x + B(2S^{-1})) \sup_{x \in \mathbb{R}^s} \nu(x + B(2T^{-1})) \prod_{j=1}^s (1 + S_j T_j).$$

**Remark.** This is Lemma 5.2 in [1]. For discrete measures the lemma is due to E. BOMBIERI and H. IWANIEC [3]. The general case follows from the discrete one by an approximation argument.

**Lemma 2.2.** Assume that each form in the real pencil of $Q_1, \ldots, Q_r$ has rank $\geq p$. Then the exponential sum (1.3) satisfies

$$(2.4) \qquad S_N(\alpha - \varepsilon) S_N(\alpha + \varepsilon) \ll \mu(|\varepsilon|)^p \quad (\alpha, \varepsilon \in \mathbb{R}^r),$$

where

$$\mu(t) = \begin{cases} 1 & 0 \leq t \leq N^{-2}, \\ t^{-1/2} N^{-1} & N^{-2} \leq t \leq N^{-1}, \\ t^{1/2} & N^{-1} \leq t \leq 1, \\ 1 & t \geq 1. \end{cases}$$

**Proof.** Set $S = S_N(\alpha - \varepsilon) S_N(\alpha + \varepsilon)$. We start with

$$S = \sum_{x, y \in \mathbb{Z}^s} w_N(x) w_N(y) e(Q_{\alpha - \varepsilon}(x) + Q_{\alpha + \varepsilon}(y))$$

$$= \sum_{m, n \in \mathbb{Z}^s \atop m \equiv n(2)} w_N(\frac{1}{2}(m-n)) w_N(\frac{1}{2}(m+n)) e(Q_{\alpha - \varepsilon}(\frac{1}{2}(m-n)) + Q_{\alpha + \varepsilon}(\frac{1}{2}(m+n)))$$

$$= \sum_{m \equiv n(2) \atop |m|, |n| \leq 8N} w_N(\frac{1}{2}(m-n)) w_N(\frac{1}{2}(m+n)) e(\frac{1}{2} Q_{\alpha}(m) + \frac{1}{2} Q_{\alpha}(n) + \langle m, Q_{\alpha}(n) \rangle).$$

To separate the variables $m$ and $n$ in the weight function write

$$(2.5) \qquad w_N(x) = \int_B h(\theta) e(-\langle \theta, x \rangle) \, d\theta,$$

where $B = (-1/2, 1/2]^s$ and $h$ denotes the (finite) Fourier series

$$h(\theta) = \sum_{k \in \mathbb{Z}^s} w_N(k) e(\langle \theta, k \rangle).$$

Since $w = p_N * p_N * p_N * p_N$ we find $h(\theta) = h_N(\theta)^2$, where

$$h_N(\theta) = \sum_{k \in \mathbb{Z}^s} p_N * p_N(k) e(\langle \theta, k \rangle).$$
Now set

\[ a(m) = e\left(\frac{1}{2}(Q_\alpha(m) - \langle \theta_1 + \theta_2, m \rangle)\right), \]
\[ b(n) = e\left(\frac{1}{2}(Q_\alpha(n) - \langle \theta_1 - \theta_2, n \rangle)\right). \]

Using (2.5) we find

\[ |S| = \left| \int_B \int_B h(\theta_1)h(\theta_2) \sum_{m \equiv n \pmod{2}} a(m)b(n)e(\langle m, Q_\epsilon n \rangle) \, d\theta_1 d\theta_2 \right| \]
\[ \leq \left( \int_B |h(\theta)| \, d\theta \right)^2 \sup_{\theta_1, \theta_2 \in B} \left| \sum_{m \equiv n \pmod{2}} a(m)b(n)e(\langle m, Q_\epsilon n \rangle) \right|. \]

Note that \( a(m) \) and \( b(n) \) are independent of \( \epsilon \). Furthermore, by Bessel’s inequality

\[ \int_B |h(\theta)| \, d\theta = \int_B |h_N(\theta)|^2 \, d\theta \leq \sum_{k \in \mathbb{Z}^s} (p_N * p_N(k))^2 \]
\[ \leq (2N + 1)^{-s} \sum_{k \in \mathbb{Z}^s} p_N * p_N(k) \leq (2N + 1)^{-s}. \]

Hence

\[ S \ll N^{-2s} \sum_{\omega \in \{0,1\}^s} \sup_{\theta_1, \theta_2 \in B} \left| \sum_{m \equiv n \pmod{2}} a(m)b(n)e(\langle m, Q_\epsilon n \rangle) \right|. \]

We are now in the position to apply Lemma 2.1. Denote by \( \lambda_1, \ldots, \lambda_s \) the eigenvalues of \( Q_\epsilon \) ordered in such a way that \( |\lambda_1| \geq \cdots \geq |\lambda_s| \). Then \( Q_\epsilon = U^T \Lambda U \), where \( U \) is orthogonal and \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_s) \). Set \( \Lambda^{1/2} = \text{diag}(|\lambda_1|^{1/2}, \ldots, |\lambda_s|^{1/2}), \ E = \text{diag}(\text{sgn}(\lambda_1), \ldots, \text{sgn}(\lambda_s)) \) and

\[ M = \{ \Lambda^{1/2}Um \mid m \in \mathbb{Z}^s, m \equiv \omega(2), |m|_\infty \leq 8N \}, \]
\[ N = \{ E\Lambda^{1/2}Um \mid m \in \mathbb{Z}^s, m \equiv \omega(2), |m|_\infty \leq 8N \}. \]

Furthermore, let \( \mu \) denote the uniform probability distribution on \( M \) and \( \nu \) the uniform probability distribution on \( N \). Choose \( S_j = T_j = 1 + 8\sqrt{s} |\lambda_j|^{1/2} N \). Then \( x \in M \) implies \( x \in B(T) \) and \( y \in N \) implies \( y \in B(S) \). If follows by (2.3) that

\[ \left| N^{-2s} \sum_{m \equiv n \pmod{2}} a(m)b(n)e(\langle m, Q_\epsilon n \rangle) \right|^2 \]
\[ \ll N^{-2s} \left( \sup_{x \in \mathbb{R}^s} A(x) \right)^2 \prod_{j=1}^s (1 + |\lambda_j|N^2), \]
where

\[ A(x) = \# \{ m \in \mathbb{Z}^s \mid |m|_\infty \leq 8N, m \equiv \omega(2), \Lambda^{1/2}Um - x \in B(2S^{-1}) \} \]
\[ \ll \# \{ z \in U\mathbb{Z}^s \mid |z|_\infty \ll N, \|\lambda_j\|^{1/2}z_j - x_j \ll S_j^{-1} \} \]
\[ \ll \prod_{j=1}^s \min(N, 1 + |\lambda_j|^{-1}N^{-1}). \]

