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1. Introduction

Let $P$ be a polynomial in $\mathbb{Z}[z_1, \ldots, z_n]$. The Mahler measure of $P$ is then defined by

$$M(P) = \exp \left\{ \int_0^1 \ldots \int_0^1 \log |P(e^{2i\pi t_1}, \ldots, e^{2i\pi t_n})| \, dt_1 \ldots dt_n \right\}.$$  \hspace{1cm} (1)

If $P$ is a one variable polynomial, $P(z) = a_0 \prod_{j=1}^d (z - \alpha_j)$, Jensen’s formula ensures that

$$M(P) = |a_0| \prod_{j=1}^d \max(1, |\alpha_j|).$$ \hspace{1cm} (2)

In this case we denote the absolute Mahler measure of $P$ i.e. $M(P)^{1/d}$ by $\mathfrak{M}(P)$. If $\alpha \in \overline{\mathbb{Q}}$, we agree that $M(\alpha)$ and $\mathfrak{M}(\alpha)$ represent respectively the Mahler measure and the absolute Mahler measure of the irreducible
polynomial of $\alpha$ with coefficients in $\mathbb{Z}$. The *Zhang-Zagier height* or simply the *height* of $\alpha$, denoted by $h_1(\alpha)$, is then defined as $h_1(\alpha) = \mathfrak{m}(\alpha)\mathfrak{m}(1-\alpha)$. From results of Zhang and of Zagier (cf. [10], [9]), we know that if $\alpha$ is an algebraic number different from the roots of $(z^2 - z)(z^2 - z + 1)$,

\begin{equation}
    h_1(\alpha) \geq \sqrt{\frac{1 + \sqrt{5}}{2}} = 1.2720196 \ldots
\end{equation}

In fact Zagier gives also the algebraic numbers which lead to an equality in (3). They are exactly the roots of $\Phi_{10}(z)\Phi_{10}(1-z)$ where $\Phi_{10}(z)$ represents the 10th cyclotomic polynomial.

Our contribution to the study of the Zhang-Zagier height consisted firstly of an improvement of (3), namely

**Theorem A.** Let $\alpha$ be an algebraic number. We suppose that $\alpha$ is different from the roots of $(z^2 - z)(z^2 - z + 1)\Phi_{10}(z)\Phi_{10}(1-z)$. Then

\[ h_1(\alpha) \geq 1.2817770214. \]

Secondly, we proved that $\mathcal{V} = \{h_1(\alpha) \mid \alpha \in \overline{\mathbb{Q}}\}$ admitted a limit point less than 1.2916674. Finally, we discovered an algebraic integer whose height, 1.2875274..., is less than the smallest previously known i.e. 1.2903349... [4]. All this work is described in [2].

From this, we are able to do a little better with the help of a new algorithm. In fact the main result of this paper is the following.

**Theorem.** The smallest limit point of $\mathcal{V} = \{h_1(\alpha) \mid \alpha \in \overline{\mathbb{Q}}\}$ is less than 1.289735.

Before we prove this theorem, we display some strange relations.

Let

\[
\begin{align*}
A_1(z) &= z, \\
A_2(z) &= z^2 - z + 1, \\
A_3(z) &= \Phi_{10}(z) = z^4 - z^3 + z^2 - z + 1, \\
A_4(z) &= z^8 - 3z^7 + 4z^6 - 2z^5 + x^4 - 2z^3 + 3z^2 - 2z + 1, \\
A_5(z) &= z^{16} - 7z^{15} + 23z^{14} - 45z^{13} + 57z^{12} - 46z^{11} + 19z^{10} \\
&\quad + 5z^9 - 9z^8 - 10z^7 + 39z^6 - 56z^5 + 52z^4 - 34z^3 + 16z^2 - 5z + 1.
\end{align*}
\]

These polynomials are remarkable according to their height. Indeed $A_1$ and $A_2$ have a trivial height, $h_1(A_3)$ is the second point of the spectrum, $A_4$ has the smallest height for the degree 8 and $h_1(A_5) = 1.2875274\ldots$ is
the smallest known height greater than $1.2720196 \ldots$ We verify that

\begin{align*}
A_2(z) &= -A_1(z)A_1(1-z) + 1, \\
A_3(z) &= A_2^2(z) + A_1(z)A_2^2(1-z), \\
A_4(z) &= A_3(z)A_3(1-z) + A_1(z)A_1^3(1-z)A_2^2(z), \\
A_5(z) &= A_4(z)A_4(1-z) + A_1(z)A_1^3(1-z)A_2^2(z)A_3(z)A_3(1-z).
\end{align*}

So we make the hypothesis that small heights come from “perturbed” polynomials of small height.

