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The analytic continuation and the order estimate of multiple Dirichlet series

par KOHJI MATSUMOTO et YOSHIO TANIGAWA

1. Introduction and statement of results

Let \( s = \sigma + it \) be a complex variable, and

\[
\varphi_k(s) = \sum_{n=1}^{\infty} \frac{a_k(n)}{n^s}
\]

be a Dirichlet series with complex coefficients \( a_k(n) \) (1 \( \leq k \leq r \)). We assume

(I) \( \varphi_k(s) \) is convergent absolutely for \( \sigma > \alpha_k (> 0) \);

(II) \( \varphi_k(s) \) can be continued meromorphically to the whole complex plane \( \mathbb{C} \), and holomorphic except for a possible pole at \( s = \alpha_k \) of order at most 1, whose residue we denote by \( R_k \);

(III) in any fixed strip \( \sigma_1 \leq \sigma \leq \sigma_2 \), the order estimate \( \varphi_k(\sigma + it) = O(|t|^A) \) holds as \( |t| \to \infty \), where \( A = A(\sigma_1, \sigma_2) \) be a non-negative constant.

In the present paper we introduce the multiple Dirichlet series

\[
\Phi_r((s_1, \ldots, s_r); (\varphi_1, \ldots, \varphi_r)) = \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} a_1(m_1) a_2(m_2) \cdots a_r(m_r) \frac{m_1^{s_1}}{(m_1 + m_2)^{s_2}} \cdots \frac{m_1 \cdots m_r^{s_r}}{(m_1 + \cdots + m_r)^{s_r}}
\]

associated with \( \varphi_1, \ldots, \varphi_r \), where \( s_k = \sigma_k + it_k \) (1 \( \leq k \leq r \)) are complex variables, and prove several basic properties. It is clear that the multiple Dirichlet series can be considered as a generalization of the classical Dirichlet series.

series (1.2) converges absolutely if \( \sigma_k > \alpha_k \) \((1 \leq k \leq r)\). Let \( \mathbb{N}, \mathbb{N}_0 \) be the sets of positive integers and non-negative integers, respectively. We first prove the following

**Theorem 1.** The multiple series \( \Phi_r((s_1, \ldots, s_r); (\varphi_1, \ldots, \varphi_r)) \) can be continued meromorphically to the whole \( \mathbb{C}^r \) space, and its possible singularities are located only on the subsets of \( \mathbb{C}^r \) each of which is defined by one of the following equations:

\[
(1.3) \quad s_j + \cdots + s_r = \alpha_j + \delta_{j+1}\alpha_{j+1} + \cdots + \delta_r\alpha_r - n, \tag{1.3}
\]

where \(1 \leq j \leq r\), \( n \in \mathbb{N}_0 \), and \( \delta_k = 0 \) or \( 1 \) \((2 \leq k \leq r)\). When \( j = r \), (1.3) is to be read as \( s_r = \alpha_r - n \). Moreover,

(i) in the case \( j = r \geq 2 \), if \( \alpha_r \in \mathbb{N} \), then the possible values of \( n \) are \( 0, 1, 2, \ldots, \alpha_r - 1 \);

(ii) in the case \( 2 \leq j \leq r - 1 \), if \( \alpha_j \in \mathbb{N} \) and \( \delta_{j+1} = \cdots = \delta_r = 1 \), then the possible values of \( n \) are \( 0, 1, 2, \ldots, \alpha_j - 1 \);

(iii) in the case \( j = 1 \), if \( r = 1 \) or if \( \delta_2 = \cdots = \delta_r = 1 \), then \( n = 0 \).

**Theorem 2.** If \( \varphi_k(s) \) is entire for \( 1 \leq k \leq r \), then

\[ \Phi_r((s_1, \ldots, s_r); (\varphi_1, \ldots, \varphi_r)) \]

is also entire.