Hence

\[ S \ll \prod_{j=1}^s \tilde{\mu}(|\lambda_j|) \]

with \( \tilde{\mu}(t) = N^{-1}(1 + t^{1/2}N) \min(N, 1 + t^{-1}N^{-1}) \). To prove (2.4) we have to consider the case \( N^{-2} \leq |\varepsilon| \leq 1 \) only. Otherwise the trivial bound \( |S_N(\alpha)| \leq 1 \) is sufficient. Since \( \lambda_j = \lambda_j(\varepsilon) \) varies continuously on \( \mathbb{R} \setminus \{0\} \) and \( \lambda_j(c\varepsilon) = c\lambda_j(\varepsilon) \) for \( c > 0 \) there exist constants \( 0 \leq c_j \leq \overline{c}_j < \infty \) such that

\[ \lambda_j(\varepsilon) \leq \overline{c}_j|\varepsilon| \quad (1 \leq j \leq s), \]
\[ c_j|\varepsilon| \leq \lambda_j(\varepsilon) \leq \overline{c}_j|\varepsilon| \quad (1 \leq j \leq p). \]

If \( N^{-2} \leq |\varepsilon| \leq 1 \) then \( |\lambda_j| \ll 1 \) and \( \tilde{\mu}(|\lambda_j|) \ll 1 \) for all \( j \leq s \). Furthermore, for \( j \leq p \) we find \( |\lambda_j| \asymp |\varepsilon| \) and \( \tilde{\mu}(|\lambda_j|) \ll \max(|\varepsilon|^{-1/2}N^{-1}, |\varepsilon|^{1/2}) \).

Altogether this yields

\[ S \ll \prod_{j=1}^p \tilde{\mu}(|\lambda_j|) \ll \max(|\varepsilon|^{-1/2}N^{-1}, |\varepsilon|^{1/2})^p \ll \mu(|\varepsilon|)^p. \]

\[ \square \]

3. The uniform bound

Lemma 3.1 (H. Davenport [5]). Let \( L_i(x) = \lambda_{i1}x_1 + \cdots + \lambda_{is}x_s \) be \( s \) linear forms with real and symmetric coefficient matrix \( (\lambda_{ij})_{1 \leq i, j \leq s} \). Denote by \( \| \cdot \| \) the distance to the nearest integer. Suppose that \( P \geq 1 \). Then the number of \( x \in \mathbb{Z}^s \) such that

\[ |x|_\infty \ll P \quad \text{and} \quad \|L_i(x)\| \ll P^{-1} \quad (1 \leq i \leq s) \]

is \( \ll (M_1 \ldots M_s)^{-1} \). Here \( M_1, \ldots, M_s \) denotes the first \( s \) of the \( 2s \) successive minima of the convex body defined by \( F(x, y) \leq 1 \), where for \( x, y \in \mathbb{R}^s \)

\[ F(x, y) = \max(P|L_1(x) - y_1|, \ldots, P|L_s(x) - y_s|, P^{-1}|x_1|, \ldots, P^{-1}|x_s|). \]
Lemma 3.2. Assume that each form in the real pencil of $Q_1, \ldots, Q_r$ is irrational. Then for any fixed $0 < T_0 < T_1 < \infty$

$$\lim_{N \to \infty} \sup_{T_0 \leq |\alpha| \leq T_1} |S_N(\alpha)| = 0.$$  

Proof. We start with one Weyl step. Using the definition of $w_N$ we find

$$|S_N(\alpha)|^2 = \sum_{x,y \in \mathbb{Z}^s} w_N(x)w_N(y)e(Q_\alpha(y) - Q_\alpha(x))$$

$$= \sum_{x \in \mathbb{Z}^s} \sum_{|z|_\infty \leq 8N} w_N(x)w_N(x+z)e(Q_\alpha(z) + 2\langle z, Q_\alpha x \rangle)$$

$$= (2N + 1)^{-8s} \sum_{m_i, n_i, z \in I(m_i, n_i, z)} e(Q_\alpha(z) + 2\langle z, Q_\alpha x \rangle).$$

Here the first sum is over all $m_1, m_2, m_3, n_1, n_2, n_3, z \in \mathbb{Z}^s$ with $|m_i|_\infty \leq N$, $|n_i|_\infty \leq N$, $|z|_\infty \leq 8N$ and $I(m_i, n_i, z)$ is the set

$$\{x \in \mathbb{Z}^s \mid |x - n_1 - n_2 - n_3|_\infty \leq N, |x + z - m_1 - m_2 - m_3|_\infty \leq N\}.$$  

It is an $s$-dimensional box with sides parallel to the coordinate axes and side length $\ll N$. By Cauchy's inequality it follows that

$$|S_N(\alpha)|^4 \ll N^{-9s} \sum_{m_i, n_i, z} \left| \sum_{x \in I(m_i, n_i, z)} e(2\langle x, Q_\alpha z \rangle) \right|^2$$

$$\ll N^{-3s} \sum_{|z|_\infty \leq 8N} \prod_{i=1}^s \min (N, \|2\langle e_i, Q_\alpha z \rangle\|^{-1})^2.$$  

Here we used the well known bound

$$\sum_{x \in I_1 \times \cdots \times I_s} e(\langle x, y \rangle) \ll \prod_{i=1}^s \min (|I_i|, \|e_i, y\|^{-1}),$$

where $I_i$ are intervals of length $|I_i| \gg 1$ and $e_i$ denotes the $i$-th unit vector. Set

$$N(\alpha) = \# \{z \in \mathbb{Z}^s \mid |z|_\infty \leq 16N, \|2\langle e_i, Q_\alpha z \rangle\| < 1/16N \text{ for } 1 \leq i \leq s\}.$$  

We claim that

$$|S_N(\alpha)|^4 \ll N^{-s}N(\alpha).$$

To see this set

$$D_m(\alpha) = \# \{z \in \mathbb{Z}^s \mid |z|_\infty \leq 8N, \frac{m_i - 1}{16N} \leq \|2\langle e_i, Q_\alpha z \rangle\| < \frac{m_i}{16N} \text{ for } i \leq s\},$$

where $\{x\}$ denotes the fractional part of $x$. Then $D_m(\alpha) \leq N(\alpha)$ for all $m = (m_1, \ldots, m_s)$ with $1 \leq m_i \leq 16N$. Note that if $z_1$ and $z_2$ are counted
in $D_m(\alpha)$ then $z_1 - z_2$ is counted in $N(\alpha)$. It follows that

$$\left| S_N(\alpha) \right|^4 \ll N^{-3s} \sum_{1 \leq m_i \leq 16N} D_m(\alpha) \prod_{i=1}^{s} \min \left( N, \frac{16N}{m_i - 1} + \frac{16N}{16N - m_i} \right)^2$$

$$\ll N^{-3s} N(\alpha) \sum_{1 \leq m_i \leq 8N} \prod_{i=1}^{s} \frac{N^2}{m_i^2}$$

$$\ll N^{-s} N(\alpha).$$

To estimate $N(\alpha)$ we use Lemma 3.1 with $P = 16N$ and $L_t(x) = 2 \langle e_i, Q_\alpha x \rangle$ for $1 \leq t \leq s$. This yields

(3.2) \quad $N(\alpha) \ll (M_{1,\alpha} \cdots M_{s,\alpha})^{-1},$

where $M_{1,\alpha} \leq \cdots \leq M_{s,\alpha}$ are the first $s$ from the $2s$ successive minima of the convex body defined in Lemma 3.1.

