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On the $n$-torsion subgroup of the Brauer group of a number field

par HERSHY KISILEVSKY et JACK SONN

RéSUMÉ. Pour toute extension galoisienne $K$ de $\mathbb{Q}$ et tout entier positif $n$ premier au nombre de classes de $K$, il existe une extension abélienne $L$ de $K$ d’exposant $n$ telle que le $n$-sous-groupe de torsion du groupe de Brauer de $K$ est égal au groupe de Brauer relatif de $L/K$.

ABSTRACT. Given a number field $K$ Galois over the rational field $\mathbb{Q}$, and a positive integer $n$ prime to the class number of $K$, there exists an abelian extension $L/K$ (of exponent $n$) such that the $n$-torsion subgroup of the Brauer group of $K$ is equal to the relative Brauer group of $L/K$.

1. Introduction

Let $K$ be a field, $Br(K)$ its Brauer group. If $L/K$ is a field extension, then the relative Brauer group $Br(L/K)$ is the kernel of the restriction map $res_{L/K} : Br(K) \to Br(L)$. Relative Brauer groups have been studied by Fein and Schacher (see e.g. [2, 3, 4].) Every subgroup of $Br(K)$ is a relative Brauer group $Br(L/K)$ for some extension $L/K$ [2], and the question arises as to which subgroups of $Br(K)$ are algebraic relative Brauer groups, i.e. of the form $Br(L/K)$ with $L/K$ an algebraic extension. For example if $L/K$ is a finite extension of number fields, then $Br(L/K)$ is infinite [3], so no finite subgroup of $Br(K)$ is an algebraic relative Brauer group. In [1] the question was raised as to whether or not the $n$-torsion subgroup $Br_n(K)$ of the Brauer group $Br(K)$ of a field $K$ is an algebraic relative Brauer group. For example, if $K$ is a $(p$-adic) local field, then $Br(K) \cong \mathbb{Q}/\mathbb{Z}$, so $Br_n(K)$ is an algebraic relative Brauer group for all $n$. This is not surprising, since this Brauer group is “small”. A counterexample was given in [1] for $n = 2$ and $K$ a formal power series field over a local field. Somewhat surprisingly, $Br_2(\mathbb{Q})$ turned out to be an algebraic relative Brauer group.


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For number fields $K$, the problem is a purely arithmetic one, because of the fundamental local-global description of the Brauer group of a number field. In [1], it was proved that $Br_n(Q)$ is an algebraic relative Brauer group for all squarefree $n$. In this paper, we prove the following affirmative result for number fields: given any number field $K$ Galois over $Q$ and any $n$ prime to the class number of $K$, $Br_n(K)$ is an algebraic relative Brauer group, in fact of an abelian extension of $K$. In particular, $Br_n(Q)$ is an algebraic relative Brauer group for all $n$.

2. Algebraic extensions of $K$ with local degree $n$ everywhere

**Theorem 2.1.** Let $K$ be a number field Galois over $Q$ with class number $h_K$. Let $\ell$ be a prime number, relatively prime to $h_K$ and let $r$ be a positive integer. There exists an abelian extension $L/K$ of exponent $\ell^r$ such that the local degree of $L/K$ at every finite prime equals $\ell^r$. If $\ell = 2$, then $L$ can be taken to be totally complex.

**Proof.** Let $k_{\infty}$ be the cyclotomic extension of $Q$ obtained by adjoining all $\ell$-power roots of unity to $Q$. Let $s$ be a positive integer. For $\ell$ odd, let $k_s$ be the unique subfield of $k_{\infty}$ of degree $\ell^s$ over $Q$. If $\ell = 2$, there are three elements of order 2 in $Gal(Q(\mu_{2^{s+1}})/Q)$, one fixing the maximal real subfield, one fixing $Q(\mu_{2^{s+1}})$, and a third fixing a cyclic totally complex extension of $Q$ of degree $2^s$, which we define to be $k_s$. (As usual, $\mu_m$ denotes the $m$th roots of unity.) Then $\ell$ is the unique prime of $Q$ ramified in $k_s$ and it is totally ramified.

Choose $s$ such that $L_0 = Kk_s$ has degree $\ell^r$ over $K$. Then the primes $l_1, \ldots, l_\ell$ of $K$ dividing $\ell$ have isomorphic completions, and since $\ell$ is prime to $h_K$, they are all totally ramified in $L_0/K$. In the case $\ell = 2$, the real primes are also ramified in $L_0/K$.

