Binary quadratic forms and Eichler orders

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Résumé. Pour tout ordre d’Eichler $\mathcal{O}(D, N)$ de niveau $N$ dans une algèbre de quaternions indéfinie de discriminant $D$, il existe un groupe Fuchsien $\Gamma(D, N) \subseteq \text{SL}(2, \mathbb{R})$ et une courbe de Shimura $X(D, N)$. Nous associons à $\mathcal{O}(D, N)$ un ensemble $\mathcal{H}(\mathcal{O}(D, N))$ de formes quadratiques binaires ayant des coefficients semi-entiers quadratiques et développons une classification des formes quadratiques primitives de $\mathcal{H}(\mathcal{O}(D, N))$ pour rapport à $\Gamma(D, N)$. En particulier nous retrouvons la classification des formes quadratiques primitives et entières de $\text{SL}(2, \mathbb{Z})$. Un domaine fondamental explicite pour $\Gamma(D, N)$ permet de caractériser les $\Gamma(D, N)$ formes réduites.

Abstract. For any Eichler order $\mathcal{O}(D, N)$ of level $N$ in an indefinite quaternion algebra of discriminant $D$ there is a Fuchsian group $\Gamma(D, N) \subseteq \text{SL}(2, \mathbb{R})$ and a Shimura curve $X(D, N)$. We associate to $\mathcal{O}(D, N)$ a set $\mathcal{H}(\mathcal{O}(D, N))$ of binary quadratic forms which have semi-integer quadratic coefficients, and we develop a classification theory, with respect to $\Gamma(D, N)$, for primitive forms contained in $\mathcal{H}(\mathcal{O}(D, N))$. In particular, the classification theory of primitive integral binary quadratic forms by $\text{SL}(2, \mathbb{Z})$ is recovered. Explicit fundamental domains for $\Gamma(D, N)$ allow the characterization of the $\Gamma(D, N)$-reduced forms.

1. Preliminars

Let $H = \left( \begin{array}{cc} a & b \\ i & j \end{array} \right)$ be the quaternion $\mathbb{Q}$-algebra of basis $\{1, i, j, ij\}$, satisfying $i^2 = a, j^2 = b, ij = -ji$, $a, b \in \mathbb{Q}^*$. Assume $H$ is an indefinite quaternion algebra, that is, $H \otimes_\mathbb{Q} \mathbb{R} \simeq M(2, \mathbb{R})$. Then the discriminant $D_H$ of $H$ is the product of an even number of different primes $D_H = p_1 \cdots p_{2r} \geq 1$ and we can assume $a > 0$. Actually, a discriminant $D$ determines a quaternion algebra $H$ such that $D_H = D$ up to isomorphism. Let us denote by $n(\omega)$ the reduced norm of $\omega \in H$.

Fix any embedding $\Phi : H \hookrightarrow M(2, \mathbb{R})$. For simplicity we can keep in mind the embedding given at the following lemma.
Lemma 1.1. Let $H = \left( \frac{a,b}{Q} \right)$ be an indefinite quaternion algebra with $a > 0$. An embedding $\Phi : H \hookrightarrow M(2, \mathbb{R})$ is obtained by:

$$\Phi(x + yi + zj + ti) = \begin{pmatrix} x + y\sqrt{a} & z + t\sqrt{a} \\ b(z - t\sqrt{a}) & x - y\sqrt{a} \end{pmatrix}.$$ 

Given $N \geq 1$, $\gcd(D, N) = 1$, let us consider an Eichler order of level $N$, that is a $\mathbb{Z}$-module of rank 4, subring of $H$, intersection of two maximal orders. By Eichler's results it is unique up to conjugation and we denote it by $O(D, N)$.

Consider $\Gamma(D, N) := \Phi(\{\omega \in O(D, N)^* \mid n(\omega) > 0\}) \subseteq \text{SL}(2, \mathbb{R})$ a group of quaternion transformations. This group acts on the upper complex half plane $\mathbb{H} = \{x + iy \in \mathbb{C} \mid y > 0\}$. We denote by $X(D, N)$ the canonical model of the Shimura curve defined by the quotient $\Gamma(D, N) \backslash \mathbb{H}$, cf. [Shi67], [AAB01].

For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}(2, \mathbb{R})$ we denote by $P(\gamma)$ the set of fixed points in $\mathbb{C}$ of the transformation defined by $\gamma(z) = \frac{az + b}{cz + d}$.