Now suppose that there exists an $\epsilon > 0$, a sequence of real numbers $N_n \to \infty$ and $\alpha^{(n)} \in \mathbb{R}^r$ with $T_0 \leq |\alpha^{(n)}| \leq T_1$ such that

(3.3) \quad $|S_{N_n}(\alpha^{(n)})| \geq \epsilon.$

By (3.1) and (3.2) this implies

$$\epsilon^4 N_n^2 \ll \left( \prod_{i=1}^{s} M_{i,\alpha^{(n)}} \right)^{-1}.$$

Since $(16N_n)^{-1} \leq M_{1,\alpha^{(n)}} \leq M_{s,\alpha^{(n)}}$ we obtain $\epsilon^4 N_n^2 \ll N_n^{-s} M_{1,\alpha^{(n)}}^{-1}$ and this proves

$$(16N_n)^{-1} \leq M_{1,\alpha^{(n)}} \leq \cdots \leq M_{s,\alpha^{(n)}} \ll (\epsilon^4 N_n)^{-1}.$$

By the definition of the successive minima there exist $x_j^{(n)}$, $y_j^{(n)} \in \mathbb{Z}^s$ such that $(x_1^{(n)}, y_1^{(n)})$, \ldots, $(x_s^{(n)}, y_s^{(n)})$ are linearly independent and $M_{j,\alpha^{(n)}} = F(x_j^{(n)}, y_j^{(n)})$. Hence for $1 \leq i, j \leq s$

$$|L_i(x_j^{(n)} - y_{j,i}^{(n)})| \ll N_n^{-2},$$

$$|x_{j,i}^{(n)}| \ll 1.$$

Since $|\alpha^{(n)}| \leq T_1$ this inequalities imply $|y_{j,i}^{(n)}| \ll T_1$. This proves that the integral vectors

$$W_n = (x_1^{(n)}, y_1^{(n)}, \ldots, x_s^{(n)}, y_s^{(n)}) \quad (n \geq 1)$$

are contained in a bounded box. Thus there exists an infinite sequence $(n'_k)_{k \geq 1}$ with $W_{n'_k} = W_{n_k}$ for $k \geq 1$. The compactness of $\{ \alpha \in \mathbb{R}^s \mid T_0 \leq |\alpha| \leq T_1 \}$ implies that there is a subsequence $(n_k)_{k \geq 1}$ of $(n'_k)_{k \geq 1}$ with
\[ \lim_{k \to \infty} \alpha^{(n_k)} = \alpha^{(0)} \quad \text{and} \quad T_0 \leq |\alpha^{(0)}| \leq T_1. \]

Let \( x_j = x_{j}^{(n_k)} \) and \( y_j = y_{j}^{(n_k)} \) for \( 1 \leq j \leq s \). Then \( x_j \) and \( y_j \) are well defined and

\[ y_j = (L_1(x_j), \ldots, L_s(x_j)) = 2Q_{\alpha^{(0)}} x_j \quad (1 \leq j \leq s). \]

We claim that \( x_1, \ldots, x_s \) are linearly independent. Indeed, suppose that there are \( q_j \) such that \( \sum_{j=1}^{s} q_j x_j = 0 \). Then \( \sum_{j=1}^{s} q_j y_j = 0 \) by (3.4). This implies \( \sum_{j=1}^{s} q_j (x_j, y_j) = 0 \) and the linear independence of \((x_j, y_j)\) yields \( q_j = 0 \) for all \( j \). The matrix equation \( 2Q_{\alpha^{(0)}} (x_1, \ldots, x_s) = (y_1, \ldots, y_s) \) implies that \( Q_{\alpha^{(0)}} \) is rational. By our assumptions this is only possible if \( \alpha^{(0)} = 0 \), contradicting \( |\alpha^{(0)}| \geq T_0 > 0 \). This completes the proof of the Lemma.

**Lemma 3.3.** Assume that each form in the real pencil of \( Q_1, \ldots, Q_r \) is irrational and has rank \( \geq 1 \). Then there exists a function \( T_1(N) \) such that \( T_1(N) \) tends to infinity as \( N \) tends to infinity and for every \( \delta > 0 \)

\[ \lim_{N \to \infty} \sup_{N^{\delta - 2} \leq |\alpha| \leq T_1(N)} |S_N(\alpha)| = 0. \]

**Proof.** We first prove that there exist functions \( T_0(N) \leq T_1(N) \) such that \( T_0(N) \downarrow 0 \) and \( T_1(N) \uparrow \infty \) for \( N \to \infty \) and

\[ \lim_{N \to \infty} \sup_{T_0(N) \leq |\alpha| \leq T_1(N)} |S_N(\alpha)| = 0. \]

From Lemma 3.2 we know that for each \( m \in \mathbb{N} \) there exist an \( N_m \) with

\[ |S_N(\alpha)| \leq \frac{1}{m} \quad \text{for} \quad N \geq N_m \quad \text{and} \quad \frac{1}{m} \leq |\alpha| \leq m. \]

Without loss of generality we assume that \((N_m)_{m \geq 1}\) is increasing. For \( N_m \leq N < N_{m+1} \) define \( T_0(N) = \frac{1}{m} \), \( T_1(N) = m \) and for \( N < N_1 \) set \( T_0(N) = T_1(N) = 1 \). Obviously this choice satisfies (3.5). Replacing \( T_0(N) \) by \( \max(T_0(N), N^{-1}) \) we can assume that \( N^{-1} \leq T_0(N) \leq 1 \). Finally, Lemma 2.2 with \( p \geq 1 \) yields

\[ \sup_{N^{\delta - 2} \leq |\alpha| \leq T_0(N)} |S_N(\alpha)| \ll \sup_{N^{\delta - 2} \leq |\alpha| \leq T_0(N)} \mu(|\alpha|^p) \ll \max(N^{-\delta/2}, T_0(N)^{1/2})^p \to 0. \]

\[ \square \]

**4. The integration procedure**

In this section we use Lemma 2.2 to integrate \( |S_N(\alpha)| \). It is here where we need the assumption \( p > 8r \).
Lemma 4.1. For $0 < U \leq T$ set $B(U, T) = \{ \alpha \in \mathbb{R}^r \mid U \leq |\alpha| \leq T \}$ and define
\[
\gamma(U, T) = \sup_{\alpha \in B(U, T)} |S_N(\alpha)|.
\]