Let $E$ be the extension of $K$ obtained by adjoining $L_0$ and the $\ell^r$th roots of all the units of $K$ (including the $\ell^r$th roots of unity). Let $S$ be the (infinite) set of primes of $K$ which split completely in $E$. For $p \in S$ consider the $\ell$-ray class field $R_p$ with conductor $p$, i.e. the $\ell$-primary part of the ray class field with conductor $p$. Since the class number $h_K$ of $K$ is prime to $\ell$, $p$ is totally ramified in $R_p$. Furthermore, the $\ell$-ray class group is isomorphic to the $\ell$-part of $K^*_p = (O_K/pO_K)^*$ (the multiplicative group of the residue field) modulo the image of the unit group of $K$. By choice of $p \in S$, the absolute norm $N(p)$ is congruent to 1 modulo $\ell^r$ and all units are $\ell^r$th powers in $K^*_p$. Hence $K^*_p$ has a (unique cyclic) quotient of order $\ell^r$. We define $L^p$ to be the corresponding cyclic subextension of degree $\ell^r$ of $R_p$.

Let $l = l_1$ be one of the prime divisors of $\ell$ in $K$. Consider the condition $l$ splits completely in $L^p$ ($p \in S$). This is equivalent to $l$ being an $\ell^r$th...
power in the ray class group mod $p$. But since $\ell$ is prime to $h = h_K$, this
is equivalent to the principal ideal $I^h = (a)$, $a \in K^*$, being an $\ell^r$th power
in the ray class group mod $p$. Since all the units of $K$ are $\ell^r$th powers
modulo $p$, this is equivalent to $a$ being an $\ell^r$th power in $\overline{K}^*_p$, which for
$p \in S$ is equivalent to $p$ splitting completely in $K(\mu_{\ell^r}, \sqrt[\ell^r]{a})$. Denote the $a$
corresponding to $l_i$ by $a_i$.

Let $S' \subset S$ be the set of primes of $K$ that split completely in $E' = E(\sqrt[\ell^r]{a_1}, \ldots, \sqrt[\ell^r]{a_l})$. The prime divisors $l_i$ of $\ell$ in $K$ split completely in $L^p$
if $p \in S'$.

We now define recursively a subsequence $S_0$ of primes of $S'$. We begin
with any prime $p_1 \in S'$ such that $\mathcal{N}(p_1) > \mathcal{N}(l_i)$ for $l_i$ dividing $\ell$. (As
above, $\mathcal{N}$ denotes the absolute norm.)

We claim there exists a prime $p_2 \in S'$ with $\mathcal{N}(p_2) > \mathcal{N}(p_1)$ satisfying:

(a) $p_2$ splits completely in $L^{p_1}$;
(b) $p_1$ splits completely in $L^{p_2}$;
(c) $q$ is inert in $L^{p_2}$ for all primes $q \neq p_1, l_i$ of $K$ with absolute norm
$\mathcal{N}(q) \leq \mathcal{N}(p_1)$.

To prove the claim, we reduce it to an application of Chebotarev’s density
theorem. Arguing as above, (b) is equivalent to $p_1$ being an $\ell^r$th power in
the ray class group mod $p_2$. But since $\ell$ is prime to $h = h_K$, this is equivalent
to the principal ideal $p_1^h = (c)$, $c \in K^*$, being an $\ell^r$th power in the ray class
group mod $p_2$. Since all the units of $K$ are $\ell^r$th powers modulo $p_2$, this is equivalent
to $c$ being an $\ell^r$th power in $\overline{K}^*_{p_2}$, which is equivalent to $p_2$
splitting completely in $K(\mu_{\ell^r}, \sqrt[\ell^r]{c})$. Thus (b) is a Chebotarev condition
compatible with (a).