Let us denote by $E(H, F)$ the set of embeddings of a quadratic field $F$ into the quaternion algebra $H$. Assume there is an embedding $\varphi \in E(H, F)$. Then, all the quaternion transformations in $\Phi(\varphi(F^*)) \subseteq \text{GL}(2, \mathbb{R})$ have the same set of fixed points, which we denote by $P(\varphi)$. In the case that $F$ is an imaginary quadratic field it yields to complex multiplication points, since $P(\varphi) \cap \mathcal{H}$ is just a point, $z(\varphi)$.

Now, we take in account the arithmetic of the orders. Let us consider the set of optimal embeddings of quadratic orders $\Lambda$ into quaternion orders $O$,

$$E^*(O, \Lambda) := \{\varphi \mid \varphi : \Lambda \hookrightarrow O, \varphi(F) \cap O = \varphi(\Lambda)\}.$$ 

Any group $G \leq \text{Nor}(O)$ acts on $E^*(O, \Lambda)$, and we can consider the quotient $E^*(O, \Lambda)/G$. Put $\nu(O, \Lambda; G) := \# E^*(O, \Lambda)/G$. We will also use the notation $\nu(D, N, d, m; G)$ for an Eichler order $O(D, N) \subseteq H$ of level $N$ and the quadratic order of conductor $m$ in $F = \mathbb{Q}(\sqrt{d})$, which we denote $A(d, m)$.

Since further class numbers in this paper will be related to this one, we include next theorem (cf. [Eic55]). It provides the well-known relation between the class numbers of local and global embeddings, and collects the formulas for the class number of local embeddings given in [Ogg83] and [Vig80] in the case $G = O^*$. Consider $\psi_p$ the multiplicative function given by $\psi_p(p^k) = p^k(1 + \frac{1}{p})$, $\psi_p(a) = 1$ if $p \nmid a$. Put $h(d, m)$ the ideal class number of the quadratic order $\Lambda(d, m)$.

**Theorem 1.2.** Let $O = O(D, N)$ be an Eichler order of level $N$ in an indefinite quaternion $\mathbb{Q}$-algebra $H$ of discriminant $D$. Let $\Lambda(d, m)$ be the quadratic order of conductor $m$ in $\mathbb{Q}(\sqrt{d})$. Assume that $E(H, \mathbb{Q}(\sqrt{d})) \neq \emptyset$
and $\gcd(m, D) = 1$. Then,

$$\nu(D, N, d, m; O^*) = h(d, m) \prod_{p|DN} \nu_p(D, N, d, m; O^*).$$

The local class numbers of embeddings $\nu_p(D, N, d, m; O^*)$, for the primes $p|DN$, are given by

(i) If $p|D$, then $\nu_p(D, N, d, m; O^*) = 1 - \left(\frac{D}{p}\right)$.

(ii) If $p \parallel N$, then $\nu_p(D, N, d, m; O^*)$ is equal to $1 + \left(\frac{D}{p}\right)$ if $p \nmid m$, and equal to $2$ if $p|m$.

(iii) Assume $N = p^ru_1$, with $p \nmid u_1$, $r \geq 2$. Put $m = p^ku_2$, $p \nmid u_2$.

(a) If $r \geq 2k + 2$, then $\nu_p(D, N, d, m; O^*)$ is equal to $2\psi_p(m)$ if $\left(\frac{D}{p}\right) = 1$, and equal to $0$ otherwise.

(b) If $r = 2k + 1$, then $\nu_p(D, N, d, m; O^*)$ is equal to $2\psi_p(m)$ if $\left(\frac{D}{p}\right) = 1$, equal to $p^k$ if $\left(\frac{D}{p}\right) = 0$, and equal to $0$ if $\left(\frac{D}{p}\right) = -1$.

(c) If $r = 2k$, then $\nu_p(D, N, d, m; O^*) = p^{k-1}(p + 1 + p^{-1})$.

(d) If $r \leq 2k - 1$, then $\nu_p(D, N, d, m; O^*)$ is equal to $p^{k/2} + p^{k/2-1}$ if $k$ is even, and equal to $2p^{k-1/2}$ if $k$ is odd.

2. Classification theory of binary forms associated to quaternions

Given $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2, \mathbb{R})$, we put $f_\alpha(x, y) := cx^2 + (d - a)xy - by^2$. It is called the binary quadratic form associated to $\alpha$.