Furthermore, let $h$ be a measurable function with $0 \leq h(\alpha) \leq (1 + |\alpha|)^{-k}$, $k > r$. If each form in the real pencil generated by $Q_1, \ldots, Q_r$ has rank $\geq p$ with $p > 8r$ and if $\gamma(U, T) \geq 4^{p/(8r)}N^{-p/4}$ then
\[
\int_{B(U, T)} |S_N(\alpha)| h(\alpha) \, d\alpha \ll N^{-2r} \min(1, U^{-(k-r)}) \gamma(U, T)^{1-\frac{8r}{p}}.
\]

Proof. Set $B = B(U, T)$ and $\gamma = \gamma(U, T)$. For $l \geq 0$ define
\[
B_l = \{ \alpha \in B \mid 2^{-l-1} \leq |S_N(\alpha)| \leq 2^{-l} \}.
\]

If $L$ denotes the least non negative integer such that $\gamma \geq 2^{-L-1}$ then $|S_N(\alpha)| \leq \gamma \leq 2^{-L}$ and for any $M \geq L$
\[
B = \bigcup_{l=L}^{M} B_l \cup D_M,
\]
where $D_M = \{ \alpha \in B \mid |S_N(\alpha)| \leq 2^{-M-1} \}$.

By Lemma 2.2
\[
|S_N(\alpha)S_N(\alpha + \epsilon)| \leq C\mu(|\epsilon|^p)
\]
with some constant $C$ depending on $Q_1, \ldots, Q_r$. By considering $C^{-1/2}S_N(\alpha)$ instead of $S_N(\alpha)$ we may assume $C = 1$. If $\alpha \in B_l$ and $\alpha + \epsilon \in B_l$ it follows that
\[
4^{-l-1} \leq |S_N(\alpha)S_N(\alpha + \epsilon)| \leq \mu(|\epsilon|^p).
\]

If $|\epsilon| \leq N^{-1}$ this implies $|\epsilon| \leq N^{-2}2^{4(l+1)/p} = \delta$, say, and if $|\epsilon| \geq N^{-1}$ this implies $|\epsilon| \geq 2^{-4(l+1)/p} = \rho$, say. Note that $\delta \leq \rho$ if $2^{8(l+1)/p} \leq N^2$, and this is true for all $l \leq M$ if
\[
M + 1 \leq \log(N^{p/4})/\log 2.
\]

We choose $M$ as the largest integer less or equal to $\log(N^{2r\gamma \rho^{-1}})/\log 2 - 1$. Then the assumption $\gamma \geq 4^{p/(8r)}N^{-p/4}$ implies $L \leq M$, (4.1) and
\[
2^{-M} \ll N^{-2r}\gamma^{1-8r/p}.
\]

To estimate the integral over $B_l$ we split $B_l$ in a finite number of subsets. If $B_l \neq \emptyset$ choose any $\beta_1 \in B_l$ and set $B_l(\beta_1) = \{ \alpha \in B_l \mid |\alpha - \beta_1| \leq \delta \}$. If $\alpha \in B_l \setminus B_l(\beta_1)$ then $|\alpha - \beta_1| \geq \rho$. If $B_l \setminus B_l(\beta_1) \neq \emptyset$ choose $\beta_2 \in B_l \setminus B_l(\beta_1)$ and set $B_l(\beta_2) = \{ \alpha \in B_l \setminus B_l(\beta_1) \mid |\alpha - \beta_2| \leq \delta \}$. Then $|\alpha - \beta_1| \geq \rho$ and $|\alpha - \beta_2| \geq \rho$ for all $\alpha \in B_l \setminus \{ B_l(\beta_1) \cup B_l(\beta_2) \}$. Especially $|\beta_1 - \beta_2| \geq \rho$. In this way we construct a sequence $\beta_1, \ldots, \beta_m$ of points in $B_l$ with $|\beta_i - \beta_j| \geq \rho$ for $i \neq j$. This construction terminates after finitely many
steps. To see this note that the balls $K_{\rho/2}(\beta_i)$ with center $\beta_i$ and radius $\rho/2$ are disjoint and contained in a ball with center 0 and radius $T + \rho/2$. Thus $m \text{vol}(K_{\rho/2}) \leq \text{vol}(K_{T+\rho/2})$ and this implies $m \ll (1 + T/\rho)^r$. Since $B_l \subseteq \bigcup_{i=1}^m B_l(\beta_i) \subseteq \bigcup_{i=1}^m K_{\delta}(\beta_i)$ we obtain

$$\int_{B_l} |S_N(\alpha)| h(\alpha) \, d\alpha \leq 2^{-l} \sum_{i=1}^m \int_{K_{\delta}(\beta_i)} (1 + |\alpha|)^{-k} \, d\alpha$$

$$\ll 2^{-l} \sum_{i \leq m \atop |\beta_i| \leq 1} \delta^r + 2^{-l} \sum_{i \leq m \atop |\beta_i| > 1} \left(\frac{\delta}{\rho}\right)^r \int_{K_{\rho/2}(\beta_i)} |\alpha|^{-k} \, d\alpha.$$

Note that $|\alpha| \asymp |\beta_i|$ for $\alpha \in K_{\rho}(\beta_i)$ if $|\beta_i| \geq 1$. If $U > 1$ the first sum is empty and the second sum is $\ll (\delta/\rho)^r \int_{|\alpha| > U/2} |\alpha|^{-k} \, d\alpha \ll (\delta/\rho)^r U^{-(k-r)}$. If $U \leq 1$ then the first sum contains $\ll \rho^{-r}$ summands; Thus both sums are bounded by $(\delta/\rho)^r$. This yields

$$\int_{B_l} |S_N(\alpha)| h(\alpha) \, d\alpha \ll 2^{-l} \left(\frac{\delta}{\rho}\right)^r \min(1, U^{-(k-r)}).$$

Altogether we obtain by (4.2) and the definition of $\delta$, $\rho$, $L$

$$\int_{B_l} |S_N(\alpha)| h(\alpha) \, d\alpha \ll \sum_{l=L}^M 2^{-l} \left(\frac{\delta}{\rho}\right)^r \min(1, U^{-(k-r)}) + 2^{-M} \int_{|\alpha| \geq U} h(\alpha) \, d\alpha$$

$$\ll \left(N^{-2r} \sum_{l=L}^M 2^{-l(1-8r/p)} + 2^{-M} \right) \min(1, U^{-(k-r)})$$

$$\ll \left(N^{-2r} 2^{-L(1-8r/p)} + 2^{-M} \right) \min(1, U^{-(k-r)})$$

$$\ll N^{-2r} \gamma^{1-8r/p} \min(1, U^{-(k-r)}).$$

5. Proof of Theorem 1.1

We apply a variant of the Davenport-Heilbronn circle method to count weighted solutions of (1.1). Without loss of generality we may assume $\epsilon = 1$. Otherwise apply Theorem 1.1 to the forms $\epsilon^{-1} Q_i$. We choose an even probability density $\chi$ with support in $[-1, 1]$ and $\chi(x) \geq 1/2$ for $|x| \leq 1/2$. By choosing $\chi$ sufficiently smooth we may assume that its Fourier transform satisfies $\hat{\chi}(t) = \int \chi(x) e(t x) \, dx \ll (1 + |t|)^{-r-3}$. Set