The phenomenon seems quite general since it occurs also for the Mahler measure \cite{6} and for the spectrum of $M(\alpha)M(1/(1-\alpha))M(1-1/\alpha)$ \cite{3}. Therefore it is quite reasonable to test polynomials $A_6(z)$ such that

\begin{align*}
A_6(z) &= A_5(z)A_5(1-z) \\
&\pm A_1^{a_1}(z)A_1^{a_2}(1-z)A_2^{a_3}(z)A_3^{a_4}(z)A_5^{a_5}(1-z)A_4(z)^{a_6}A_4^{a_7}(1-z)
\end{align*}

with the $a_i$'s in $\mathbb{N}$ verifying $a_1 + a_2 + 2a_3 + 4(a_4 + a_5) + 8(a_6 + a_7) \leq 31$. The results are quite disappointing since we get only a handful of good polynomials; the best corresponding to the choice

\begin{align*}
A_6(z) &= A_5(z)A_5(1-z) - A_1^5(z)A_1^5(1-z)A_2^5(z)A_3(z)A_3(1-z)
\end{align*}

having $\delta(A_6) = 1.2906235 \ldots$ We also notice that $A_6(z)$ is invariant under the map $z \mapsto 1-z$. Now, every polynomial of even degree symmetric under $z \mapsto 1-z$ can be expressed in terms of

\begin{align*}
X = z(1-z).
\end{align*}

For instance $\Phi_1(z)\Phi_1(1-z) = X^4 - 2X^3 + 4X^2 - 3X + 1$.

After many unfruitful computations we decided to consider only polynomials in the new variable $X$ and no longer in $z$. As a consequence we can increase the number of perturbing factors. So we put

\begin{align*}
P_1(X) &= X, \\
P_2(X) &= 1-X, \\
P_3(X) &= X^3 + X^2 - 2X + 1, \\
P_4(X) &= X^4 - 2X^3 + 4X^2 - 3X + 1,
\end{align*}
Then we choose a starting polynomial \( P(X) \) of degree \( d \), we take integers and ask simply that

\[
9
\]

At this point, for each combination of verifying (4) we estimate by the method of Graef Re [1]. If this evaluation is rather small we compute precisely its Zhang-Zagier height. Before we did this search, we knew only 9 polynomials whose height is less than 1.29. Now, just for the degree 28 we have more than 120 polynomials with a height less than 1.29. Figures 1 and 2 help us to see the gap we filled.

![Figure 1. Known heights previously.](image1)

![Figure 2. Known heights at present.](image2)

and \( d_i = \deg P_i \). Note that \( P_1(X) = A_1(z)A_1(1-z) \), \( P_2(X) = A_2(z) \), \( P_4(X) = A_3(z)A_3(1-z) \), \( P_5(X) = A_4(z)A_4(1-z) \), \( P_9(X) = A_5(z)A_5(1-z) \).

Then we choose a starting polynomial \( P(X) \) of degree \( d \), we take integers \( (a_i)_{1 \leq i \leq 9} \) and ask simply that

\[
9 \sum_{i=1}^{9} a_id_i \leq d - 1.
\]

At this point, for each combination of \((a_i)_{1 \leq i \leq 9}\) verifying (4) we estimate

\[
\delta \left( P(z(1-z)) \pm P_1A_1P_2A_2P_3A_3P_4A_4P_5A_5P_6A_6P_7A_7P_8A_8P_9A_9(z(1-z)) \right)
\]

by the method of Graeffe [1]. If this evaluation is rather small we compute precisely its Zhang-Zagier height. Before we did this search, we knew only 9 polynomials whose height is less than 1.29. Now, just for the degree 28 we have more than 120 polynomials with a height less than 1.29. Figures 1 and 2 help us to see the gap we filled.

\[
\begin{array}{ccccccc}
1.287 & 1.288 & 1.289 & 1.29 & 1.291 & 1.292 \\
\end{array}
\]

Figure 1. Known heights previously.

\[
\begin{array}{ccccccc}
1.287 & 1.288 & 1.289 & 1.29 & 1.291 & 1.292 \\
\end{array}
\]

Figure 2. Known heights at present.
Table 1 shows the improvements made, if any, for each degree.