For a binary quadratic form $f(x, y) := Ax^2 + Bxy + Cy^2 = (A, B, C)$, we consider the associated matrix $A(f) = \begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix}$, and the determinants $\det_1(f) = \det A(f)$ and $\det_2(f) = 2^2 \det A(f) = -(B^2 - 4AC)$. Denote by $\mathcal{P}(f)$ the set of solutions in $\mathbb{C}$ of $Ax^2 + Bz + C = 0$. If $f$ is (positive or negative) definite, then $\mathcal{P}(f) \cap \mathbb{R}$ is just a point which we denote by $\tau(f)$.

The proof of the following lemma is straightforward.

**Lemma 2.1.** Let $\alpha \in M(2, \mathbb{R})$.

(i) For all $\lambda, \mu \in \mathbb{Q}$, we have $f_{\lambda\alpha} = \lambda f_\alpha$ and $f_{\alpha + \mu 1_d} = f_\alpha$; in particular, $\mathcal{P}(f_{\lambda\alpha + \mu 1_d}) = \mathcal{P}(f_\alpha)$.

(ii) $z \in \mathbb{C}$ is a fixed point of $\alpha$ if and only if $z \in \mathcal{P}(f_\alpha)$, that is, $\mathcal{P}(f_\alpha) = \mathcal{P}(\alpha)$.

(iii) Let $\gamma \in \text{GL}(2, \mathbb{R})$. Then $A(f_{\gamma^{-1}\alpha\gamma}) = (\det \gamma^{-1})^4 A(f_\alpha)\gamma$; in particular, if $\gamma \in \text{SL}(2, \mathbb{R})$, $z \in \mathcal{P}(f_\alpha)$ if and only if $\gamma^{-1}(z) \in \mathcal{P}(f_{\gamma^{-1}\alpha\gamma})$.

**Definition 2.2.** For a quaternion $\omega \in H^*$, we define the binary quadratic form associated to $\omega$ as the binary quadratic form $f_{\Phi(\omega)}$. 

Given a quaternion algebra $H$ denote by $H_0$ the pure quaternions. By using lemma 2.1 it is enough to consider the binary forms associated to pure quaternions:

$$\mathcal{H}(a, b) = \{ f_{\Phi(\omega)} : \omega \in H_0 \}, \quad \mathcal{H}(O) = \{ f_{\Phi(\omega)} : \omega \in O \cap H_0 \}.$$

**Definition 2.3.** Let $O$ be an order in a quaternion algebra $H$. We define the denominator $m_O$ of $O$ as the minimal positive integer such that $m_O \cdot O \subseteq \mathbb{Z}[1, i, j, ij]$. Then the ideal $(m_O)$ is the conductor of $O$ in $\mathbb{Z}[1, i, j, ij]$.

Properties for these binary forms are collected in the following proposition, easy to be verified.

**Proposition 2.4.** Consider an indefinite quaternion algebra $H = \left( \begin{array}{cc} a & b \\ \bar{b} & \bar{a} \end{array} \right)$, and an order $O \subseteq H$. Fix the embedding $\Phi$ as in lemma 1.1. Then:

(i) There is a bijective mapping $H_0 \rightarrow \mathcal{H}(a, b)$ defined by $\omega \mapsto f_{\Phi(\omega)}$. Moreover $\det_1(f_{\Phi(\omega)}) = n(\omega)$.

(ii) $\mathcal{H}(a, b) = \{ (b(\lambda_2 + \lambda_3 \sqrt{a}), \lambda_1 \sqrt{a}, -\lambda_2 + \lambda_3 \sqrt{a}) | \lambda_1, \lambda_2, \lambda_3 \in \mathbb{Q} \}$

$$= \{ (\beta_\alpha, \alpha, -\beta) | \alpha, \beta \in \mathbb{Q}(\sqrt{a}), \text{tr}(\alpha) = 0 \}.$$

(iii) the binary quadratic forms of $\mathcal{H}(O)$ have coefficients in $\mathbb{Z} \left[ \frac{1}{m_O}, \sqrt{a} \right]$

Given a quaternion order $O$ and a quadratic order $A$, put $\mathcal{F}_w = \mathcal{F} = -n(\omega)$. Then $\varphi_w(\sqrt{d}) = \omega$ defines an embedding $\varphi_w \in \mathcal{E}(H, F_w)$. By considering $\Lambda_w := \varphi_w^{-1}(O) \cap F_w$, we have $\varphi_w \in \mathcal{E}^*(O, \Lambda_w)$. Therefore, by construction, it is clear that $P(f_{\Phi(\omega)}) = P(\Phi(\omega)) = P(\varphi_w)$. In particular, if we deal with quaternions of positive norm, we obtain definite binary forms, imaginary quadratic fields and a unique solution $\tau(f_{\Phi(\omega)}) = z(\varphi_w) \in \mathcal{H}$. The points corresponding to these binary quadratic forms are in fact the complex multiplication points.