$$K(v_1, \ldots, v_r) = \prod_{i=1}^r \chi(v_i).$$
Then \( \hat{K}(\alpha) = \prod_{i=1}^{r} \chi(\alpha_i) \). By Fourier inversion we obtain for an integer parameter \( N \geq 1 \)

\[
A(N) := \sum_{x \in \mathbb{Z}^s} w_N(x) K(Q_1(x), \ldots, Q_r(x))
\]

\[
= \sum_{x \in \mathbb{Z}^s} w_n(x) \int_{\mathbb{R}^r} e(\alpha_1 Q_1(x) + \cdots + \alpha_r Q_r(x)) \hat{K}(\alpha) \, d\alpha_1 \ldots d\alpha_r
\]

\[
= \int_{\mathbb{R}^r} S_N(\alpha) \hat{K}(\alpha) \, d\alpha.
\]

Our aim is to prove for \( N \geq N_0 \), say,

(5.1) \[
A(N) \geq cN^{-2r}
\]

with some constant \( c > 0 \). This certainly implies the existence of a non-trivial solution of (1.1), since the contribution of the trivial solution \( x = 0 \) to \( A(N) \) is \( \ll N^{-s} \) and \( s \geq p > 8r \). To prove (5.1) we divide \( \mathbb{R}^r \) in a major arc, a minor arc and a trivial arc. For \( \delta > 0 \) set

\[
\mathcal{M} = \{ \alpha \in \mathbb{R}^r \mid |\alpha| < N^{\delta-2} \},
\]

\[
\mathcal{M} = \{ \alpha \in \mathbb{R}^r \mid N^{\delta-2} \leq |\alpha| \leq T_1(N) \},
\]

\[
\mathcal{M} = \{ \alpha \in \mathbb{R}^r \mid |\alpha| > T_1(N) \},
\]

where \( T_1(N) \) denotes the function of Lemma 3.3. Using the bound \( \hat{K}(\alpha) \ll (1 + |\alpha|)^{-\tau-3} \), Lemma 4.1 (with the choice \( U = T_1(N) \) and the trivial estimate \( \gamma(T_1(N), \infty) \leq 1 \)) implies

\[
\int_{\mathcal{M}} S_N(\alpha) \hat{K}(\alpha) \, d\alpha = O(N^{-2r} T_1(N)^{-3}) = o(N^{-2r}).
\]

Furthermore, Lemma 4.1 with \( U = N^{\delta-2} \) and \( T = T_1(N) \), together with Lemma 3.3 yield

\[
\int_{\mathcal{M}} S_N(\alpha) \hat{K}(\alpha) \, d\alpha = O(N^{-2r} \gamma(N^{\delta-2}, T_1(N))^{1-\frac{8r}{p}}) = o(N^{-2r}).
\]

Thus (5.1) follows if we can prove that the contribution of the major arc is

(5.2) \[
\int_{\mathcal{M}} S_N(\alpha) \hat{K}(\alpha) \, d\alpha \gg N^{-2r}.
\]

6. The major arc

Lemma 6.1. Assume that each form in the real pencil of \( Q_1, \ldots, Q_r \) has rank \( \geq p \). Let \( g, h : \mathbb{R}^s \to \mathbb{C} \) be measurable functions with \( |g| \leq 1 \) and \( |h| \leq 1 \). Then for \( N \geq 1 \)

\[
N^{-2k} \int_{[-N,N]^s} \int_{[-N,N]^s} g(x) h(y) e(\langle x, Q_\alpha y \rangle) \, dx \, dy \ll (|\alpha|^{-1/2} N^{-1})^p.
\]
Proof. Note that the bound is trivial for $|\alpha| \leq N^{-2}$. Hence we assume $|\alpha| \geq N^{-2}$. Denote by $\lambda_1, \ldots, \lambda_s$ the eigenvalues of $Q_\alpha$ ordered in such a way that $|\lambda_1| \geq \cdots \geq |\lambda_s|$. Then $Q_\alpha = U^T \Lambda U$, where $U$ is orthogonal and $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_s)$. Write $x = (\bar{x}, \overline{x})$, where $\bar{x} = (x_1, \ldots, x_p)$ and $\overline{x} = (x_{p+1}, \ldots, x_s)$. Then

$$N^{-2s} \int_{[-N,N]^s} \int_{[-N,N]^s} g(x) h(y) e(\langle x, Q_\alpha y \rangle) \, dx \, dy = N^{-2s} \int_{U[-N,N]^s} \int_{U[-N,N]^s} g(U^{-1} x) h(U^{-1} y) e(\langle x, \Lambda y \rangle) \, dx \, dy$$

$$= N^{-2(s-p)} \int_{|\bar{x}| \leq \sqrt{s} N \atop |\overline{x}| \leq \sqrt{s} N} \int_{|\bar{y}| \leq \sqrt{s} N \atop |\overline{y}| \leq \sqrt{s} N} e \left( \sum_{i=p+1}^s \lambda_i \bar{x}_i \overline{y}_i \right) J(\bar{x}, \overline{y}) \, d\bar{x} \, d\overline{y} ,$$

(6.1)

where

$$J(\bar{x}, \overline{y}) = N^{-2p} \int_{-\sqrt{s} N}^{\sqrt{s} N} \int_{-\sqrt{s} N}^{\sqrt{s} N} \tilde{g}(\bar{x}) \tilde{h}(\overline{y}) e \left( \sum_{i=1}^p \lambda_i \bar{x}_i \overline{y}_i \right) \, d\bar{x} \, d\overline{y} .$$

Here $\tilde{g}(\bar{x}) = g(U^{-1} x) I_A(\bar{x})(\bar{x})$ with

$$A(\bar{x}) = \{ x \in \mathbb{R}^p \mid (x, \bar{x}) \in U[-N, N]^s \} \subseteq [-\sqrt{s} N, \sqrt{s} N]^p ,$$

and $\tilde{h}$ is defined similarly. If $|\alpha| \geq N^{-2}$ then by (2.6) $|\lambda_i| \asymp |\alpha| \gg N^{-2}$ for $i \leq p$. Now we apply the double large sieve bound (2.3). For $1 \leq j \leq p$ set $S_j = T_j = \sqrt{s} |\lambda_j| N$. Let $\mu = \nu$ be the continuous uniform probability distribution on $\prod_{j=1}^p [-T_j, T_j]$ and set $\tilde{g}(\bar{x}) = \tilde{g}(|\lambda_1|^{-1/2} x_1, \ldots, |\lambda_p|^{-1/2} x_p)$ and $\tilde{h}(\overline{y}) = \tilde{h}(\text{sgn}(\lambda_1)|\lambda_1|^{-1/2} x_1, \ldots, \text{sgn}(\lambda_p)|\lambda_p|^{-1/2} x_p)$. Then

