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Congruences modulo ℓ between ε factors for cuspidal representations of GL(2)


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Congruences modulo ℓ between ε factors for cuspidal representations of $GL(2)$

par MARIE-FRANCE VIGNÉRAS

Pour Jacques Martinet

RÉSUMÉ. Titre français : Congruences modulo ℓ entre facteurs ε des représentations cuspidales de $GL(2)$
Soient ℓ ≠ p deux nombres premiers distincts, F un corps local non archimédien de caractéristique résiduelle p, $Q_ℓ$, une clôture algébrique du corps des nombres ℓ-adiques, et $F_ℓ$ le corps résiduel de $Q_ℓ$.
On conjecture que la correspondance locale de Langlands pour $GL(n, F)$ sur $Q_ℓ$ respecte les congruences modulo ℓ entre les facteurs $L$ et ε de paires, et que la correspondance locale de Langlands sur $F_ℓ$ est caractérisée par des identités entre de nouveaux facteurs $L$ et ε. Nous allons le démontrer lorsque $n = 2$.

ABSTRACT. Let ℓ ≠ p be two different prime numbers, let F be a local non archimedean field of residual characteristic p, and let $Q_ℓ$, $Z_ℓ$, $F_ℓ$ be an algebraic closure of the field of ℓ-adic numbers $Q_ℓ$, the ring of integers of $Q_ℓ$, the residual field of $Z_ℓ$.
We proved the existence and the unicity of a Langlands local correspondence over $F_ℓ$ for all $n ≥ 2$, compatible with the reduction modulo ℓ in [V5], without using $L$ and ε factors of pairs.
We conjecture that the Langlands local correspondence over $Q_ℓ$ respects congruences modulo ℓ between $L$ and ε factors of pairs, and that the Langlands local correspondence over $F_ℓ$ is characterized by identities between new $L$ and ε factors. The aim of this short paper is prove this when $n = 2$.

Introduction

The Langlands local correspondence is the unique bijection between all irreductible $Q_ℓ$-representations of $GL(n, F)$ and certain ℓ-adic representations of an absolute Weil group $W_F$ of dimension $n$, for all integers $n ≥ 1$,
which is induced by the reciprocity law of local class field theory

\[ W^{ab}_F \simeq F^* \]

when \( n = 1 \) (\( W^{ab}_F \) is the biggest abelian Hausdorff quotient of \( W_F \)), and which respects \( L \) and \( \varepsilon \) factors of pairs [LRS], [HT], [H2].

Let \( \psi : F \to \mathbb{Z}_\ell \) be a non trivial character. We denote by \( \text{Cusp}_R \text{GL}(n, F) \) the set of isomorphism classes of irreducible cuspidal \( R \)-representations of \( \text{GL}(n, F) \). When \( \pi \in \text{Cusp}_{\mathbb{Q}_\ell} \text{GL}(n, F) \), Henniart [H1] showed that \( \pi \) is characterized by the epsilon factors of pairs \( \varepsilon(\pi, \sigma) \) for all \( \sigma \in \text{Cusp}_{\mathbb{Q}_\ell} \text{GL}(m, F) \) and for all \( m \leq n - 1 \) (note that \( L(\pi, \sigma) = 1 \)), using the theory of Jacquet, Piatetski-Shapiro, and Shalika [JPS1].

Does this remain true for cuspidal irreducible \( \overline{F}_\ell \)-representations of \( \text{GL}(n, F) \)? We need first to define the epsilon factors of pairs.

Let \( \pi \in \text{Cusp}_{\overline{Q}_\ell} \text{GL}(n, F) \). It is known that the constants of the epsilon factors of pairs \( \varepsilon(\pi, \sigma) \) belong to \( \mathbb{Z}_\ell \) for all \( \sigma \in \text{Cusp}_{\overline{Q}_\ell} \text{GL}(m, F) \) and for all \( m \leq n - 1 \), and that the conductor does not change by reduction modulo \( \ell \) (this is proved by Deligne [D] for the irreducible representations of the Weil group, and by the local Langlands correspondence over \( \mathbb{Q}_\ell \) is true for cuspidal representations).