We now consider (c). We want all $q \neq p_1, l_1, \ldots, l_\ell$ with $\mathcal{N}(q) \leq \mathcal{N}(p_1)$
(these are finitely many) to be inert in $L^{p_2}$. For this it suffices that $q$ be
inert in $M^{p_2}$, where $M^{p_2}$ is the subextension of $L^{p_2}$ of degree $\ell$ over $K$.
As above, if $q^h = (b)$, $b \in K^*$, this means that $b$ is not an $\ell$th power in
the ray class group mod $p_2$, i.e. $b$ is not an $\ell$th power in $\overline{K}^*_{p_2}$ (again since
all units are $\ell^r$th powers in $\overline{K}^*_{p_2}$), i.e. $p_2$ is nonsplit in $K(\mu_{\ell}, \sqrt{\beta})$. Since
$p_2$ splits in $K(\mu_{\ell^r})$, this is equivalent to $p_2$ being nonsplit in $K(\mu_{\ell}, \sqrt{\beta})$.
For this Chebotarev condition to be compatible with (a) and (b), it suffices
that the fields $L^{p_1}E'$ and $\{K(\mu_{\ell^r}, \sqrt{\beta}) : q^h = (b), q \neq p_1, l_i, \mathcal{N}(q) \leq \mathcal{N}(p_1)\}$
be linearly disjoint over $K(\mu_{\ell^r})$.

Let $q_1, \ldots, q_u$ be the primes of $K$ distinct from $p_1, l_1, \ldots, l_\ell$ of absolute
norm less than or equal to that of $p_1$, and let $q_i^h = (b_i)$, $i = 1, \ldots, u$. Set
$K' = K(\mu_{\ell^r})$. We show first that the fields $\{K'(\sqrt[b_i]{\beta}) : i = 1, \ldots, u\}$
are linearly disjoint over $K'$. If not, then by Kummer theory we have
an equation $\prod_i q_i^{e_i} = x^\ell$ with $x \in K'$, and not all the $e_i$ divisible by
Taking ideals in $K'$, we have $\prod_i^n (b_i)^{e_i} = (x)^\ell$. Since $h_K$ is prime to $\ell$, and the primes $q_i$ are unramified in $K'$, we see that $\ell$ must divide all the $e_i$, contradiction. Set $F_1 := L\pi\mathcal{E}'(\sqrt[\ell]{c})$, $F_2 := K'(\sqrt[\ell]{b_1}, \ldots, \sqrt[\ell]{b_n})$. It remains to show that $F_1 \cap F_2 = K'$. If not, then there is a common cyclic subextension $F_3 \subseteq F_1 \cap F_2$ with $[F_3 : K'] = \ell$. On the one hand, $F_3$ is of the form $K'(\sqrt[\ell]{\prod_i^n b_i^{e_i}})$ with not all $e_i$ divisible by $\ell$. For such an $i$, the prime divisors of $q_i$ in $K'$ ramify in $F_3$. But the only primes ramifying in $F_1$ are divisors of $p_1, l_1, \ldots, l_t$, contradiction. Thus the disjointness assertion is proved.

This shows the existence of $p_2$ satisfying (a),(b),(c).

We now assume inductively that $n > 2$ and $p_1, \ldots, p_{n-1} \in S'$ with $\mathcal{N}(p_i) < \mathcal{N}(p_{i+1})$, $i = 1, \ldots, n - 2$, have been chosen such that

(a) $p_i$ splits completely in $L^{p_j}$ for all $j < i$, $i = 2, \ldots, n - 1$

(b) $p_j$ splits completely in $L^{p_i}$ for all $j < i$, $i = 2, \ldots, n - 1$

(c) $q$ is inert in $L^{p_i}$ for all primes $q$ satisfying $\mathcal{N}(p_{i-2}) < \mathcal{N}(q) \leq \mathcal{N}(p_{i-1})$, $q \neq p_{i-1}$, $i = 2, \ldots, n - 1$ (take $p_0 = 1$)

(Note (a) and (b) together say $p_i$ splits completely in $L^{p_j}$ for all $i \neq j$, $1 \leq i, j \leq n - 1$.)

Claim: There exists a prime $p_n \in S$ satisfying (a$_n$),(b$_n$),(c$_n$).

The argument is similar to that for $p_2$: (a$_n$) is satisfied if and only if $p_n$ splits completely in the composite $L^{p_1} \cdots L^{p_{n-1}}$.

