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The distribution of the values of a rational function modulo a big prime

par ALEXANDRU ZAHARESCU

RÉSUMÉ. Étant donné un grand nombre premier p et une fonction rationnelle $r(X)$ définie sur $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, on évalue la grandeur de l'ensemble $\{x \in \mathbb{F}_p : \tilde{r}(x) > \tilde{r}(x+1)\}$, où $\tilde{r}(x)$ et $\tilde{r}(x+1)$ sont les plus petits représentants de $r(x)$ et $r(x+1)$ dans $\mathbb{Z}$ modulo $p\mathbb{Z}$.

ABSTRACT. Given a large prime number p and a rational function $r(X)$ defined over $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, we investigate the size of the set $\{x \in \mathbb{F}_p : \tilde{r}(x) > \tilde{r}(x+1)\}$, where $\tilde{r}(x)$ and $\tilde{r}(x+1)$ denote the least positive representatives of $r(x)$ and $r(x+1)$ in $\mathbb{Z}$ modulo $p\mathbb{Z}$.

1. Introduction

Several problems on the distribution of points satisfying various congruence constraints have been investigated recently. Given a large prime number p , for any $a \in \{1, 2, \dots, p-1\}$ let $\bar{a} \in \{1, 2, \dots, p-1\}$ be such that $a\bar{a} \equiv 1 \pmod{p}$. A question raised by D.H. Lehmer (see Guy [4, Problem F12]) asks to say something nontrivial about the number, call it $N(p)$, of those a for which a and $\bar{a}$ are of opposite parity. The problem was studied by Wenpeng Zhang in [8], [9] and [10] who proved that

$$(1) \quad N(p) = \frac{p}{2} + O\left(p^{1/2} \log^2 p\right)$$

and then generalized (1) to the case when p is replaced by any odd number q . In [2] it is obtained a generalization of (1), in which the pair $(a, \bar{a})$ is replaced by a point lying on a more general irreducible curve defined mod p . Zhang also studied the problem of the distribution of distances $|a - \bar{a}|$, where $a, \bar{a}$ run over the set of integers in $\{1, \dots, n-1\}$ which are relatively prime to n . He proved in [11] that for any integer $n \geq 2$ and any $0 < \delta \leq 1$ one has

$$(2) \quad \left| \{a: 1 \leq a \leq n-1, (a, n) = 1, |a - \bar{a}| < \delta n\} \right| \\ = \delta(2 - \delta)\varphi(n) + O\left(n^{\frac{1}{2}}d^2(n)\log^3 n\right),$$

where $\varphi(n)$ is the Euler function and $d(n)$ denotes the number of divisors of n . In [12] Zhiyong Zheng investigated the same problem, with $(a, \bar{a})$ replaced by a pair (x, y) satisfying a more general congruence. Precisely, let p be a prime number and let $f(x, y)$ be a polynomial with integer coefficients of total degree $d \geq 2$, absolutely irreducible modulo p . Then it is proved in [12] that for any $0 < \delta \leq 1$ one has:

$$\left| \{(x, y) \in \mathbb{Z}^2 : 0 \leq x, y < p, f(x, y) \equiv 0 \pmod{p}, |x - y| < \delta p\} \right| \\ = \delta(2 - \delta)p + O_d\left(p^{\frac{1}{2}}\log^2 p\right).$$

A generalization of this problem, where the pair (x, y) is replaced by a point lying on an irreducible curve in a higher dimensional affine space over the field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, has been obtained in [3].

There are different ways to measure the randomness of the distribution of a given set. B. Z. Moroz showed in [5] that the squares (or the l -th powers, if l divides $p-1$) are randomly distributed among the values $\{i_p(f(0)), \dots, i_p(f(p-1))\}$ of a fixed irreducible polynomial $f(X)$ in $\mathbb{Z}[X]$ modulo a prime p , as $p \rightarrow \infty$ (here i_p stands for the reduction modulo p).