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<th>New record</th>
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</tr>
<tr>
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<td>1.2916397...</td>
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<td>12</td>
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<tr>
<td>17</td>
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<tr>
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<td>—</td>
</tr>
</tbody>
</table>

Table 1.

It is hard to explain precisely why this algorithm gives good results. Nevertheless here is a kind of heuristic argument. In [2], we observed the importance of the resultant for the Zhang-Zagier height. Namely, a polynomial \( P \) with \( f_j(P) \) small usually has a small resultant with each of the \( P_i \). Now it is obvious that

\[
\text{Res} \left( P \pm P_1^{a_1} P_2^{a_2} P_3^{a_3} P_4^{a_4} P_5^{a_5} P_6^{a_6} P_7^{a_7} P_8^{a_8} P_9^{a_9}, P_i \right) = \text{Res}(P, P_i)
\]

for any \( P_i \). So \( P \pm P_1^{a_1} P_2^{a_2} P_3^{a_3} P_4^{a_4} P_5^{a_5} P_6^{a_6} P_7^{a_7} P_8^{a_8} P_9^{a_9} \) are good candidates. These results convince us of the existence of a limit point of \( \mathcal{V} \) less than 1.29 which leads us to the Theorem. To prove it, we use the same techniques as in [2].
2. Proof of the Theorem

With the above notations, we shall use $P_1(X), P_2(X), P_4(X), P_6(X)$ and $P_8(X)$. We introduce also $Q(X) = Q_1(X)Q_2(X)$ with

$$Q_1(X) = X^{28} - 7X^{27} + 30X^{26} - 97X^{25} + 269X^{24} - 679X^{23} + 1612X^{22} - 3618X^{21} + 7646X^{20} - 15180X^{19} + 28457X^{18} - 50741X^{17} + 86189X^{16} - 138288X^{15} + 206152X^{14} - 279897X^{13} + 339335X^{12} - 360911X^{11} + 331775X^{10} - 260367X^9 + 172556X^8 - 95554X^7 + 43677X^6 - 16221X^5 + 4786X^4 - 1084X^3 + 178X^2 - 19X + 1,$$

and

$$Q_2(X) = X^{28} - 7X^{27} + 30X^{26} - 96X^{25} + 255X^{24} - 586X^{23} + 1212X^{22} - 2360X^{21} + 4573X^{20} - 9148X^{19} + 18749X^{18} - 37783X^{17} + 71770X^{16} - 124910X^{15} + 195848X^{14} - 273368X^{13} + 335981X^{12} - 359545X^{11} + 331349X^{10} - 260271X^9 + 172542X^8 - 95553X^7 + 43677X^6 - 16221X^5 + 4786X^4 - 1084X^3 + 178X^2 - 19X + 1.$$

We verify that

$$J(P_1) = J(P_2) = 1, \quad J(P_4) = 1.272019650\ldots, \quad J(P_6) = 1.297431163\ldots, \quad J(P_8) = 1.289442541\ldots,$$

$$J(Q_1) = 1.288275954\ldots, \quad J(Q_2) = 1.288646007\ldots.$$

Finally, we need the following lemma proved in [2].

**Lemma.** Let $P$ be a polynomial in two variables $y$ and $z$, such that $\deg_z P > 0$. Let $\zeta_n$ be $\frac{\ln n}{n}$ and assume that for all $n$ and all $k$, $P(\zeta_n^k, z)$ is not identically zero. We then have

$$M(P(y, z)) = \lim_{n \to \infty} M\left(\prod_{k=1}^{n} P(\zeta_n^k, z)\right).$$