Our \( \Phi_r((s_1, \ldots, s_r); (\varphi_1, \ldots, \varphi_r)) \) includes various interesting special cases. The function \( \varphi_k(s) \) can be the Riemann zeta-function \( \zeta(s) \), the Dirichlet L-function \( L(s, \chi) \), the automorphic L-function \( L(s, f) \) attached to a certain cusp form, etc. When \( \varphi_k(s) = \zeta(s) \) \((1 \leq k \leq r)\), our \( \Phi_r \) is nothing but the well-known Euler-Zagier sum, whose analytic continuation was recently established by various methods (Arakawa and Kaneko [3], Zhao [13], Akiyama, Egami and Tanigawa [1], Matsumoto [10]; see also Goncharov [6]). Our proof of Theorems 1 and 2 is a generalization of the method given in Matsumoto [10].

When \( \varphi_k(s) = L(s, \chi_k) \) \((1 \leq k \leq r)\), where \( \chi_1, \ldots, \chi_r \) are Dirichlet characters, the corresponding \( \Phi_r \) may be called the multiple Dirichlet L-function. The special values at positive integer arguments of these kinds of multiple L-functions have been studied by Goncharov, Arakawa and Kaneko [4] and others. It is also to be noted that a different type of multiple Dirichlet L-functions of the form

\[
\sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \frac{\chi_1(m_1) \chi_2(m_1 + m_2) \cdots \chi_r(m_1 + \cdots + m_r)}{m_1^{s_1} (m_1 + m_2)^{s_2} \cdots (m_1 + \cdots + m_r)^{s_r}}
\]

has been studied by Goncharov [5], Akiyama and Ishikawa [2], and Ishikawa [8, 9].
By Theorem 2, the multiple Dirichlet $L$-function
\[ \Phi((s_1, \ldots, s_r); (L(\cdot, \chi_1), \ldots, L(\cdot, \chi_r))) \]
is entire if $\chi_1, \ldots, \chi_r$ are non-principal. Also, the multiple automorphic
$L$-function
\[ \Phi((s_1, \ldots, s_r); (L(\cdot, f_1), \ldots, L(\cdot, f_r))) \]
is entire, where $f_1, \ldots, f_r$ are cusp forms. On the other hand, in the present
paper we assume that the order of the pole at $s = \alpha_k$ is at most 1, but this
is only for simplicity. We may consider the case when $\varphi_k(s)$ has the pole
of higher order by the same method.

A natural next problem is to study the order estimate of $\Phi_r$. In this
paper we treat the simplest case $\Phi_2$. By the assumption (III) there exists
a non-negative constant $\theta_k(\sigma)$ for any $\sigma$ such that
\begin{equation}
\varphi_k(\sigma + it) = O(|t|^{\theta_k(\sigma)})
\end{equation}
holds as $|t| \to \infty$. We shall prove

**Theorem 3.** Let $\eta$ be a small positive number. The estimate
\[ \Phi_2((s_1, s_2); (\varphi_1, \varphi_2)) \ll (1 + |t_2|)^{-\alpha_2} (1 + |t_1 + t_2|)^{\theta_1(\sigma_1 + \sigma_2 - \alpha_2)} R_2 
+ (1 + |t_2|)^{\theta_2(\eta) + \frac{1}{2} + \max\{0, \frac{1}{2} - \sigma_2 + \eta\}} \max\{1 + |t_2|, 1 + |t_1 + t_2|\}^{\theta_1(\sigma_1 + \sigma_2 - \eta)} \]
holds in the region \{$(s_1, s_2) | \sigma_1 + \sigma_2 > \alpha_1 + \eta, \sigma_2 > \eta$\}, except the points
near the set of singularities.

Therefore, if some non-trivial estimate of $\varphi_k(s)$ is known, then Theorem 3
gives a non-trivial estimate of $\Phi_2$. For instance, when $\varphi_k(s) = \zeta(s), L(s, \chi)$,
or the automorphic $L$-function $L(s, f)$ (see Good [7]), sharp values of $\theta_k(\sigma)$
are known.

It is clearly an interesting problem to generalize Theorem 2 to the case
of $\Phi_r, r \geq 3$.

**2. Proof of Theorems 1 and 2**

We prove the theorems by induction. The argument is similar to that
developed in the last section of [10], or in [11], [12].