In the present paper we study what happens with the order of residue classes mod p when they are transformed through a rational function $r(X) \in \mathbb{F}_p(X)$. For any $y \in \mathbb{F}_p$ denote by $j(y)$ the least positive representative of y in $\mathbb{Z}$ modulo $p\mathbb{Z}$. To any rational function $r(X) \in \mathbb{F}_p(X)$ we associate the map $\tilde{r} : \mathbb{F}_p \rightarrow \{0, 1, \dots, p-1\}$ given by $\tilde{r}(x) = j(r(x))$ if $x \in \mathbb{F}_p$ is not a pole of $r(X)$, and $\tilde{r}(x) = 0$ if x is a pole of $r(X)$. As the degree of $r(X)$ will be assumed to be small in terms of p in what follows, the contribution of the poles of $r(X)$ in our asymptotic results will be negligible. If we count those $x \in \mathbb{F}_p$ for which $\tilde{r}(x+1) < \tilde{r}(x)$, respectively those x for which $\tilde{r}(x+1) > \tilde{r}(x)$, there should be no bias towards any one of these inequalities. In other words one would expect that for about half of the elements $x \in \mathbb{F}_p$, $\tilde{r}(x+1)$ is larger than $\tilde{r}(x)$ and for about half of the elements $x \in \mathbb{F}_p$, $\tilde{r}(x+1)$ is smaller than $\tilde{r}(x)$.

In order to handle the above problem, we fix nonzero positive integers a, b and study the distribution of the set $\{b\tilde{r}(x+1) - a\tilde{r}(x) : x \in \mathbb{F}_p\}$. For any real number t consider the set $\mathcal{M}(a, b, p, r, t) = \{x \in \mathbb{F}_p : b\tilde{r}(x+1) - a\tilde{r}(x) < tp\}$ and denote by $D(a, b, p, r, t)$ the number of elements of $\mathcal{M}(a, b, p, r, t)$. Our aim is to provide an asymptotic formula for $D(a, b, p, r, t)$.

We now introduce a function $G(t, a, b)$ which will play an important role in the estimation of $D(a, b, p, r, t)$.

$$G(t, a, b) = \begin{cases} 0, & \text{if } t < -a \\ \frac{(t+a)^2}{2ab}, & \text{if } -a \leq t \leq W \\ \left(1 - \frac{(W+a)^2}{ab}\right) \frac{t-W}{Z-W} + \frac{(W+a)^2}{2ab}, & \text{if } W < t < Z \\ 1 - \frac{(t-b)^2}{2ab}, & \text{if } Z \leq t < b \\ 1, & \text{if } b \leq t \end{cases}$$

where $W = \min\{0, b - a\}$ and $Z = \max\{0, b - a\}$. We will prove the following

Theorem 1.1. *For any positive integers a, b, d , any prime number p , any real number t and any rational function $r(X) = \frac{f(X)}{g(X)}$ which is not a linear polynomial, with $f, g \in \mathbb{F}_p[X]$, $\deg f, \deg g \leq d$, one has*

$$(3) \quad D(a, b, p, r, t) = pG(t, a, b) + O_{a,b,d} \left(p^{1/2} \log^2 p \right).$$

As a consequence of Theorem 1.1 we show that the inequality $\tilde{r}(x) > \tilde{r}(x+1)$ holds indeed for about half of the values of x in $\mathbb{F}_p$.

Corollary 1.2. *Let p be a prime number, d a positive integer and let $r(X) = \frac{f(X)}{g(X)}$ be a rational function which is not a linear polynomial, with $f, g \in \mathbb{F}_p[X]$ and $\deg f, \deg g \leq d$. Then one has*

$$\#\{x \in \mathbb{F}_p : \tilde{r}(x) > \tilde{r}(x+1)\} = \frac{p}{2} + O_d \left(p^{1/2} \log^2 p \right).$$

As another application of Theorem 1.1 we obtain an asymptotic result for all the even moments of the distance between $\tilde{r}(x+1)$ and $\tilde{r}(x)$.