Now let \( \pi \in \text{Cusp}_{\overline{F}_\ell} \text{GL}(n, F) \). Then \( \pi \) lifts to \( \text{Cusp}_{\overline{Q}_\ell} \text{GL}(n, F) \) [V1, III.5.10]. By reduction modulo \( \ell \), one can define epsilon factors of pairs \( \varepsilon(\pi, \sigma) \) for all \( \sigma \in \text{Cusp}_{\overline{F}_\ell} \text{GL}(m, F) \) and for all \( m \leq n - 1 \). Let \( q \) be the order of the residual field of \( F \). We expect that \( \pi \) is characterized by the epsilon factors \( \varepsilon(\pi, \sigma) \) for all \( \sigma \), when the multiplicative order of \( q \) modulo \( \ell \) is \( > n - 1 \); otherwise, \( \pi \) should be characterized by less naive but natural epsilon factors. The same should be true when \( \pi \) is replaced by an \( \overline{F}_\ell \)-irreducible representation of the Weil group \( W_F \).

The existence [V4] of an integral Kirillov model for \( \pi \in \text{Cusp}_{\overline{Q}_\ell} \text{GL}(n, F) \) seems to be an adequate tool to solve the problem. The description of the representation \( \pi \) on the Kirillov model is given by the central character \( \omega_\pi \) and by the action of the symmetric group \( S_n \) (the Weyl group of \( GL(n, F) \)). The action of \( S_n \) is related with the \( \varepsilon(\pi, \sigma) \) for all \( \sigma \) as above [GK, see the end of paragraph 7]. When \( n = 2 \) Jacquet and Langlands [JL] described the action of \( S_2 \) on the Kirillov model in terms of \( \varepsilon(\pi, \chi) = \varepsilon(\pi \otimes \chi) \) for all \( \overline{Q}_\ell \)-characters \( \chi \) of \( F^* \), using the Fourier transform on \( F^* \).

In the case \( n = 2 \) and only in this case, we will prove that two integral \( \pi, \pi' \in \text{Cusp}_{\overline{Q}_\ell} \text{GL}(2, F) \) have the same reduction modulo \( \ell \) if and only if their central characters have the same reduction modulo \( \ell \) and the factors \( \varepsilon(\pi \otimes \chi), \varepsilon(\pi' \otimes \chi) \) have the same reduction modulo \( \ell \) for integral \( \overline{Q}_\ell \)-characters \( \chi \) of \( F^* \) when \( \ell \) does not divide \( q - 1 \). When \( \ell \) divides \( q - 1 \) this remains true with new epsilon factors taking into account the natural
congruences modulo \( \ell \) satisfied by the \( \varepsilon(\pi \otimes \chi) \) for all \( \chi \). By reduction modulo \( \ell \), we get that the local Langlands \( \mathbf{F}_\ell \)-correspondence for \( n = 2 \) is characterized by the equality on \( L \) and new \( \varepsilon \) factors of pairs. The field \( \mathbf{F}_\ell \) can be replaced by any algebraically closed field \( R \) of characteristic \( \ell \).

The case \( n = 3 \) could be treated probably, but the general case \( n \geq 4 \) remains an open and interesting question.

1. Integral Kirillov model

The definition of the \( L \) and \( \varepsilon \) factors of pairs [JPS1] uses the Whittaker model, or what is equivalent the Kirillov model. We showed [V4] that these models are compatible with the reduction modulo \( \ell \).