Theorem 4.53 in [AB04] states a bijective mapping $f$ from the set $\mathcal{E}(O, \Lambda)$ of embeddings of a quadratic order $\Lambda$ into a quaternion order $O$ onto the set $\mathcal{H}(\mathbb{Z} + 2O, \Lambda)$ of binary quadratic forms associated to the orders $\mathbb{Z} + 2O$ and $\Lambda$. By using optimal embeddings, a definition of primitivity for the forms in $\mathcal{H}(\mathbb{Z} + 2O, \Lambda)$ was introduced. We denote by $\mathcal{H}^*(\mathbb{Z} + 2O, \Lambda)$ the corresponding subset of $(O, \Lambda)$-primitive binary forms. Then equivalence of embeddings yields to equivalence of forms.

**Corollary 2.5.** Given orders $O$ and $\Lambda$ as above, for any $G \subseteq O^*$ consider $\Phi(G) \subseteq \text{GL}(2, \mathbb{R})$. There is a bijective mapping between $\mathcal{E}^*(O, \Lambda)/G$ and $\mathcal{H}^*(\mathbb{Z} + 2O, \Lambda)/\Phi(G)$. 

Montserrat Alsina
Fix \( \mathcal{O} = \mathcal{O}(D, N), \Lambda = \Lambda(d, m) \) and \( G = \mathcal{O}^* \). We use the notation
\[
h(D, N, d, m) := \# \mathcal{H}^*(\mathbb{Z} + 2\mathcal{O}(D, N), \Lambda(d, m)) / \Gamma_{\mathcal{O}^*}.
\]
Thus, \( h(D, N, d, m) = \nu(D, N, d, m; \mathcal{O}^*) \), which can be computed explicitly by Eichler results (cf. theorem 1.2).

3. Generalized reduced binary forms

Fix an Eichler order \( \mathcal{O}(D, N) \) in an indefinite quaternion algebra \( \mathcal{H} \). Consider the associated group \( \Gamma(D, N) \) and the Shimura curve \( X(D, N) \).

For a quadratic order \( \Lambda(d, m) \), consider the set \( \mathcal{H}^*(\mathbb{Z} + 2\mathcal{O}(2p, N), \Lambda) \) of binary quadratic forms. As above, for a definite binary quadratic form \( f = Ax^2 + Bxy + Cy^2 \), denote by \( T(f) \) the solution of \( Az^2 + Bz + C = 0 \) in \( \mathcal{H} \).

**Definition 3.1.** Fix a fundamental domain \( \mathcal{D}(D, N) \) for \( \Gamma(D, N) \) in \( \mathcal{H} \). Make a choice about the boundary in such a way that every point in \( \mathcal{H} \) is equivalent to a unique point of \( \mathcal{D}(D, N) \). A binary form \( f \in \mathcal{H}^*(\mathbb{Z} + 2\mathcal{O}(D, N), \Lambda) \) is called \( \Gamma(D, N) \)-reduced form if \( T(f) \in \mathcal{D}(D, N) \).

**Theorem 3.2.** The number of positive definite \( \Gamma(D, N) \)-reduced forms in \( \mathcal{H}^*(\mathbb{Z} + 2\mathcal{O}(D, N), \Lambda(d, m)) \) is finite and equal to \( h(D, N, d, m) \).

**Proof.** We can assume \( d < 0 \), in order \( \mathcal{H}^*(\mathbb{Z} + 2\mathcal{O}(D, N), \Lambda(d, m)) \) consists on definite binary forms. By lemma 2.1 (iii), we have that \( \Gamma(D, N) \)-equivalence of forms yields to \( \Gamma(D, N) \)-equivalence of points. Note that \( \tau(f) = \tau(-f) \), but \( f \) is not \( \Gamma(D, N) \)-equivalent to \(-f \). Thus, in each class of \( \Gamma(D, N) \)-equivalence of forms there is a unique reduced binary form.