Corollary 1.3. *Let k be a positive integer and let $p, d, r(X)$ be as in the statement of Corollary 1. Then we have*

$$\begin{aligned} M(p, r, 2k) &:= \sum_{x \in \mathbb{F}_p} (\tilde{r}(x+1) - \tilde{r}(x))^{2k} \\ &= \frac{p^{2k+1}}{(k+1)(2k+1)} + O_{k,d} \left(p^{2k+1/2} \log^2 p \right). \end{aligned}$$

In particular, for $k = 1$ one has

$$M(p, r, 2) = \frac{p^3}{6} + O_d(p^{5/2} \log^2 p).$$

This says that in quadratic average $|\tilde{r}(x+1) - \tilde{r}(x)|$ is $\sim \frac{p}{\sqrt{6}}$.

2. Proof of Theorem 1.1

We will need the following lemma, which is a consequence of the Riemann Hypothesis for curves defined over a finite field (see [7], [6], [1]).

Lemma 2.1. *Let p be a prime number and $\mathbb{F}_p$ the field with p elements. Let ψ be a nontrivial character of the additive group of $\mathbb{F}_p$ and let $R(X)$ be a nonconstant rational function. Then*

$$\sum_{a \in \mathbb{F}_p} \psi(R(a)) = O(\sqrt{p}),$$

where the poles of $R(X)$ are excluded from the summation, and the implicit O -constant depends at most on the degrees of the numerator and denominator of $F(X)$.

Let now p be a prime number, let a, b, d be positive integers less than p , let t be a real number and let $r(X) = \frac{f(X)}{g(X)}$, $r(X)$ not a linear polynomial, with $f(X), g(X) \in \mathbb{F}_p[X]$, $\deg f(X), \deg g(X) \leq d$. For any $y, z \in \{0, 1, \dots, p-1\}$ we set

$$(4) \quad H(y, z) = H(t, y, z, a, b) = \begin{cases} 1, & \text{if } bz - ay < tp \\ 0, & \text{if } bz - ay \geq tp \end{cases}$$

Then we may write $D(a, b, p, r, t)$ in the form

$$\begin{aligned} D(a, b, p, r, t) &= \sum_{x \in \mathbb{F}_p} H(\tilde{r}(x), \tilde{r}(x+1)) \\ &= \sum_{0 \leq y, z \leq p-1} H(y, z) \# \{x \in \mathbb{F}_p : \tilde{r}(x) = y, \tilde{r}(x+1) = z\}. \end{aligned}$$

Next, we write $D(a, b, p, r, t)$ in terms of exponential sums mod p . Denote as usual $e_p(w) = e^{\frac{2\pi i w}{p}}$ for any w . Using the equalities

$$\sum_{0 \leq m \leq p-1} e_p(m(y - \tilde{r}(x))) = \begin{cases} p, & \text{if } \tilde{r}(x) = y \\ 0, & \text{else} \end{cases}$$

and

$$\sum_{0 \leq n \leq p-1} e_p(n(z - \tilde{r}(x+1))) = \begin{cases} p, & \text{if } \tilde{r}(x+1) = z \\ 0, & \text{else} \end{cases}$$

we find that

$$\begin{aligned} (5) \quad D(a, b, p, r, t) &= \frac{1}{p^2} \sum_{0 \leq y, z \leq p-1} H(y, z) \\ &\times \sum_{x \in \mathbb{F}_p} \sum_{0 \leq m \leq p-1} e_p(m(y - \tilde{r}(x))) \sum_{0 \leq n \leq p-1} e_p(n(z - \tilde{r}(x+1))) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{p^2} \sum_{0 \leq m, n \leq p-1} \sum_{0 \leq y, z \leq p-1} H(y, z) e_p(my + nz) \sum_{x \in \mathbb{F}_p} e_p(-m\tilde{r}(x) - n\tilde{r}(x+1)) \\
&= \frac{1}{p^2} \sum_{0 \leq m, n \leq p-1} \check{H}(m, n) S(-m, -n, r, p),
\end{aligned}$$