We denote by \( O_F \) the ring of integers of \( F \). Let \( R \) be an algebraically closed field of characteristic \( \neq p \), and let \( \psi : F \to R^* \) be a character such that \( O_F \) is the biggest ideal on which \( \psi \) is trivial. We extend \( \psi \) to a \( R \)-character of the group \( N \) of strictly upper triangular matrices of \( G = GL(n, F) \) by \( \psi(n) = \psi(\sum n_{i,i+1}) \) for \( n = (n_{i,j}) \in N \). The mirabolic subgroup \( P \) of \( G \) is the semi-direct product of the group \( GL(n-1, F) \) embedded in \( GL(n, F) \) by

\[
g \to \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}
\]

and of the group \( F^{n-1} \) embedded in \( GL(n, F) \) by

\[
x \to \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.
\]

The representation \( \tau_R := \text{ind}_{P,N} \psi \) of the mirabolic subgroup \( P \) (compact induction) is called mirabolic. It is irreducible (this is a corollary of [V4 prop.1]), but it is not admissible when \( n \geq 2 \).

**Lemma.** \( \text{End}_{RP} \tau_R \simeq R \).

**Proof.** This is a general fact: the representation \( \tau_R \) is absolutely irreducible [V1, I.6.10], hence \( \text{End}_{RP} \tau_R \simeq R \). From the Schur’s lemma [V1, I.6.9] \( \text{End}_{RP} \tau_R \simeq R \) when the cardinal of \( R \) is strictly bigger than \( \dim_R \tau_R \) (countable dimension). There exists an algebraically closed field \( R' \) which contains \( R \) and of uncountable cardinal. Two \( RP \)-endomorphisms of \( \tau_R \) which are proportional over \( R' \) are proportional over \( R \). \( \square \)

**Theorem.** An irreducible \( R \)-representation \( \pi \) of \( G \) is cuspidal if and only if extends the mirabolic representation \( \tau_R \).

**Proof.** This results from [BZ] and [V1]. Suppose that \( \pi \) is cuspidal. Then \( \pi|_P \) is the mirabolic representation: when \( R = \overline{Q}_\ell \simeq C \) see [BZ, 5.13 & 5.20], when \( R = \mathbf{F}_\ell \), \( \pi \) lifts to \( \overline{Q}_\ell \) [V1, III.5.10] where it is true then reduce. Conversely, suppose \( \pi|_P = \tau_R \) and \( R = \overline{Q}_\ell \) or \( \mathbf{F}_\ell \). Then \( \pi \) is cuspidal [V1,
The case of a general $R$ is deduced from this two cases by the next lemma.

Let $G$ be the group of rational points of a reductive connected group over $F$. We denote by $\text{Irr}_R G$ the set of isomorphism classes of irreducible $R$-representations of $G$.

**Lemma.** (1) A non zero homomorphism of algebraically closed fields $f : R \to R'$ gives a natural injective map $\pi \mapsto f_*(\pi) : \text{Irr}_R G \to \text{Irr}_{R'} G$ which respects cuspidality.

(2) Let $\pi' \in \text{Cusp}_{R'} G$. Then there exists an unramified character $\chi$ of $G$ such that $\pi' \otimes \chi = f_*(\pi)$ with $\pi \in \text{Cusp}_R G$.

**Proof.** This results from [VI].

(1) $f_*$ respects irreducibility [VI, II.4.5], and commutes with the parabolic restriction. Hence it respects cuspidality. The linear independence of characters [VI, I.6.13] shows that if $\pi, \pi' \in \text{Irr}_R G$ are not isomorphic then $f_* \pi, f_* \pi'$ are not isomorphic.

(2) Let $Z$ be the center of $G$. The group of rational characters $X(Z)$ is a subgroup of finite index in the group $X(G)$. This implies that there exists an unramified character $\chi$ of $G$ such that the quotient $Z/Z_0$ of $Z$ by the kernel $Z_0$ of the central character $\omega$ of $\pi' \otimes \chi$ is profinite. Hence the values of $\omega$ are roots of unity. We deduce that $\pi' \otimes \chi$ has a model on $R$ [VI, II.4.9].