Consider \( G = \{ \omega \in \mathcal{O}^* \mid n(\omega) > 0 \} \) in order to get \( \Phi(G) = \Gamma(D, N) \). The group \( G \) has index 2 in \( \mathcal{O}^* \) and the number of classes of \( \Gamma(D, N) \)-equivalence in \( \mathcal{H}^*(\mathbb{Z} + 2\mathcal{O}(D, N), \Lambda(d, m)) \) is \( 2h(D, N, d, m) \). In that set, positive and negative definite forms were included, thus the number of classes of positive definite forms is exactly \( h(D, N, d, m) \). \( \square \)

4. Non-ramified and small ramified cases

**Definition 4.1.** Let \( \mathcal{H} \) be a quaternion algebra of discriminant \( D \). We say that \( \mathcal{H} \) is nonramified if \( D = 1 \), that is \( \mathcal{H} \simeq \mathbb{M}(2, \mathbb{Q}) \). We say \( \mathcal{H} \) is small ramified if \( D = pq \); in this case, we say it is of type A if \( D = 2p, p \equiv 3 \) mod 4, and we say it is of type B if \( D_H = pq, q \equiv 1 \) mod 4 and \( \left( \frac{2}{q} \right) = -1 \).

It makes sense because of the following statement.
Proposition 4.2. For \( H = \left( \frac{p \cdot q}{\mathbb{Q}} \right) \), \( p, q \) primes, exactly one of the following statements holds:

(i) \( H \) is nonramified.
(ii) \( H \) is small ramified of type A.
(iii) \( H \) is small ramified of type B.

We are going to specialize above results for reduced binary forms for each one of these cases.

4.1. Nonramified case. Consider \( H = \text{M}(2, \mathbb{Q}) \) and take the Eichler order

\[ \mathcal{O}_0(1, N) := \left\{ \left( \begin{array}{cc} a & b \\ cN & d \end{array} \right) \mid a, b, c, d \in \mathbb{Z} \right\}. \]

Then \( \Gamma(1, N) = \Gamma_0(N) \) and the curve \( X(1, N) \) is the modular curve \( X_0(N) \).

To unify results with the ramified case, it is also interesting to work with the Eichler order \( \mathcal{O}(1, N) := \mathbb{Z} \left[ 1, \frac{i+j}{2}, N \frac{-j+i}{2}, \frac{1-i}{2} \right] \) in the nonramified quaternion algebra \( \left( \frac{1-i}{\mathbb{Q}} \right) \).

Proposition 4.3. Consider the Eichler order \( \mathcal{O} = \mathcal{O}_0(1, N) \subseteq \text{M}(2, \mathbb{Q}) \) and the quadratic order \( \Lambda = \Lambda(d, m) \). Then:

(i) \( \mathcal{H}^*(\mathbb{Z} + 2\mathcal{O}, \Lambda) \simeq \{ f = (Na, b, c) \mid a, b, c \in \mathbb{Z}, \det_2(f) = -D_\Lambda \} \).
(ii) The \( (\mathcal{O}, \Lambda) \)-primitivity condition is \( \gcd(a, b, c) = 1 \).
(iii) If \( d \neq 0 \), the number of \( \Gamma_0(N) \)-reduced positive definite primitive binary quadratic forms in \( \mathcal{H}^*(\mathbb{Z} + 2\mathcal{O}, \Lambda) \) is equal to \( h(1, N, d, m) \).

For \( N = 1 \), the well-known theory on reduced integer binary quadratic forms is recovered. In particular, the class number of \( \text{SL}(2, \mathbb{Z}) \)-equivalence is \( h(d, m) \).

For \( N > 1 \), a general theory of reduced binary forms is obtained. For \( N \) equal to a prime, let us fix the symmetrical fundamental domain

\[ \mathcal{D}(1, N) = \{ z \in \mathcal{H} \mid |\text{Re}(z)| \leq 1/2, \left| z - \frac{k}{N} \right| > \frac{1}{N}, k \in \mathbb{Z}, 0 < |k| \leq \frac{N-1}{2} \} \]

given at [AB04]; a detailed construction can be found in [Als00]. Then a positive definite binary form \( f = (Na, b, c), a > 0 \), is \( \Gamma_0(N) \)-reduced if and only if \( |b| \leq Na \) and \( |\tau(f) - \frac{k}{N}| > \frac{1}{N} \) for \( k \in \mathbb{Z}, 0 < |k| \leq \frac{N-1}{2} \).