where

$$(6) \quad \check{H}(m, n) = \sum_{0 \leq y, z \leq p-1} H(y, z) e_p(my + nz)$$

and

$$(7) \quad S(-m, -n, r, p) = \sum_{x \in \mathbb{F}_p} e_p(-m\tilde{r}(x) - n\tilde{r}(x+1)).$$

Note that for $m = n = 0$ one has

$$(8) \quad S(0, 0, r, p) = p.$$

Next, we claim that if $(m, n) \neq (0, 0)$ then the rational function $h(X) = mr(X) + nr(X+1) \in \mathbb{F}_p(X)$ is nonconstant. Indeed, if $n = 0$ then $m \neq 0$ and $h(X) = mr(X)$ is nonconstant by the hypotheses from the statement of the theorem. The same conclusion holds if $m = 0$ and $n \neq 0$. Let now $m \neq 0$, $n \neq 0$ and assume that

$$(9) \quad mr(X) + nr(X+1) = c$$

for some $c \in \mathbb{F}_p$. Suppose first that $r(X)$ is not a polynomial and choose a root $\alpha \in \overline{\mathbb{F}_p}$ of the denominator of $r(X)$, where $\overline{\mathbb{F}_p}$ denotes the algebraic closure of $\mathbb{F}_p$. Since α is a pole of $r(X)$, from (9) it follows that α is also a pole of $r(X+1)$, that is $\alpha+1$ is a pole of $r(X)$. By repeating the above reasoning with α replaced by $\alpha+1$ we see that $\alpha+2, \alpha+3, \dots, \alpha+p-1$ are poles of $r(X)$. This forces $\deg g(X)$ to be $\geq p$, so $d \geq p$, in which case (3) becomes trivial. Let us suppose now that $r(X)$ is a polynomial, say

$$r(X) = a_l X^l + a_{l-1} X^{l-1} + \dots + a_1 X + a_0$$

with $a_0, \dots, a_l \in \mathbb{F}_p$, $a_l \neq 0$. Then by the hypotheses of Theorem 1.1 it follows that $l \geq 2$. Looking at the coefficient of X^l in (9) we deduce that $m+n=0$ in $\mathbb{F}_p$. But then, the coefficient of X^{l-1} on the left side of (9) equals lna_l , which is nonzero in $\mathbb{F}_p$, contradicting (9). This proves our claim that $h(X)$ is nonconstant in $\mathbb{F}_p(X)$. By Lemma 2.1 it follows that

$$(10) \quad |S(-m, -n, r, p)| = O_d(\sqrt{p})$$

for any $(m, n) \neq (0, 0)$.

Next, we proceed to evaluate the coefficients $\check{H}(m, n)$. We calculate explicitly $\check{H}(0, 0)$ and provide upper bounds for $|\check{H}(m, n)|$ for $(m, n) \neq (0, 0)$. There are four cases.

I. $m = 0, n \neq 0$. We have

$$\check{H}(0, n) = \sum_{0 \leq y, z \leq p-1} H(y, z) e_p(nz).$$

By the definition of $H(y, z)$ it follows that for each $y \in \{0, 1, \dots, p-1\}$ we have a sum of $e_p(nz)$ with z running over a subinterval of $\{0, 1, \dots, p-1\}$, that is a sum of a geometric progression with ratio $e_p(n)$. The absolute value of such a sum is $\leq \frac{2}{|e_p(n)-1|}$ and consequently

$$(11) \quad |\check{H}(0, n)| \leq \frac{2p}{|e_p(n) - 1|} = \frac{p}{\sin \frac{n\pi}{p}} \leq \frac{p}{2 \left\| \frac{n}{p} \right\|},$$

where $\|\cdot\|$ denotes the distance to the nearest integer.