Let $\pi \in \text{Cusp}_R GL(n, F)$ of central character $\omega$. The realisation of $\pi$ on the mirabolic representation $\tau_R$ is called the Kirillov model $K(\pi)$ of $\pi$. It is sometimes useful to use the Whittaker model instead of the Kirillov model. By adjonction and the theorem $\text{Hom}_{RG}(\pi, \text{Ind}_{G,N} \psi) \simeq R$ (the unicity of the Whittaker model); the Whittaker model $W(\pi)$ is the unique realisation of $\pi$ in $\text{Ind}_{G,N} \psi$. By definition

$$W(g) = (\pi(g)W)(1)$$

for all $g \in G$ and for all Whittaker functions $W \in W(\pi)$. We denote by $\Gamma(j)$ the subgroup of matrices $k \in GL(n, O_F)$ of the form

$$k = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a \in GL(n-1, O_F), d \in O_F^*, c \in T^*, O_F$$

for any integer $j > 0$. The smallest $j > 0$ such that $\pi$ contains a non-zero vector transforming under $\Gamma(j)$ according to the one dimensional character $\omega_j(k) = \omega(d)$

for $k \in \Gamma(j)$ as above, is called the conductor of $\pi$ and denoted $f$. 

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III.1.8. The case of a general $R$ is deduced from this two cases by the next lemma.
Theorem. Let $\pi \in \text{Cusp}_R \text{GL}(n,F)$ of central character $\omega = \omega$ and conductor $f$.

(1) The restriction from $G$ to $P$ induces a $G$-equivariant isomorphism

$$W \rightarrow W|_P : W(\pi) \simeq K(\pi)$$

from the Whittaker model to the Kirillov model.

(2) Let $\pi' \in \text{Cusp}_R \text{GL}(n,F)$. There is a natural isomorphism $W \rightarrow W'$ of $R$-vector spaces defined by the condition $W|_P = W'|_P$.

(3) There is unique function $W_\pi \in W(\pi)$ such that

$$W_\pi|_{\text{GL}(n-1,F)} = 1_{\text{GL}(n-1,O_F)}.$$ 

The function $W_\pi$ is called the new vector of $\pi$ and generates the space of vectors of $\pi$ transforming under $\Gamma(f)$ according to $\omega_f$.

(4) $W(\pi)$ is contained in the compactly induced representation $\text{Ind}_{G,NZ} \psi \otimes \omega_\pi$.

Proof. (1) There exists $W \in W(\pi)$ with $W(1) \neq 0$, and $f : W \rightarrow W_P$ is a non zero $P$-equivariant map from $\pi$ to $\text{Ind}_N^P \psi$. The map $f$ is injective of image $\text{ind}_N^P \psi$, because $\text{End}_R \tau_R \simeq R$. We get also (2).

(3) The space of $\tau_R$ is isomorphic by restriction to $G' = \text{GL}(n-1,F)$, to the space of $\text{ind}_{N',G'} \psi$ where $N' = N \cap G'$. As $\psi$ is trivial on $O_F$, the characteristic function of $\text{GL}(n-1,O_F)$ belongs to $\text{ind}_{N'}^{G'} \psi$. For the conductor [JPS2].

(4) Let $W \in W(\pi)$. The function $x \rightarrow W(xg)$ on the parabolic standard subgroup $PZ$ is locally constant of compact support modulo $NZ$ for all $g \in G$. As $G = PZ\text{GL}(n,O_F)$, the function $W$ is of compact support modulo $NZ$. 