Figure 4.1 shows the 46 points corresponding to reduced binary forms in \( \mathcal{H}^*(\mathbb{Z} + 2\mathcal{O}_0(1, 23), \Lambda) \) for \( D_\Lambda = 7, 11, 19, 23, 28, 43, 56, 67, 76, 83, 88, 91, 92 \), which occurs in an special graphical position. In fact these points are exactly the special complex multiplication points of \( X(1, 23) \), characterized by the existence of elements \( \alpha \in \Lambda(d, m) \) of norm \( DN \) (cf. [AB04]). The table describes the \( n = h(1, 23, d, m) \) inequivalent points for each quadratic order \( \Lambda(d, m) \).
Note that for these symmetrical domains it is easy to implement an algorithm to decide if a form in this set is reduced or not, by using isometric circles.

**Figure 4.1.** The points $\tau(f)$ for some $f$ reduced binary forms corresponding to quadratic orders $\Lambda(d, m)$ in a fundamental domain for $X(1, 23)$.

<table>
<thead>
<tr>
<th>$(d, m)$</th>
<th>$n$</th>
<th>$\tau(f)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-7, 1)</td>
<td>2</td>
<td>$\tau_1 = \frac{-19 + \sqrt{74}}{46}$, $\tau_2 = \frac{19 + \sqrt{74}}{46}$</td>
</tr>
<tr>
<td>(-7, 2)</td>
<td>2</td>
<td>$\tau_3 = \frac{-4 + \sqrt{74}}{23}$, $\tau_4 = \frac{4 + \sqrt{74}}{23}$</td>
</tr>
<tr>
<td>(-11, 1)</td>
<td>2</td>
<td>$\tau_5 = \frac{-9 + \sqrt{114}}{46}$, $\tau_6 = \frac{9 + \sqrt{114}}{46}$</td>
</tr>
<tr>
<td>(-14, 1)</td>
<td>8</td>
<td>$\tau_7 = \frac{-20 + \sqrt{144}}{46}$, $\tau_8 = \frac{-26 + \sqrt{144}}{69}$, $\tau_9 = \frac{-20 + \sqrt{144}}{69}$, $\tau_{10} = \frac{-3 + \sqrt{144}}{23}$, $\tau_{11} = \frac{3 + \sqrt{144}}{23}$, $\tau_{12} = \frac{20 + \sqrt{144}}{69}$, $\tau_{13} = \frac{26 + \sqrt{144}}{69}$, $\tau_{14} = \frac{20 + \sqrt{144}}{46}$</td>
</tr>
<tr>
<td>(-19, 1)</td>
<td>2</td>
<td>$\tau_{15} = \frac{-21 + \sqrt{194}}{46}$, $\tau_{16} = \frac{21 + \sqrt{194}}{46}$</td>
</tr>
<tr>
<td>(-19, 2)</td>
<td>6</td>
<td>$\tau_{17} = \frac{-25 + \sqrt{194}}{92}$, $\tau_{18} = \frac{-21 + \sqrt{194}}{92}$, $\tau_{19} = \frac{-2 + \sqrt{194}}{23}$, $\tau_{20} = \frac{2 + \sqrt{194}}{23}$, $\tau_{21} = \frac{21 + \sqrt{194}}{92}$, $\tau_{22} = \frac{25 + \sqrt{194}}{92}$</td>
</tr>
<tr>
<td>(-22, 1)</td>
<td>4</td>
<td>$\tau_{23} = \frac{-22 + \sqrt{224}}{46}$, $\tau_{24} = \frac{-1 + \sqrt{224}}{23}$, $\tau_{25} = \frac{1 + \sqrt{224}}{23}$, $\tau_{26} = \frac{22 + \sqrt{224}}{46}$</td>
</tr>
<tr>
<td>(-23, 1)</td>
<td>3</td>
<td>$\tau_{27} = \frac{-23 + \sqrt{234}}{92}$, $\tau_{28} = \frac{23 + \sqrt{234}}{92}$, $\tau_{29} = \frac{-23 + \sqrt{234}}{46}$, $\tau_{30} = \frac{23 + \sqrt{234}}{46}$</td>
</tr>
<tr>
<td>(-23, 2)</td>
<td>3</td>
<td>$\tau_{31} = \frac{-23 + \sqrt{234}}{25}$, $\tau_{32} = \frac{23 + \sqrt{234}}{69}$</td>
</tr>
<tr>
<td>(-43, 1)</td>
<td>2</td>
<td>$\tau_{33} = \frac{-7 + \sqrt{434}}{46}$, $\tau_{34} = \frac{7 + \sqrt{434}}{46}$</td>
</tr>
<tr>
<td>(-67, 1)</td>
<td>2</td>
<td>$\tau_{35} = \frac{-5 + \sqrt{674}}{46}$, $\tau_{36} = \frac{5 + \sqrt{674}}{46}$</td>
</tr>
<tr>
<td>(-83, 1)</td>
<td>6</td>
<td>$\tau_{37} = \frac{-49 + \sqrt{834}}{138}$, $\tau_{38} = \frac{-43 + \sqrt{834}}{138}$, $\tau_{39} = \frac{-3 + \sqrt{834}}{46}$, $\tau_{40} = \frac{3 + \sqrt{834}}{46}$, $\tau_{41} = \frac{43 + \sqrt{834}}{138}$, $\tau_{42} = \frac{49 + \sqrt{834}}{138}$</td>
</tr>
<tr>
<td>(-91, 1)</td>
<td>4</td>
<td>$\tau_{43} = \frac{-47 + \sqrt{914}}{230}$, $\tau_{44} = \frac{-1 + \sqrt{914}}{46}$, $\tau_{45} = \frac{1 + \sqrt{914}}{46}$, $\tau_{46} = \frac{47 + \sqrt{914}}{230}$</td>
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4.2. Small ramified case of type A. Let us consider $H_A(p) := \left( \frac{p-1}{Q} \right)$ and the Eichler order $O_A(2p, N) := \mathbb{Z} \left[ 1, i, N, \frac{1+i+i+j+i}{2} \right]$, for $N | \frac{p-1}{2}$, $N$ square-free. The elements in the group $\Gamma_A(2p, N)$ are $\gamma = \frac{1}{2} \left( \begin{array}{cc} \alpha' & \beta' \\ -\beta & \alpha \end{array} \right)$ such that $\alpha, \beta \in \mathbb{Z}[\sqrt{p}], \alpha \equiv \beta \equiv \alpha \sqrt{p} \mod 2$, $\det \gamma = 1$, $N|\left( \text{tr}(\beta) - \frac{\beta' - \beta}{\sqrt{p}} \right)$. We denote by $X_A(2p, N)$ the Shimura curve of type A defined by $\Gamma_A(2p, N)$.