II. $m \neq 0, n = 0$. Similarly, as in case I, we have

$$(12) \quad |\check{H}(m, 0)| \leq \frac{p}{2 \left\| \frac{m}{p} \right\|}.$$

III. $m \neq 0, n \neq 0$. We need the following lemma.

Lemma 2.2. *Let $h, k \not\equiv 0 \pmod{p}$, L, T and $u \geq 0$ be integers. Let $S = \sum_{y=0}^L \sum_{z=0}^{uy+T} e_p(hy) e_p(kz)$. Then one has*

$$|S| = O \left(\frac{1}{\left\| \frac{k}{p} \right\|} \min \left\{ L, \frac{1}{\left\| \frac{h+uk}{p} \right\|} \right\} + \frac{1}{\left\| \frac{k}{p} \right\|} \cdot \frac{1}{\left\| \frac{h}{p} \right\|} \right).$$

Proof. One has

$$\begin{aligned} S &= \sum_{y=0}^L e_p(hy) \sum_{z=0}^{uy+T} e_p(kz) \\ &= \sum_{y=0}^L e_p(hy) \frac{1 - e_p(k(uy + T + 1))}{1 - e_p(k)} \\ &= \frac{1}{1 - e_p(k)} \sum_{y=0}^L e_p(hy) - \frac{e_p(k(T + 1))}{1 - e_p(k)} \sum_{y=0}^L e_p((h + ku)y). \end{aligned}$$

Thus

$$|S| \leq \frac{1}{|1 - e_p(k)|} \left| \sum_{y=0}^L e_p(hy) \right| + \frac{1}{|1 - e_p(k)|} \left| \sum_{y=0}^L e_p((h + ku)y) \right|.$$

Note that

$$\frac{1}{|1 - e_p(k)|} = \frac{1}{\left|1 - e^{\frac{2\pi i k}{p}}\right|} = \frac{1}{\left|e^{-\frac{\pi i k}{p}} - e^{\frac{\pi i k}{p}}\right|} = \frac{1}{\left|2 \sin \frac{\pi k}{p}\right|} = O\left(\frac{1}{\left\|\frac{k}{p}\right\|}\right).$$

Also,

$$\left|\sum_{y=0}^L e_p(hy)\right| = \frac{|1 - e_p(h(L+1))|}{|1 - e_p(h)|} = O\left(\frac{1}{\left\|\frac{h}{p}\right\|}\right).$$

Lastly, if $h + ku$ is not a multiple of p , then

$$\left|\sum_{y=0}^L e_p((h + ku)y)\right| = \frac{|1 - e_p((h + ku)(L+1))|}{|1 - e_p(h + ku)|} = O\left(\frac{1}{\left\|\frac{h+ku}{p}\right\|}\right).$$

We also have the bound

$$\left|\sum_{y=0}^L e_p((h + ku)y)\right| \leq L + 1,$$

which is valid for any h, k and u . Putting the above bounds together, Lemma 2.2 follows.

We now return to the estimation of $\check{H}(m, n)$. Writing

$$\check{H}(m, n) = \sum_{\substack{0 \leq y, z \leq p-1 \\ bz - ay < tp}} e_p(my + nz)$$

as a sum of b sums according to the residue of y modulo b , one arrives at sums as in Lemma 2.2, with $h = mb$, $k = n$, $u = a$. It follows that

$$(13) \quad |\check{H}(m, n)| = O_{a,b} \left(\frac{1}{\left\|\frac{n}{p}\right\|} \min \left\{ p, \frac{1}{\left\|\frac{mb+an}{p}\right\|} \right\} + \frac{1}{\left\|\frac{n}{p}\right\|} \cdot \frac{1}{\left\|\frac{mb}{p}\right\|} \right).$$