Let $\pi \in \text{Irr}_Q \text{GL}(n,F)$, $E/O_\ell$ be an extension contained in a finite extension of the maximal unramified extension of $Q_\ell$. Example: the extension $E/Q_\ell$ generated by the values of $\psi$. The ring of integers $O_E$ is principal. An $O_E$-free module $L$ with an action of $G$ such that $L$ is a finite type $O_EG$-module and such that $\overline{Q_\ell} \otimes_{O_E} L \simeq \pi$ is called an $O_E$-integral structure of $\pi$. If such an $L$ exists, $\pi$ is called integral, the representation $r_\ell L = L \otimes_{O_E} \overline{F_\ell}$ is of finite length. One calls $\overline{Z_\ell} \otimes_{O_E} L$ an integral structure of $\pi$. When $L,L'$ are two integral structures of $\pi$, then the semi-simplifications of $r_\ell L, r_\ell L'$ are isomorphic (see [V1, II.5.11.b] when $E/Q_\ell$ is finite, and [Vig4, proof of theorem 2, page 416] in general). When $\pi \in \text{Cusp}_E \text{GL}(n,F)$ is integral, $r_\ell L = L \otimes_{O_E} \overline{F_\ell}$ is irreducible; the isomorphism class $r_\ell \pi$ of $r_\ell L$ is called the reduction of $\pi$; any irreducible cuspidal $\overline{F_\ell}$-representation of $G$ is the reduction of an integral irreducible cuspidal $Q_\ell$-representation of $G$. For all these facts see [V1, III.5.10].
A function with values in $\mathbf{Q}_\ell$ is called integral, when its values belong to $\mathbf{Z}_\ell$. We denote by $K(\pi, \mathbf{Z}_\ell)$, resp. $W(\pi, \mathbf{Z}_\ell)$, the set of integral functions in the Kirillov model, resp. Whittaker model, of $\pi \in \text{Cusp}_{\mathbf{Q}_\ell} G$. Let $\Lambda$ be the maximal ideal of $\mathbf{Z}_\ell$. The reduction modulo $\ell$ of an integral function $f$ is the function $r_\ell f$ with values in $\mathbf{Z}_\ell/\Lambda \simeq \mathbf{F}_\ell$ deduced from $f$.

**Theorem.** (A) Let $\pi \in \text{Cusp}_{\mathbf{Q}_\ell} G$ with central character $\omega_\pi$. Then the following properties are equivalent:

(A.1) $\omega_\pi$ is integral.

(A.2) $\pi$ is integral.

(A.3) $K(\pi, \mathbf{Z}_\ell)$ is a $\mathbf{Z}_\ell$-structure of $\pi$, called the integral Kirillov model.

(A.4) $W(\pi, \mathbf{Z}_\ell)$ is a $\mathbf{Z}_\ell$-structure of $\pi$, called the integral Whittaker model.

(B) When $\pi$ is integral, we have

(B.1) The restriction to $\mathfrak{p}$ from $W(\pi, \mathbf{Z}_\ell)$ to $K(\pi, \mathbf{Z}_\ell)$ is an isomorphism.

(B.2) The integral Kirillov model is $\mathbf{Z}_\ell \mathfrak{p}$-generated by any function $f$ with $f(1) = 1$. The integral Whittaker model $W(\pi, \mathbf{Z}_\ell)$ is $\mathbf{Z}_\ell \mathfrak{p}$ generated by the new vector.

(B.3) $F_\ell \otimes_{\mathbf{Z}_\ell} K(\pi, \mathbf{Z}_\ell) = K_r(\pi, \mathbf{F}_\ell)$ is the Kirillov model, and $F_\ell \otimes_{\mathbf{Z}_\ell} W(\pi, \mathbf{Z}_\ell) = W_r(\pi, \mathbf{F}_\ell)$ is the Whittaker model of $r_\ell \pi$.

**Proof.** The equivalence of (A1) (A2) [V1, II.4.12]; for the rest [V4 th.2] and the last theorem. \qed

**Corollary.** Let $\pi, \pi' \in \text{Cusp}_{\mathbf{Q}_\ell} G$ integral, with central character $\omega_\pi, \omega_{\pi'}$. Then $r_\ell \pi = r_\ell \pi'$ if and only if

\[ r_\ell \omega_\pi = r_\ell \omega_{\pi'}, \quad r_\ell \pi(w)(f) = r_\ell \pi'(w)(f) \]

for all $w \in S_n$, and for all $f$ in the integral Kirillov model.