$$|J(\bar{x}, \overline{y})|^2 \ll \left( \int \int \tilde{g}(\bar{x}) \tilde{h}(\overline{y}) \, d\mu(\bar{x}) \, d\nu(\overline{y}) \right)^2$$

$$\ll \prod_{j=1}^p (1 + |\lambda_j| N^2) (|\lambda_j|^{-1} N^{-2})^2$$

$$\ll |\alpha|^{-p} N^{-2p} .$$

Together with (6.1) this proves the lemma. \qed

For $\alpha \in \mathfrak{M}$ we want to approximate $S_N(\alpha)$ by

$$G_0(\alpha) = \int \sum_{x \in \mathbb{Z}^s} w_N(x) e(Q_\alpha(x + z)) \, d\pi(z) ,$$

where $\pi = I_B * I_B * I_B * I_B$ is the fourfold convolution of the continuous uniform distribution on $B = (-1/2, 1/2]^8$. Set $g(u) = e(Q_\alpha(u))$. Denote by
$g_{u_1}$ the directional derivative of $g$ in direction $u_1$, and set $g_{u_1u_2} = (g_{u_1})_{u_2}$. We use the Taylor series expansions

$$f(1) = f(0) + \int_0^1 f'(\tau) \, d\tau,$$

$$f(1) = f(0) + f'(0) + \frac{1}{2} f''(0) + \frac{1}{2} \int_0^1 (1 - \tau)^2 f'''(\tau) \, d\tau,$$

$$f(1) = f(0) + f'(0) + \frac{1}{2} f''(0) + \frac{1}{2} \int_0^1 (1 - \tau)^2 f'''(\tau) \, d\tau.$$

Applying the third of these relations to $f(\tau) = g(x + \tau u_1)$, the second to $f(\tau) = g_{u_1}(x + \tau u_2)$ and the first to $f(\tau) = g_{u_1u_1}(x + \tau u_3)$ we find for $u_1, u_2, u_3 \in \mathbb{R}^s$

$$g(x + u_1) = g(x) + g_{u_1}(x) + \frac{1}{2} g_{u_1u_1}(x) + \frac{1}{2} \int_0^1 (1 - \tau)^2 g_{u_1u_1u_1}(x + \tau u_1) \, d\tau,$$

$$g_{u_1}(x + u_2) = g_{u_1}(x) + g_{u_1u_2}(x) + \int_0^1 (1 - \tau) g_{u_1u_2u_2}(x + \tau u_2) \, d\tau,$$

$$g_{u_1u_1}(x + u_3) = g_{u_1u_1}(x) + \int_0^1 g_{u_1u_1u_3}(x + \tau u_3) \, d\tau.$$

Together we obtain the expansion

$$g(x) = g(x + u_1) - g_{u_1}(x + u_2) - \frac{1}{2} g_{u_1u_1}(x + u_3) + g_{u_1u_2}(x + u_3) + \int_0^1 \left\{ - g_{u_1u_2u_3}(x + \tau u_3) + \frac{1}{2} g_{u_1u_1u_3}(x + \tau u_3) + (1 - \tau) g_{u_1u_2u_2}(x + \tau u_2) - \frac{1}{2} (1 - \tau)^2 g_{u_1u_1u_1}(x + \tau u_1) \right\} \, d\tau.$$

Multiplying with $w_N(x)$, summing over $x \in \mathbb{Z}^s$, and integrating $u_1, u_2, u_3$ with respect to the probability measure $\pi$ yields

$$S_N(\alpha) = G_0(\alpha) + G_1(\alpha) + G_2(\alpha) + G_3(\alpha) + R(\alpha),$$

where $G_0(\alpha)$ is defined by (6.2),

$$G_1(\alpha) = - \int \int \sum_{x \in \mathbb{Z}^s} w_N(x) g_u(x + z) \, d\pi(u) \, d\pi(z),$$

$$G_2(\alpha) = \frac{1}{2} \int \int \sum_{x \in \mathbb{Z}^s} w_N(x) g_{uu}(x + z) \, d\pi(u) \, d\pi(z),$$

$$G_3(\alpha) = \int \int \int \sum_{x \in \mathbb{Z}^s} w_N(x) g_{uv}(x + z) \, d\pi(u) \, d\pi(v) \, d\pi(z),$$

and

$$R(\alpha) \ll \sup_{|u|_\infty, |v|_\infty, |w|_\infty, |z|_\infty \leq 1} \left| \sum_{x \in \mathbb{Z}^s} w_N(x) g_{uvw}(x + z) \right|.$$
An elementary calculation yields
\[ g_u(x) = 4\pi i e(Q_\alpha(x)) \langle x, Q_\alpha u \rangle, \]
\[ g_{uv}(x) = (4\pi i)^2 e(Q_\alpha(x)) \langle x, Q_\alpha u \rangle \langle x, Q_\alpha v \rangle + 4\pi i e(Q_\alpha(x)) \langle u, Q_\alpha v \rangle, \]
\[ g_{uvw}(x) = (4\pi i)^3 e(Q_\alpha(x)) \langle x, Q_\alpha u \rangle \langle x, Q_\alpha v \rangle \langle x, Q_\alpha w \rangle + (4\pi i)^2 e(Q_\alpha(x)) \times \]
\[ \langle x, Q_\alpha v \rangle \langle u, Q_\alpha w \rangle + \langle x, Q_\alpha u \rangle \langle v, Q_\alpha w \rangle + \langle x, Q_\alpha w \rangle \langle u, Q_\alpha v \rangle. \]

Since \( g_u \) and \( g_{uv} \) are sums of odd functions (in at least one of the components of \( u \)) we infer \( G_1(\alpha) = 0 \) and \( G_3(\alpha) = 0 \). Furthermore, the trivial bound \( g_{uvw}(x) \ll |\alpha|^3 N^3 + |\alpha|^2 N \) for \(|x|_\infty \ll N \) yields
\[ R(\alpha) \ll |\alpha|^3 N^3 + |\alpha|^2 N. \]

This is sharp enough to prove
\[
\int_{\mathbb{R}} |R(\alpha) \tilde{K}(\alpha)| \, d\alpha \ll \int_{|\alpha| \leq N^{\delta-2}} |\alpha|^3 N^3 + |\alpha|^2 N \, d\alpha
\ll \int_0^{N^{\delta-2}} u^{r+2} N^3 + u^{r+1} N \, du
\ll N^{3-(2-\delta)(r+3)} + N^{1-(2-\delta)(r+2)}
\ll N^{-2r-3+\delta(r+3)} = o(N^{-2r}).
\]