The case $r = 1$ is clear from the assumption (II). Consider the case $\Phi_r$,
assuming that the theorems are true for $\Phi_{r-1}$. First assume $\sigma_k > \alpha_k$ ($1 \leq k \leq r$). Recall the classical Mellin-Barnes formula
\begin{equation}
\Gamma(s)(1 + \lambda)^{-s} = \frac{1}{2\pi i} \int_{(c)} \Gamma(s + z)\Gamma(-z)\lambda^zdz,
\end{equation}
where $\Re s > 0$, $|\arg \lambda| < \pi$, $\lambda \neq 0$, $-\Re s < c < 0$, and the path of integration is the vertical line $\Re z = c$. Put $s = s_r$ and

$$\lambda = \frac{m_r}{m_1 + \cdots + m_{r-1}}$$

in (2.1); we may assume $-\sigma_r < c < -\alpha_r$. Then multiply the both sides by $a_1(m_1) \cdots a_r(m_r)m_1^{-s_1} \cdots (m_1 + \cdots + m_{r-2})^{-s_{r-2}}(m_1 + \cdots + m_{r-1})^{-s_{r-1} - s_r}$ and sum up with respect to $m_1, \ldots, m_r$. We obtain

$$(2.2) \quad \Phi_r((s_1, \ldots, s_r); (\varphi_1, \ldots, \varphi_r)) = \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(s_r + z)\Gamma(-z)}{\Gamma(s_r)}$$

$$\times \Phi_{r-1}((s_1, \ldots, s_{r-2}, s_{r-1} + s_r + z); (\varphi_1, \ldots, \varphi_{r-1})) \varphi_r(-z)dz.$$  

Next we shall shift the path to $\Re z = M - \eta$, where $M$ is a non-negative integer and $\eta$ is a small positive number. The function $\varphi_r(-z)$ is of polynomial order with respect to $\Re z$ by the assumption (III), while

$$\Phi_{r-1}((s_1, \ldots, s_{r-2}, s_{r-1} + s_r + z); (\varphi_1, \ldots, \varphi_{r-1})) = O(1)$$

in the region $c \leq \Re z \leq M - \eta$ because of the expression (1.2). Hence the integrand on the right-hand side of (2.2) tends to zero when $|\Re z| \to \infty$, so this shifting is possible. Counting the residues of the poles at $s = -\alpha_r, 0, 1, 2, \ldots, M - 1$, we obtain

$$(2.3) \quad \Phi_r((s_1, \ldots, s_r); (\varphi_1, \ldots, \varphi_r)) =$$

$$= \frac{\Gamma(s_r - \alpha_r)\Gamma(\alpha_r)}{\Gamma(s_r)} \Phi_{r-1}((s_1, \ldots, s_{r-2}, s_{r-1} + s_r - \alpha_r); (\varphi_1, \ldots, \varphi_{r-1})) R_r$$

$$+ \sum_{l=0}^{M-1} \left(-\frac{s_r}{l}\right) \Phi_{r-1}((s_1, \ldots, s_{r-2}, s_{r-1} + s_r + l); (\varphi_1, \ldots, \varphi_{r-1})) \varphi_r(-l)$$

$$+ \frac{1}{2\pi i} \int_{(M-\eta)} \frac{\Gamma(s_r + z)\Gamma(-z)}{\Gamma(s_r)}$$

$$\times \Phi_{r-1}((s_1, \ldots, s_{r-2}, s_{r-1} + s_r + z); (\varphi_1, \ldots, \varphi_{r-1})) \varphi_r(-z)dz.$$  

The first two terms on the right-hand side are meromorphic in the whole $\mathbb{C}^r$ space by the induction assumption, while we can see that the last integral term is holomorphic in the region

$$\{(s_1, \ldots, s_r) \mid \sigma_k + \cdots + \sigma_r > \alpha_k + \cdots + \alpha_{r-1} - M + \eta \quad (1 \leq k \leq r-1) \}$$

$$\sigma_r > -M + \eta$$

by using the induction assumption on the location of singularities of $\Phi_{r-1}$. Since $M$ is arbitrary, this implies the meromorphic continuation of $\Phi_r$ to the whole $\mathbb{C}^r$ space.