**Proof.** Use (B.3) and End$_{F_\ell} F_\ell \simeq F_\ell$. \qed

**Questions.** Can one define an integral Kirillov or Whittaker model for $\pi \in \text{Irr}_{\mathbf{Q}_\ell} G$ integral and not cuspidal? What is the action of $S_n$ in the Kirillov model?

2. The case $n = 2$

We can go further in the case $n = 2$. Let $\pi \in \text{Cusp}_{\mathbf{Q}_\ell} G$ where $G = GL(2, F)$. The restriction of $GL(2, F)$ to $GL(1, F) = F^*$ gives an isomorphism from $K(\pi)$ to the space $C_c^\infty(F^*, \mathbf{Q}_\ell)$ of locally constant functions $F^* \to \mathbf{Q}_\ell$ with compact support, which respects the natural $\mathbf{Z}_\ell$-structures $K(\pi, \mathbf{Z}_\ell) \simeq C_c^\infty(F^*, \mathbf{Z}_\ell)$. The unique non trivial element of $S_2$ is represented by

\[ w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \]
The action of $\pi(w)$ on the Kirillov model was described by Jacquet and Langlands [JL, Prop. 2.10 p. 46], using Fourier transform for complex representations.

We choose a $\mathbb{Q}_\ell$-Haar measure $dx$ on $F^*$. The Fourier transform of $f \in C_c^\infty(F^*, \overline{Q}_\ell)$ with respect to $dx$ is

$$\hat{f}(\chi) := \int_{F^*} f(x) \chi(x) dx$$

for any character $\chi : F^* \to \overline{Q}_\ell$.

We choose a uniformizing parameter $p_F$ of $F$. A function $f \in C_c^\infty(F^*, \overline{Q}_\ell)$ is determined by the set of functions $f_n \in C_c^\infty(O_F^*, \overline{Q}_\ell)$ defined by $f_n(x) := f(p_F^{-n}x)$ for all $n \in \mathbb{Z}$. The functions $f_n$ depend on the choice of $p_F$. Extension by zero allows to consider $C_c^\infty(O_F^*, \overline{Q}_\ell)$ as a subspace of $C_c^\infty(F^*, \overline{Q}_\ell)$, because $O_F^*$ is open in $F^*$. We have

$$\hat{f}(\chi) = \sum_n \hat{f}_n(\chi)(p_F^{-n}).$$

For a given character $\chi$, the sum is finite. The functions $\hat{f}_n(\chi)$ depend only on the restriction of $\chi$ to $O_F^*$. Set $\hat{O}_F^* := \text{Hom}(O_F^*, \overline{Q}_\ell)$. One introduces the formal series

$$f(x, X) := \sum_{n \in \mathbb{Z}} f_n(x) X^n, \quad \hat{f}(\chi, X) := \sum_{n \in \mathbb{Z}} \hat{f}_n(\chi) X^n$$

for all $x \in O_F^*$ and for all $\chi \in \hat{O}_F^*$.

Jacquet and Langlands [JL Prop. 2.10 page 46] proved that the action of $\pi(w)$ on the Kirillov model is given by:

$$(\pi(w)f)_n(\chi) = c(\pi \otimes \chi^{-1}) \hat{f}_m(\chi^{-1} \omega_\pi^{-1})$$

for all $\chi \in \hat{O}_F^*$, all integers $n \in \mathbb{Z}$, where $m = -n - f(\pi \otimes \chi^{-1})$, for some constant $c(?) \in \overline{Q}_\ell$ and some integer $f(?) \in \mathbb{Z}$. The formula and $c(\pi \otimes \chi^{-1})$ are independent of the choice of $dx$. The formula is equivalent to