(b$_n$) is satisfied if $p_n$ splits completely in $K'(\sqrt[\ell]{c_1}, \ldots, \sqrt[\ell]{c_{n-1}})$, where $p_i^h = (c_i)$, $i = 1, \ldots, n - 1$

(c$_n$) is satisfied if $p_n$ remains inert in $K'(\sqrt[\ell]{b})$ for each $q^h = (b)$, $q \neq p_{n-1}$, with $\mathcal{N}(p_{n-2}) < \mathcal{N}(q) \leq \mathcal{N}(p_{n-1})$. In order to apply the Chebotarev theorem we need the linear disjointness of $L^{p_1} \cdots L^{p_{n-1}} \cdot \mathcal{E}'(\sqrt[\ell]{c_1}, \ldots, \sqrt[\ell]{c_{n-1}})$ and the $K'(\sqrt[\ell]{b})$ over $K = K(\mu_{\ell^r})$, for all the above $b$'s. Since the (c$_i$)'s and the (b)'s are distinct prime ideals raised to the power $h$, the previous argument goes through, proving the claim.

We therefore have an infinite sequence $S_0 = \{p_n\}_{n=1}^\infty$ of primes of $S'$ satisfying

(i) $p_i$ splits completely in $L^{p_j}$ for all $i \neq j$, and

(ii) $q$ is inert in $L^{p_n}$ for all $q \neq p_{n-1}$ with $\mathcal{N}(p_{n-2}) < \mathcal{N}(q) \leq \mathcal{N}(p_{n-1})$.

Now take $L$ to be the composite of $L_0$ and all the $L^{p_n}$. We check the local degrees of $L/K$:

For $p = p_i \in S_0$, $L$ contains $L^{p_i}$ which is totally ramified of degree $\ell^r$ at $p$. $p$ splits completely in $L_0$, and by (i), $p$ splits completely in $L^{p_j}$ for $j \neq i$, so $[L_p : K_p] = \ell^r$. 

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For $p \notin S_0$, $p$ not dividing $\ell$, $p$ is unramified in $L$, so $L_p/K_p$ is cyclic of exponent dividing $\ell^r$. There exists a positive integer $n$ such that $\mathcal{N}(p_{n-1}) < \mathcal{N}(p_n)$. By (ii), $p$ is inert in $L^{p_n+1}$ hence $[L_p : K_p] = \ell^r$. For $p$ dividing $\ell$, $p$ is totally ramified in $L_0$ and splits completely in all the $L_{p_i}$, hence $[L_p : K_p] = \ell^r$. $L_0$ is totally complex, hence so is $L$. This completes the proof of Theorem 2.1.

**Remark 2.2.** The hypothesis that $K$ is Galois over $\mathbb{Q}$ guarantees that all primes $p$ dividing $\ell$ in $K$ have isomorphic completions, which is all that is needed in the proof. Also we can have $L/K$ unramified at the infinite primes by choosing the maximal real subfield of $\mathbb{Q}(\mu_{2^{n+2}})$ in place of $k_s$.

### 3. The $n$-torsion subgroup of the Brauer group of $K$

**Theorem 3.1.** Given a number field $K$ Galois over $\mathbb{Q}$ and a positive integer $n$ prime to the class number of $K$, there exists an abelian extension $L/K$ (of exponent $n$) such that the $n$-torsion subgroup of the Brauer group of $K$ is equal to the relative Brauer group of $L/K$.

**Proof.** Consider the case $n = \ell^r$, $\ell$ prime. By Theorem 2.1, there exists an abelian $\ell$-extension $L/K$ whose local degree at every finite prime is $\ell^r$, and is 2 at the real primes if $\ell = 2$. It follows from the fundamental theorem of class field theory on the Brauer group of a number field that $L$ splits every algebra class of order dividing $\ell^r$, and conversely, any algebra class split by $L$ has order dividing $\ell^r$. For general $n$, the theorem follows from a straightforward reduction to the prime power case (see [1]).

**Remark 3.2.** For $K = \mathbb{Q}$, Theorem 3.1 says that $Br_n(\mathbb{Q})$ is an algebraic relative Brauer group for all $n$. The proof of Theorem 2.1 is more concrete in this case because the ray class fields involved are simply the degree $\ell^r$ subfields of $\mathbb{Q}(\mu_p)$ with $p \equiv 1$ (mod $\ell^r$). Theorem 3.1 was proved in [1] for the case $n$ squarefree, $K = \mathbb{Q}$. The case $n = 2$ was proved there by constructing $L/\mathbb{Q}$ with local degree 2 everywhere except perhaps at the prime 2. We are grateful to Romyar Sharifi and David Ford (independently) for a construction of $L/\mathbb{Q}$ with local degree 2 everywhere, including 2, the idea of which was instrumental in the proof of Theorem 2.1.

### References


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