The location of singularities of $\Phi_r$ can also be seen from (2.3). In fact, the factor $\Gamma(s_r - \alpha_r)$ is singular on $s_r = \alpha_r - n$ ($n \in \mathbb{N}_0$). If $\alpha_r \in \mathbb{N}$,
then this singularity is cancelled by the factor $\Gamma(s_r)$ when $n \geq \alpha_r$. By the induction assumption, the factor

$$\Phi_{r-1}((s_1, \ldots, s_{r-2}, s_{r-1} + s_r - \alpha_r); (\varphi_1, \ldots, \varphi_{r-1}))$$

is singular on

$$s_j + \cdots + s_r - \alpha_r = \alpha_j + \delta_{j+1} \alpha_{j+1} + \cdots + \delta_{r-1} \alpha_{r-1} - n,$$

where $1 \leq j \leq r-1$ and $n \in \mathbb{N}_0$, but

(i) if $j \geq 2$, $\alpha_j \in \mathbb{N}$ and $\delta_{j+1} = \cdots = \delta_{r-1} = 1$ then $0 \leq n \leq \alpha_j - 1$, and

(ii) if $j = 1$, and if $r = 2$ or $\delta_2 = \cdots = \delta_{r-1} = 1$ then $n = 0$.

Similarly the factor

$$\Phi_{r-1}((s_1, \ldots, s_{r-2}, s_{r-1} + s_r + l); (\varphi_1, \ldots, \varphi_{r-1})) \quad (l \in \mathbb{N}_0)$$

is singular on

$$s_j + \cdots + s_r + l = \alpha_j + \delta_{j+1} \alpha_{j+1} + \cdots + \delta_{r-1} \alpha_{r-1} - n$$

$(1 \leq j \leq r-1, n \in \mathbb{N}_0)$, with the same restrictions as above. Collecting the above information we obtain the assertion on the location of singularities of $\Phi_r$.

Lastly, if $\varphi_r(s)$ is entire then $R_r = 0$, hence the first term on the right-hand side of (2.3) vanishes. Therefore inductively we can show that $\Phi_r$ is entire. All the assertions of Theorems 1 and 2 are now established.

3. Estimation of certain integrals

In this section, as a preparation for the proof of Theorem 3, we consider the integral of the form

$$(3.1) \quad J(u, v) = \int_{-\infty}^{\infty} (1 + |u + y|)^p (1 + |v + y|)^q (1 + |y|)^r \times \exp(A|u + y| + B|y|) \, dy,$$

where $u, v, p, q, r, A, B$ are real numbers with $A + B < 0$. To evaluate this integral, we use the estimate

$$(3.2) \quad \int_{-\infty}^{\infty} (1 + |y|)^\rho \exp(A|u + y| + B|y|) \, dy = O\left((1 + |u|)^{\rho + \delta} e^{B|u|} + e^{A|u|}\right),$$

where $u, \rho, A, B$ are real numbers with $A + B < 0, \delta = 1$ or 0 according as $A = B$ or $A \neq B$, and the implied constant depends only on $\rho, A$ and $B$. This estimate is a special case of Lemma 3 in Matsumoto [11].
Temporarily we assume $|u| \geq |v|$. Divide the integral (3.1) as
\[
\int_{|y|>2|u|} + \int_{2|u| \geq |y| > 2|v|} + \int_{|y| \leq 2|v|},
\]
and estimate the factor $(1 + |u + y|)^p (1 + |v + y|)^q (1 + |y|)^r$ in the integrand as
\[
\begin{cases}
O((1 + |y|)^{p+q+r}) & \text{if } |y| > 2|u|, \\
O(\max\{1, U^p\}(1 + |y|)^{q+r}) & \text{if } 2|u| \geq |y| > 2|v|, \\
O(\max\{1, U^p\} \max\{1, V^q\}(1 + |y|)^r) & \text{if } |y| \leq 2|v|,
\end{cases}
\]
where $U = 1 + |u|$ and $V = 1 + |v|$. Then replace the intervals of all integrals by $(-\infty, \infty)$, and apply (3.2). The result is that
\[
J(u, v) \ll U^{p+q+r+\delta} e^{B|u|} + e^{A|u|}
\]
\[
+ \max\{1, U^p\}\left\{U^{q+r+\delta} e^{B|u|} + e^{A|u|}\right\}
\]
\[
+ \max\{1, U^p\} \max\{1, V^q\} \left\{U^{r+\delta} e^{B|u|} + e^{A|u|}\right\},
\]
hence

**Lemma 1.** Using the notations as above, we have
\[
(3.3) \quad J(u, v) \ll (1 + U^p)(1 + U^q + V^q) U^{r+\delta} e^{B|u|}
\]
\[
\quad \quad \quad + (1 + U^p)(1 + V^q)e^{A|u|}.
\]

This is proved under the assumption $|u| \geq |v|$, but the case $|u| < |v|$ can be treated similarly, and the same conclusion (3.3) holds.