$$(\pi(w)f)(\chi, X) = \epsilon(\pi \otimes \chi^{-1}) \hat{f}(\chi^{-1} \omega_\pi^{-1}, X^{-1})$$

for all $\overline{Q}_\ell$-characters $\chi$ of $O_F^*$, where the epsilon factor is

$$\epsilon(\pi \otimes \chi^{-1}) = c(\pi \otimes \chi^{-1}) X^{f(\pi \otimes \chi^{-1})}.$$
We suppose that \( dx \) is a \( \mathbb{Z}_\ell \)-Haar measure on \( F^* \) which is not divisible by \( \ell \). Let
\[
\mathcal{L} = \text{the Fourier transform of } C^\infty_c(O^*_F, \mathbb{Z}_\ell).
\]
We have \( \mathcal{L} \subset C^\infty_c(\hat{O}^*_F, \mathbb{Z}_\ell) \) and \( \mathcal{L} = C^\infty_c(\hat{O}^*_F, \overline{\mathbb{Q}}_\ell) \) if and only if \( q \neq 1 \mod \ell \) [V2]. In general, we separate the \( \ell \)-regular part \( X \) of \( O^*_F \) from the \( \ell \)-part \( Y \) of \( O^*_F \), which is a cyclic group of order \( m = \ell^a \). The volume of \( X \) for \( dx \) should be a unit in \( \mathbb{Z}_\ell^* \); we can suppose it is equal to 1. The group of \( \mathbb{Q}_\ell \)-characters satisfy \( \hat{O}^*_F \simeq \hat{X} \times \hat{Y} \). A general character in \( \hat{O}^*_F \) is now written as \( \chi \mu \) where \( \chi \in \hat{X} \) and \( \mu \in \hat{Y} \), and a function \( v : \hat{O}^*_F \to \overline{\mathbb{Q}}_\ell \) is thought as a function \( v : \hat{X} \to C(\hat{Y}, \overline{\mathbb{Q}}_\ell) \) with \( v(\chi)(\mu) := v(\chi \mu) \).

The \( \mathbb{Z}_\ell \)-module \( \mathcal{L} \) consists of all functions \( v : \hat{X} \to L \) with compact support, where
\[
L \subset C^\infty_c(\hat{Y}, \mathbb{Z}_\ell)
\]
is the free \( \mathbb{Z}_\ell \)-module with basis the characters \( y : \mu \to \mu(y^{-1}) \) of \( \hat{Y} \) for all \( y \in Y \).

We need some elementary linear algebra. The \( \mathbb{Z}_\ell \)-module \( L \) is the set of functions \( v \in C^\infty_c(\hat{Y}, \overline{\mathbb{Q}}_\ell) \) such that
\[
y \mapsto <v, y> := |Y|^{-1} \sum_{\mu \in \hat{Y}} v(\mu)\mu(y)
\]
belongs to \( C(Y, \mathbb{Z}_\ell) \). The orthogonality formula of characters gives
\[
v = \sum_{y \in Y} <v, y> y
\]
for all \( v \in C(\hat{Y}, \overline{\mathbb{Q}}_\ell) \). For the usual product, \( C^\infty_c(\hat{Y}, \overline{\mathbb{Q}}_\ell) \) is an algebra.

**Lemma.** Let \( v \in C^\infty_c(\hat{Y}, \overline{\mathbb{Q}}_\ell) \).

(i) The inclusion \( vL \subset L \) is equivalent to \( v \in L \).

(ii) The equality \( vL = L \) is equivalent to \( v \in L \) and \( v(\mu) \in \mathbb{Z}_\ell^* \) for all \( \mu \in \hat{Y} \).

(iii) The inclusion \( vL \subset \Lambda L \) is equivalent to \( <v, y> \in \Lambda \) for all \( y \in Y \)
(\( \Lambda \) is the maximal ideal of \( \mathbb{Z}_\ell \)).