IV. $m, n = 0$. By definition, we have

$$\check{H}(0, 0) = \sum_{0 \leq y, z \leq p-1} H(y, z).$$

Let $\mathcal{D}$ be the set of real points from the square $[0, p) \times [0, p)$ which lie below the line $bz - ay = tp$. Then $\check{H}(0, 0)$ equals the number of integer points (y, z) from $\mathcal{D}$. Therefore

$$\check{H}(0, 0) = \text{Area}(\mathcal{D}) + O(\text{length}(\partial\mathcal{D})).$$

An easy computation shows that $\text{Area}(\mathcal{D})$ equals $p^2 G(t, a, b)$ with $G(t, a, b)$ defined as in the Introduction, while the length of the boundary $\partial\mathcal{D}$ is $\leq 4p$. Hence

$$\check{H}(0, 0) = p^2 G(t, a, b) + O(p).$$

By (5) we know that

$$\left| D(a, b, p, r, t) - \frac{1}{p^2} \check{H}(0, 0) S(0, 0, r, p) \right| \leq D_1 + D_2 + D_3 ,$$

where

$$\begin{aligned} D_1 &= \frac{1}{p^2} \sum_{m=1}^{p-1} |\check{H}(m, 0)| |S(-m, 0, r, p)| , \\ D_2 &= \frac{1}{p^2} \sum_{n=1}^{p-1} |\check{H}(0, n)| |S(0, -n, r, p)| , \\ D_3 &= \frac{1}{p^2} \sum_{m=1}^{p-1} \sum_{n=1}^{p-1} |\check{H}(m, n)| |S(-m, -n, r, p)| . \end{aligned}$$

One has

$$\frac{1}{p^2} \check{H}(0, 0) S(0, 0, r, p) = \frac{\check{H}(0, 0)}{p} = pG(t, a, b) + O(1).$$

By (11) and (10) we have

$$D_2 = O_d \left(\frac{1}{p^2} \sum_{n=1}^{p-1} \frac{p}{\left\| \frac{n}{p} \right\|} \sqrt{p} \right) = O_d (\sqrt{p} \log p) .$$

Similarly one has

$$D_1 = O_d (\sqrt{p} \log p) .$$

In order to estimate D_3 we first use (10) and (13) to obtain

$$\begin{aligned} (14) \quad D_3 &= O_{a,b,d} \left(\frac{1}{p^{3/2}} \sum_{m=1}^{p-1} \sum_{n=1}^{p-1} \frac{1}{\left\| \frac{n}{p} \right\|} \min \left\{ p, \frac{1}{\left\| \frac{mb+an}{p} \right\|} \right\} \right. \\ &\quad \left. + \frac{1}{p^{3/2}} \sum_{m=1}^{p-1} \sum_{n=1}^{p-1} \frac{1}{\left\| \frac{n}{p} \right\|} \cdot \frac{1}{\left\| \frac{mb}{p} \right\|} \right) \end{aligned}$$

The first double sum in (14) is

$$\begin{aligned} &\sum_{m=1}^{p-1} \sum_{n=1}^{p-1} \frac{1}{\left\| \frac{n}{p} \right\|} \min \left\{ p, \frac{1}{\left\| \frac{mb+an}{p} \right\|} \right\} \\ &\leq \sum_{n=1}^{p-1} \frac{1}{\left\| \frac{n}{p} \right\|} \sum_{mb+an \equiv 0 \pmod{p}}^{p-1} p + \sum_{n=1}^{p-1} \frac{1}{\left\| \frac{n}{p} \right\|} \sum_{mb+an \not\equiv 0 \pmod{p}}^{p-1} \frac{1}{\left\| \frac{mb+an}{p} \right\|} \end{aligned}$$

$$\leq p \sum_{n=1}^{\frac{p-1}{2}} \frac{p}{n} + \sum_{n=1}^{p-1} \frac{1}{\left\| \frac{n}{p} \right\|} \sum_{m'=1}^{p-1} \frac{1}{\left\| \frac{m'}{p} \right\|} \leq p^2(1 + \log p) + 4p^2(1 + \log p)^2,$$