To deal with \( G_0 \) and \( G_2 \) we need a bound for
\[
\tilde{G}_j(\alpha, u) = \int_{\mathbb{R}^2} \sum_{x \in \mathbb{Z}^2} w_N(x) L(x + z)^j e(Q_\alpha(x + z)) \, d\pi(z),
\]
where \( L(x) = \langle x, Q_\alpha u \rangle \) and \( 0 \leq j \leq 2 \). Using the definition of \( w_N \) and \( \pi \) we find that \( \tilde{G}_j(\alpha, u) \) is equal to
\[
\int_{B^4} \sum_{x_1, \ldots, x_4 \in \mathbb{Z}^4} \left( \prod_{i=1}^4 p_N(x_i) L \left( \sum_{i=1}^4 (x_i + z_i) \right)^j e \left( Q_\alpha \left( \sum_{i=1}^4 (x_i + z_i) \right) \right) \right) \, dx_1 \ldots dx_4
= (2N+1)^{-4s} \int_{|x_1|_{\infty}, \ldots, |x_4|_{\infty} \leq N^{1/2}} L \left( \sum_{i=1}^4 x_i \right)^j e \left( Q_\alpha \left( \sum_{i=1}^4 x_i \right) \right) \, dx_1 \ldots dx_4.
\]
Expanding \( L(x_1 + x_2 + x_3 + x_4) \) and \( Q_\alpha(x_1 + x_2 + x_3 + x_4) \) this can be bounded by
\[
\max_{l_1 + l_2 + l_3 + l_4 = j} N^{-4s} \left| \int \left( \prod_{i=1}^4 L(x_i)^{l_i} e(Q_\alpha(x_i)) \right) e(2 \sum_{i < j} \langle x_i, Q_\alpha x_j \rangle) \, dx_1 \ldots dx_4 \right|
\ll \max_{l_1 + l_2 + l_3 + l_4 = j} \frac{(|\alpha| N)^j}{N^{4s}} \left| \int \left( \prod_{i=1}^4 h_i(x_i) \right) e(2 \sum_{i < j} \langle x_i, Q_\alpha x_j \rangle) \, dx_1 \ldots dx_4 \right|.
\]
Here

\[ h_i(x_i) = L(x_i) \hat{h}(Q_\alpha(x_i)) (|\alpha|N)^{-\ell_1} I_{\{|x_i| \leq N^{1/2}\}} \ll 1. \]

Applying Lemma 6.1 to the double integral over \(x_1\) and \(x_2\) and estimating the integral over \(x_3\) and \(x_4\) trivially we obtain uniformly in \(|u| \ll 1\)

\[ \tilde{G}_j(\alpha, u) \ll (|\alpha|N)^2|\alpha|^{-p/2}N^{-p}. \]

Setting

\[ H_j(N) = \int_{\mathbb{R}} G_j(\alpha) \hat{K}(\alpha) \, d\alpha \]

we conclude for sufficiently small \(\delta > 0\) and \(p > 8r\) \((G_0(\alpha) = \tilde{G}_0(\alpha, 0))\)

\[
\int_{\mathfrak{M}} G_0(\alpha) \hat{K}(\alpha) \, d\alpha = H_0(N) - \int_{|\alpha| \geq N^{\delta-2}} \tilde{G}_0(\alpha, 0) \hat{K}(\alpha) \, d\alpha
\]

\[
= H_0(N) + O(N^{-p} \left( \int_{N^{\delta-2} \leq |\alpha| \leq 1} |\alpha|^{-p/2} \, d\alpha + 1 \right))
\]

\[
= H_0(N) + O(N^{-p - (2-\delta)(r-p/2)}) + O(N^{-p})
\]

\[
= H_0(N) + o(N^{-2r}).
\]

Similarly, the explicit expression of \(g_{uu}(x)\) and the definition of \(\tilde{G}_2(\alpha, u)\) yield

\[
\int_{\mathfrak{M}} G_2(\alpha) \hat{K}(\alpha) \, d\alpha
\]

\[
= H_2(N) + O \left( \sup_{|u| \leq 2} \int_{|\alpha| \geq N^{\delta-2}} |\tilde{G}_2(\alpha, u) \hat{K}(\alpha)| + |\alpha||\tilde{G}_0(\alpha, u) \hat{K}(\alpha)| \, d\alpha \right)
\]

\[
= H_2(N) + O \left( N^{2-p} \left( \int_{N^{\delta-2} \leq |\alpha| \leq 1} |\alpha|^{-p/2} \, d\alpha + 1 \right) \right)
\]

\[
+ O \left( N^{-p} \left( \int_{N^{\delta-2} \leq |\alpha| \leq 1} |\alpha|^{1-p/2} \, d\alpha + 1 \right) \right)
\]

\[
= H_2(N) + o(N^{-2r}).
\]

Hence

\[
\int_{\mathfrak{M}} S_N(\alpha) \hat{K}(\alpha) \, d\alpha = H_0(N) + H_2(N) + o(N^{-2r}).
\]

Altogether we have proved that for \(p > 8r\)

(6.3) \[ A(N) = H_0(N) + H_2(N) + o(N^{-2r}). \]
7. Analysis of the terms $H_0(N)$ and $H_2(N)$

Lemma 7.1. Denote by $\pi_N$ the fourfold convolution of the continuous uniform probability distribution on $B_N = (-N - 1/2, N + 1/2]^8$ and by $f_N$ the density of $\pi_N$. Then

$$H_0(N) = \int K(Q_1(x), \ldots, Q_r(x)) f_N(x) \, dx$$

and

$$H_2(N) = -\frac{1}{6} \int K(Q_1(x), \ldots, Q_r(x)) \Delta f_N(x) \, dx,$$

where $\Delta f_N(x) = \sum_{i=1}^8 \frac{\partial^2 f_N}{\partial x_i^2}(x)$. Furthermore, $\Delta f_N(x) \ll N^{-s-2}$.

Proof. By Fourier inversion and the definition of $w_N$ and $\pi = \pi_0$ we find

$$H_0(N) = \int_{\mathbb{R}^r} G_0(\alpha) \tilde{K}(\alpha) \, d\alpha$$

$$= \int \sum_{x \in \mathbb{Z}^r} w_N(x) \int_{\mathbb{R}^r} e(Q_\alpha(x + z)) \tilde{K}(\alpha) \, d\alpha \, d\pi(z)$$

$$= \int \sum_{x \in \mathbb{Z}^r} w_N(x) K(Q_1(x + z), \ldots, Q_r(x + z)) \, d\pi(z)$$

$$= \int K(Q_1(x), \ldots, Q_r(x)) \, d\pi_N(x).$$

This proves the first assertion of the Lemma. Similarly,

$$-2G_2(\alpha) = \int \int g_{uu}(x) \, d\pi(u) \, d\pi_N(x).$$

This implies

$$-2H_2(N) = -2 \int G_2(\alpha) \tilde{K}(\alpha) \, d\alpha = \int \int \int g_{uu}(x) \tilde{K}(\alpha) \, d\alpha \, d\pi(u) \, d\pi_N(x).$$