**Proposition 4.4.** Consider the Eichler order $O_A(2p, N)$ and the quadratic order $\Lambda = \Lambda(d, m)$.

(i) The set $\mathcal{H}(\mathbb{Z} + 2O_A(2p, N), \Lambda)$ of binary forms is equal to

$$\left\{ f = (a + b\sqrt{p}, 2c\sqrt{p}, a - b\sqrt{p}) : a, b, c \in \mathbb{Z}, \ a \equiv b \equiv c \mod 2, \ N | (a + b), \ \det_1(f) = -D_\lambda \right\}.$$

(ii) The $(O_A(2p, N), \Lambda)$-primitivity condition for these binary quadratic forms is $\gcd \left( \frac{a+b}{2}, \frac{a+b}{2N}, b \right) = 1$.

(iii) If $d < 0$, the number of $\Gamma_A(2p, N)$-reduced positive definite primitive binary forms in $H^*(\mathbb{Z} + 2O_A(2p, N), \Lambda)$ is equal to $h(2p, N, d, m)$.

For example, consider the fundamental domain $D(6, 1)$ for the Shimura curve $X_A(6, 1)$ in the Poincaré half plane defined by the hyperbolic polygon of vertices $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ at figure 4.2 (cf. [AB04]). The table contains the corresponding reduced binary quadratic forms $f \in \mathcal{H}^*(\mathbb{Z} + 2O_A(6, 1), \Lambda(d, 1))$ and the associated points $\tau(f)$ for $\det_1(f) = 4, 3, 24, 40$, that is $d = -1, -3, -6, -10$. Since the vertices are elliptic points of order 2 or 3, they are the associated points to forms of determinant 4 or 3, respectively. We put $n = h(6, 1, d, 1)$ the number of such reduced forms for each determinant.

4.3. Small ramified case of type B. Consider $H_B(p, q) := \left( \frac{b}{q} \right)$ and the Eichler order $O_B(pq, N) := \mathbb{Z} \left[ 1, N, \frac{1+i+i+j+i}{2} \right]$, where $N | \frac{q-1}{4}$, $N$ square-free and $\gcd(N, p) = 1$. Then the group of quaternion transformations is

$$\Gamma_B(pq, N) = \left\{ \gamma = \frac{1}{2} \left( \begin{array}{cc} \alpha' & \beta' \\ -\beta & \alpha \end{array} \right) : \alpha, \beta \in \mathbb{Z}[\sqrt{p}], \ \alpha \equiv \beta \mod 2, \ N | \frac{\alpha' + \beta + \beta'}{2\sqrt{p}}, \ \det \gamma = 1 \right\}.$$

We denote by $X_B(pq, N)$ the corresponding Shimura curve of type B.
Proposition 4.5. Consider the Eichler order $O_B(pq, N)$ in $H_B(p, q)$ and the quadratic order $A = A(d, r_m).