In particular, if $A = B = -\frac{\pi}{2}$ then $\delta = 1$, then (3.3) implies the following

**Lemma 2.** We have
\[
\int_{-\infty}^{\infty} (1 + |u + y|)^p (1 + |v + y|)^q (1 + |y|)^r \exp\left(-\frac{\pi}{2}|u + y| - \frac{\pi}{2}|y|\right) dy
\]
\[
= O\left( (1 + U^p)(1 + U^q + V^q)(1 + U^{r+1}) \exp\left(-\frac{\pi}{2}|u|\right) \right),
\]
with the implied constant depending only on $p, q$ and $r$.

4. Proof of Theorem 3

Putting $r = 2$ and $M = 0$ in (2.3), we have
\[
\Phi_2((s_1, s_2); (\varphi_1, \varphi_2)) = \frac{\Gamma(s_2 - \alpha_2) \Gamma(\alpha_2)}{\Gamma(s_2)} \varphi_1(s_1 + s_2 - \alpha_2) R_2
\]
\[
+ \frac{1}{2\pi i} \int_{(-\eta)} \frac{\Gamma(s_2 + z) \Gamma(-z)}{\Gamma(s_2)} \varphi_1(s_1 + s_2 + z) \varphi_2(-z) dz,
\]
and the last integral is holomorphic in the region

$$\{(s_1,s_2) \mid \sigma_1 + \sigma_2 > \alpha_1 + \eta, \ \sigma_2 > \eta\}.$$

Hence in this region, by using Stirling’s formula and (1.4), we have

$$\Phi_2((s_1,s_2);(\varphi_1,\varphi_2)) \ll (1 + |t_2|)^{-\alpha_2} (1 + |t_1 + t_2|)^{\theta_1(\sigma_1 + \sigma_2 - \alpha_2) R_2} + (1 + |t_2|)^{-\sigma_2 + \frac{1}{2} \log |t_2|} \int_{-\infty}^{\infty} (1 + |t_2 + y|)^{\sigma_2 - \eta - \frac{1}{2}} (1 + |t_1 + t_2 + y|)^{\theta_1(\sigma_1 + \sigma_2 - \eta)}$$

$$\times \left(1 + |y|\right)^{\theta_2(\eta) + \eta - \frac{1}{2}} \exp \left(-\frac{\pi}{2} |t_2 + y| - \frac{\pi}{2} |y|\right) \, dy.$$

We apply Lemma 2 with $u = t_2, v = t_1 + t_2, p = \sigma_2 - \eta - \frac{1}{2}, q = \theta_1(\sigma_1 + \sigma_2 - \eta)$, and $r = \theta_2(\eta) + \eta - \frac{1}{2}$. Since $q \geq 0$ and $r + 1 \geq 0$, the estimate of Lemma 2 can be written as

$$O \left((1 + U^p)U^{r+1} \max\{U, V\}^q \exp \left(-\frac{\pi}{2} |u|\right)\right).$$

Hence

$$\Phi_2((s_1,s_2);(\varphi_1,\varphi_2)) \ll (1 + |t_2|)^{-\alpha_2} (1 + |t_1 + t_2|)^{\theta_1(\sigma_1 + \sigma_2 - \alpha_2) R_2} + (1 + |t_2|)^{\theta_2(\eta) - \sigma_2 + \eta + 1}$$

$$\times \left\{1 + (1 + |t_2|)^{\sigma_2 - \eta - \frac{1}{2}} \right\} \max\left\{1 + |t_2|, 1 + |t_1 + t_2|\right\}^{\theta_1(\sigma_1 + \sigma_2 - \eta)},$$

which implies the assertion of Theorem 3.

References


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