At present let $(q_1, q_2, \ldots, q_5) \in \mathbb{Q}_+^5$ and $b$ a denominator of the $q_i$'s. Then it is clear that

$$J\left(\left(\prod_{i=1}^{6} P_i^{bq_i} P_2^{bq_2} P_4^{bq_4} P_6^{bq_6} P_8^{bq_8}(z(1-z))\right)^n - \left(Q^b(z(1-z))\right)^n\right)$$

gives rise to a limit point of $\{J(P) \mid P \in \mathbb{Z}[z]\}$ when $n$ tends to infinity. Factorizing the polynomial over $\mathbb{C}[z]$, we see that the limit of (5) when $n$
tends to infinity is equal to
\[
\lim_{n \to \infty} \mathcal{M} \left( \prod_{k=1}^{n} \left( P_1^{b_{q_1}} P_2^{b_{q_2}} P_4^{b_{q_3}} P_6^{b_{q_4}} P_8^{b_{q_5}} (z(1 - z)) - \zeta_n^k Q^b (z(1 - z)) \right) \right)^2.
\]

As the degree in $z$ of the polynomial in the product of (6) is
\[
D(b) = 2b \max(56, q_1 + 2q_2 + 4q_3 + 8q_4 + 8q_5),
\]
the Lemma asserts that (6) equals
\[
M \left( P_1^{b_{q_1}} P_2^{b_{q_2}} P_4^{b_{q_3}} P_6^{b_{q_4}} P_8^{b_{q_5}} (z(1 - z)) - yQ^b (z(1 - z)) \right)^{2/D(b)}.
\]

Now recall (1) and put $\chi(s) = e^{2\pi t}(1 - e^{2\pi t})$ so that (7) is
\[
\left[ \exp \left\{ \int_0^1 \int_0^1 \log \left| \left( P_1^{q_1} P_2^{q_2} P_4^{q_3} P_6^{q_4} P_8^{q_5} (\chi(s)) \right) \right| \right| \right| dsdt \right]^{2/D(b)}
\]
and finally by $b$ changes of variable $t \mapsto t + j/b$, for $j \in \{1, b\}$, we get
\[
\left[ \exp \left\{ \int_0^1 \int_0^1 \log \left| \left( P_1^{q_1} P_2^{q_2} P_4^{q_3} P_6^{q_4} P_8^{q_5} (\chi(s)) \right) \right| \right| \right| dsdt \right]^{2b/D(b)}
\]
The trick of considering $z$ as a simple parameter together with formula (2) imply that (8) is nothing but
\[
\left[ M \left( Q(z(1 - z)) \right) \times \exp \left\{ \int_0^1 \log^+ \left| \frac{P_1^{q_1} P_2^{q_2} P_4^{q_3} P_6^{q_4} P_8^{q_5} (\chi(s))}{Q(\chi(s))} \right| ds \right\} \right]^2/B(D(b))
\]
where $\log^+(w)$ is $\max(0, \log |w|)$ as usual. Then we search the best $q_i$'s in order to have the smallest limit point possible. As we can easily compute the simple integral of (9) by a Riemann sum, we test several choices of $q_i$'s. Thanks to these computations, we produce a limit point less than 1.29 thus better than the previously known $i.e.$ 1.2916674. The best combination found is
\[
q_1 = 13.1, q_2 = 10.6, q_3 = 3.2, q_4 = 1.15, q_5 = 0.24
\]
which yields a limit point less than $\ell = 1.289735$. Of course this limit point $\ell$ concerns only polynomials but fortunately we can deduce from them a subsequence of irreducible polynomials with a height less than $\ell$ (see [2, Lemmas 3, 4, 5]).
Let us say a few words about the polynomials involved in (9). While the selection of $P_1, P_2, P_4, P_5, P_8$ among $P_1, \ldots, P_8$ is forced by the search of the minimum of (9), the choice of the polynomials to be perturbed is more arbitrary. In fact, we take $Q_1(X)$ and $Q_2(X)$ according to their very small height and also because we remarked that a product of $P_i$'s, namely $P_1^7(X)P_2^5(X)P_4(X)$, divides $Q_1(X) - Q_2(X)$. This point seems determinant, however we do not understand why it is so important.

The results confirm that our previous algorithm was far from being exhaustive since a great number of heights were missed. Nevertheless, no new height less than 1.2875274 appears, so that we still do not know the second non-trivial point, if any, of $V$. It would be desirable to improve our knowledge of the set of $f_1(\alpha)$ as it was done for the Mahler measure of non-reciprocal polynomials [7] and for the height of totally real algebraic integers [8] and [5], but we have the feeling that this will be difficult without new ideas.

References


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