while the second double sum is

$$\sum_{m=1}^{p-1} \sum_{n=1}^{p-1} \frac{1}{\left\| \frac{n}{p} \right\|} \cdot \frac{1}{\left\| \frac{mb}{p} \right\|} = 4 \sum_{m=1}^{\frac{p-1}{2}} \frac{p}{m} \sum_{n=1}^{\frac{p-1}{2}} \frac{p}{n} \leq 4p^2(1 + \log p)^2.$$

Hence $D_3 = O_{a,b,d}(\sqrt{p} \log^2 p)$. Putting all these together, Theorem 1.1 follows.

3. Proof of the Corollaries

For the proof of the first Corollary, let us notice that

$$\#\{x \in \mathbb{F}_p : \tilde{r}(x) > \tilde{r}(x+1)\} = D(1, 1, p, r, 0).$$

Here $W = Z = 0$ and so

$$G(0, 1, 1) = \frac{(t+a)^2}{2ab} = \frac{1}{2}.$$

Thus

$$\#\{x \in \mathbb{F}_p : \tilde{r}(x) > \tilde{r}(x+1)\} = \frac{p}{2} + O_d(p^{\frac{1}{2}} \log^2 p)$$

which proves Corollary 1.2.

In order to prove Corollary 1.3 note that

$$\begin{aligned} M(p, r, 2k) &= \sum_{x \in \mathbb{F}_p} (\tilde{r}(x+1) - \tilde{r}(x))^{2k} \\ &= \sum_{-p < m < p} m^{2k} \#\{x \in \mathbb{F}_p : \tilde{r}(x+1) - \tilde{r}(x) = m\}. \end{aligned}$$

This equals

$$\begin{aligned} \sum_{-p < m < p} m^{2k} (D(\frac{m+1}{p}) - D(\frac{m}{p})) &= D(1)(p-1)^{2k} \\ &+ \sum_{-p < m < p} D(\frac{m}{p})((m-1)^{2k} - m^{2k}) \end{aligned}$$

where for any t we denote $D(t) = D(1, 1, p, r, t)$. From Theorem 1.1 it follows that

$$\begin{aligned} M(p, r, 2k) &= p^{2k+1} G(1, 1, 1) + p \sum_{-p < m < p} G(\frac{m}{p}, 1, 1)((m-1)^{2k} - m^{2k}) \\ &+ O_{k,d}(p^{2k+\frac{1}{2}} \log^2 p) + O_d(p^{1/2} \log^2 p \sum_{-p < m < p} |(m-1)^{2k} - m^{2k}|). \end{aligned}$$

Since $(m-1)^{2k} - m^{2k} = -2km^{2k-1} + O_k(p^{2k-2})$ and $0 \leq G(\frac{m}{p}, 1, 1) \leq 1$ for any m , we derive

$$M(p, r, 2k) = p^{2k+1}G(1, 1, 1) - 2kp \sum_{-p < m < p} m^{2k-1}G(\frac{m}{p}, 1, 1) + O_{k,d}(p^{2k+\frac{1}{2}} \log^2 p).$$

From the definition of G we see that

$$G(\frac{m}{p}, 1, 1) = \begin{cases} 0, & \text{if } m < -p \\ \frac{(1+\frac{m}{p})^2}{2}, & \text{if } -p \leq m \leq 0 \\ 1 - \frac{(1-\frac{m}{p})^2}{2}, & \text{if } 0 < m < p \\ 1, & \text{if } p \leq m. \end{cases}$$

Using the fact that for any positive integer r one has $\sum_{-p < m < p} m^r = \frac{2p^{r+1}}{r+1} + O_r(p^r)$ if r is even and $\sum_{-p < m < p} m^r = 0$ if r is odd, the statement of Corollary 1.3 follows after a straightforward computation.

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