With the abbreviations $L_m = 2\langle x, Q_m u \rangle$ and $\tilde{L}_m = 2\langle u, Q_m v \rangle$ the innermost integral can be calculated as

$$\int_{\mathbb{R}^r} g_{uu}(x) \tilde{K}(\alpha) \, d\alpha$$

$$= \int_{\mathbb{R}^r} e(Q_\alpha(x)) \left\{ \sum_{m,n=1}^r L_m L_n \frac{\partial^2 K}{\partial v_m \partial v_n}(\alpha) + \sum_{m=1}^r \tilde{L}_m \frac{\partial K}{\partial v_m}(\alpha) \right\} \, d\alpha$$

$$= \sum_{m,n=1}^r L_m L_n \frac{\partial^2 K}{\partial v_m \partial v_n}(Q_1(x), Q_r(x)) + \sum_{m=1}^r \tilde{L}_m \frac{\partial K}{\partial v_m}(Q_1(x), Q_r(x)).$$
Here we used the relations
\[ \frac{\partial K}{\partial u_m}(\alpha) = 2\pi i \alpha_m \hat{K}(\alpha), \]
\[ \frac{\partial^2 K}{\partial v_m \partial v_n}(\alpha) = (2\pi i)^2 \alpha_m \alpha_n \hat{K}(\alpha). \]

Since
\[ \sum_{i,j=1}^{s} u_i u_j \frac{\partial^2}{\partial x_i \partial x_j}(K(Q_1(x), \ldots, Q_r(x))) \]
\[ = \sum_{m,n=1}^{r} L_m L_n \frac{\partial^2 K}{\partial v_m \partial v_n}(Q_1(x), \ldots, Q_r(x)) + \sum_{m=1}^{r} \tilde{L}_m \frac{\partial K}{\partial v_m}(Q_1(x), \ldots, Q_r(x)) \]
we find
\[ \int_{\mathbb{R}^r} g_{uu}(x) \hat{K}(\alpha) \, d\alpha = \sum_{i,j=1}^{s} u_i u_j \frac{\partial^2}{\partial x_i \partial x_j}(K(Q_1(x), \ldots, Q_r(x))). \]

 Altogether we conclude
\[ -2H_2(N) = \int \int \sum_{i,j=1}^{s} u_i u_j \frac{\partial^2}{\partial x_i \partial x_j}(K(Q_1(x), \ldots, Q_r(x))) \, d\pi(u) \, d\pi_N(x) \]
\[ = \sum_{i=1}^{s} \int \int u_i^2 \frac{\partial^2}{\partial x_i^2}(K(Q_1(x), \ldots, Q_r(x))) \, d\pi(u) \, d\pi_N(x) \]
\[ = \left( \int u_1^2 \, d\pi(u) \right) \sum_{i=1}^{s} \int \frac{\partial^2}{\partial x_i^2}(K(Q_1(x), \ldots, Q_r(x))) \, d\pi_N(x). \]

Since \( \pi_N \) has compact support and \( f_N \) is two times continuously differentiable, partial integration yields
\[ \int \frac{\partial^2}{\partial x_i^2}(K(Q_1(x), \ldots, Q_r(x))) f_N(x) \, dx = \int K(Q_1(x), \ldots, Q_r(x)) \frac{\partial^2 f_N}{\partial x_i}(x) \, dx. \]

This completes the proof of the second assertion of the Lemma, since
\[ \int u_1^2 \, d\pi(u) = 1/3. \]

Finally, we prove
\[ \frac{\partial^2 f_N}{\partial x_i^2}(x) \ll N^{-s-2}. \]

Note that
\[ \hat{f}_N(t) = \prod_{i=1}^{s} \left( \frac{\sin(\pi t_i(2N+1))}{\pi t_i(2N+1)} \right)^4 = \hat{f}_0((2N+1)t). \]
Hence, by Fourier inversion

\[
\frac{\partial^2 f_N}{\partial x_i^2}(x) = (-2\pi i)^2 \int \widehat{f_N}(t) t_i^2 e(-\langle t, x \rangle) \, dt \\
= -(2\pi)^2 (2N + 1)^{-s-2} \int \widehat{f_0}(t) t_i^2 e(-(2N + 1)\langle t, x \rangle) \, dt \\
\ll N^{-s-2}.
\]

This completes the proof of Lemma 7.1. We remark that we used the fourfold convolution in the definition of \(w_N, \pi_N, f_N\) for the above treatment of \(H_2(N)\) only. At all other places of the argument a twofold convolution would be sufficient for our purpose. \(\square\)

**Lemma 7.2.** Assume that the system \(Q_1(x) = 0, \ldots, Q_r(x) = 0\) has a nonsingular real solution, then

\[
\lambda(\{ x \in \mathbb{R}^s \mid |Q_i(x)| \leq N^{-2}, |x|_\infty \leq 1 \}) \gg N^{-2r},
\]

where \(\lambda\) denotes the \(s\)-dimensional Lebesgue measure.

**Proof.** This is proved in Lemma 2 of [10]. Note that if a system of homogeneous equations \(Q_1(x) = 0, \ldots, Q_r(x) = 0\) has a nonsingular real solution, then it has a nonsingular real solution with \(|x|_\infty \leq 1/2\).

Now we complete the proof of Theorem 1.1 as follows. For \(c > 0\) and \(N > 0\) set

\[
A(c, N) = \lambda(\{ x \in \mathbb{R}^s \mid |Q_i(x)| \leq N^{-2}, |x|_\infty \leq c \}).
\]

Then

\[
A(c, N) = c^s A(1, cN).
\]

By Lemma 7.1

\[
H_0(N) \gg N^{-s} \int_{|x|_\infty \leq 2N} K(Q_1(x), \ldots, Q_r(x)) \, dx \\
\gg \int_{|y|_\infty \leq 2} K(N^2 Q_1(y), \ldots, N^2 Q_r(y)) \, dy \\
\gg A(2, 2N) \\
\gg A(1, 5N)
\]
With Lemma 7.2 this yields

\[ H_2(N) \ll N^{-s-2} \int_{|x|_\infty \leq 5N} K(Q_1(x), \ldots, Q_r(x)) \, dx \]

\[ \ll N^{-2} \int_{|y|_\infty \leq 5} K(N^2Q_1(y), \ldots, N^2Q_r(y)) \, dy \]

\[ \ll N^{-2} A(5, N) \]

\[ \ll N^{-2} A(1, 5N). \]

With Lemma 7.2 this yields

\[ H_0(N) + H_2(N) \gg A(1, 5N) \gg N^{-2r} \]

for \( N \geq N_0 \), say. Together with (6.3) this completes the proof of Theorem 1.1. \( \square \)

References


Wolfgang Müller
Institut für Statistik
Technische Universität Graz
8010 Graz, Austria
E-mail: w.mueller@tugraz.at