(i) The set $\mathcal{H}(Z + 2O_B(pq, N), \Lambda)$ of binary forms contains precisely the forms $f = (a + b\sqrt{p})$, $2c\sqrt{p}, -a + b\sqrt{p}$ where $a, b, c \in Z$, $2N|(c - b)$ and $\det_1(f) = -D_\Lambda$.

(ii) The $(O_B(pq, N), \Lambda)$-primitivity condition for these binary quadratic forms in (i) is $\gcd(a, b, c) = 1$.

(iii) If $d < 0$, the number of $\Gamma_B(pq, N)$-reduced positive definite primitive binary forms in $\mathcal{H}^*(Z + 2O_B(pq, N), \Lambda)$ is equal to $h(pq, N, d, m)$.

In figure 4.3 we show a fundamental domain for $\Gamma_B(10, 1)$ given by the hyperbolic polygon of vertices $\{w_1, w_2, w_3, w_4, w_5, w_6\}$. All the vertices are elliptic points of order 3; thus they are the associated points to binary
Figure 4.3. Reduced binary forms in a fundamental domain for $X_B(10, 1)$.

<table>
<thead>
<tr>
<th>$\det_1(f)$</th>
<th>$n$</th>
<th>$f$</th>
<th>$\tau(f)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>4</td>
<td>$(5 + 5\sqrt{2})x^2 + 2\sqrt{2}xy + (1 + \sqrt{2})y^2$</td>
<td>$w_1 = \frac{-\sqrt{2} + \sqrt{36}}{5(-1 + \sqrt{2})} \sim w_3$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(5 + 5\sqrt{2})x^2 + 2\sqrt{2}xy + (-1 + \sqrt{2})y^2$</td>
<td>$w_2 = \frac{-\sqrt{2} + \sqrt{36}}{5(1 + \sqrt{2})}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(35 + 25\sqrt{2})x^2 - 2\sqrt{2}xy + (-7 + 5\sqrt{2})y^2$</td>
<td>$w_4 = \frac{\sqrt{2} + \sqrt{36}}{5(7 + 5\sqrt{2})} \sim w_6$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(5 + 5\sqrt{2})x^2 + 2\sqrt{2}xy + (-1 + \sqrt{2})y^2$</td>
<td>$w_5 = \frac{-\sqrt{2} + \sqrt{36}}{5(1 + \sqrt{2})}$</td>
</tr>
<tr>
<td>8</td>
<td>2</td>
<td>$5\sqrt{2}x^2 + 2\sqrt{2}xy + \sqrt{2}y^2$</td>
<td>$\tau_1 = \frac{-1 + 2\sqrt{5}}{2}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(5\sqrt{2}x^2 + 2\sqrt{2}xy + \sqrt{2}y^2$</td>
<td>$\tau_2 = \frac{1 + 2\sqrt{5}}{2}$</td>
</tr>
<tr>
<td>20</td>
<td>2</td>
<td>$(10 + 10\sqrt{2})x^2 + (-2 + 2\sqrt{2})y^2$</td>
<td>$\tau_3 = \frac{(\sqrt{10} - \sqrt{5})\sqrt{2}}{2}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(-10 + 10\sqrt{2})x^2 + (2 + 2\sqrt{2})y^2$</td>
<td>$\tau_4 = \frac{(\sqrt{10} + \sqrt{5})\sqrt{2}}{2}$</td>
</tr>
<tr>
<td>40</td>
<td>2</td>
<td>$(40 + 30\sqrt{2})x^2 + (-8 + 6\sqrt{2})y^2$</td>
<td>$\tau_5 = \frac{(3\sqrt{5} - 2\sqrt{10})\sqrt{5}}{2}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$10\sqrt{2}x^2 + 2\sqrt{2}y^2$</td>
<td>$\tau_6 = \frac{\sqrt{54}}{5}$</td>
</tr>
</tbody>
</table>

forms of determinant 3. We also represent the points corresponding to reduced binary quadratic forms $f$ with $\det_1(f) = 40$, which correspond to special complex points. The table also contains the explicit reduced definite positive binary forms and the corresponding points for determinants 8 and 20.
References


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