**Proof.** (i) The inclusion \( vL \subset L \) is equivalent to \( <v_z, z'> = <v, z^{-1}z'> \in \mathbb{Z}_\ell \) for all \( z, z' \in Y \), which is equivalent to \( v \in L \).

(ii) \( vL = L \) means that \( v_z \) for \( z \in Y \) is a basis of \( L \). We have \( v_z = \sum_{z' \in Y} <v, z^{-1}z'> z' \), hence \( vL = L \) means that
\[
<v, z^{-1}z'>_{z, z'} \in SL(m, \mathbb{Z}_\ell).
\]
The Dedekind determinant \( \det(\langle v, z^{-1}z' \rangle_{z, z'}) \) is equal to \( \prod_{\mu \in \hat{Y}} v(\mu) \) (see [L] exercise 28 page 495).

(iii) see the proof of (i).
Let $\pi \in \text{Cusp}_{Q^*} G$ integral. As $\pi(w)$ is an isomorphism of the integral Kirillov model, the function 

$$c(\pi \otimes \chi) : \mu \in \hat{Y} \to c(\pi \otimes \chi \mu) \in Q^*_\ell$$

satisfies $c(\pi \otimes \chi)L = L$ for all character $\chi \in \hat{X}$. We apply the lemma to $c(\pi \otimes \chi)$. We define new epsilon factors 

$$\varepsilon(\pi, y) := <c(\pi), y \chi > Xf(\pi), <c(\pi), y \chi > = |Y|^{-1} \sum_{\mu \in \hat{Y}} c(\pi \otimes \mu) \mu(y),$$

for all $y \in Y$. As have $f(\pi) \geq 2$ for $\pi \in \text{Cusp}_{Q^*} G$, we have $f(\pi) = f(\pi \otimes \mu) \geq 2$ for all $\mu \in \hat{Y}$. When $Y$ is trivial (i.e. $q \equiv 1 \text{ mod } \ell$), they are simply the usual ones.

**Theorem.** (1) Let $\pi \in \text{Cusp}_{Q^*} G$ integral. Then the constant of the epsilon factor is a unit $c(\pi) \in Z^*_\ell$ and the new constants $<c(\pi), y \chi>$ in $Z^*_\ell$ are integral, for all $y \in Y$.

(2) Let $\pi, \pi' \in \text{Cusp}_{Q^*} G$ integral with central characters $\omega_\pi, \omega_{\pi'}$. Then $r_\ell \pi = r_\ell \pi'$ if and only if $r_\ell \omega_\pi = r_\ell \omega_{\pi'}$ and their new epsilon factors have the same reduction modulo $\ell$: the conductors $f(\pi \otimes \chi) = f(\pi' \otimes \chi)$ are equal, and the new constants have the same reduction modulo $\ell$:

$$r_\ell <c(\pi \otimes \chi), y > = r_\ell <c(\pi' \otimes \chi), y >$$

for all $y \in Y$, and all $Q^*_\ell$-characters $\chi \in \hat{X}$.

**Proof.** With the last corollary of the paragraph (1), $r_\ell \pi = r_\ell \pi'$ if and only if $r_\ell \omega_\pi = r_\ell \omega_{\pi'}$ and

$$c(\pi \otimes \chi) f_n(x^{-1} \omega_\pi^{-1}) = c(\pi' \otimes \chi) f'_n(x^{-1} \omega_{\pi'}^{-1}) \text{ modulo } \Lambda L$$

for all $f_n \in C_c(O^*_\ell, \hat{Z}_\ell)$ and all $n \in Z$. With the lemma, we deduce the theorem. \qed

We apply now the theorem to representations over $F_\ell$. Any $\pi \in \text{Cusp}_{F_\ell} G$ lifts to $Q^*_\ell$ and we can define epsilon factors

$$\varepsilon(\pi \otimes \chi, y) := <c(\pi \otimes \chi), y \chi > Xf(\pi \otimes \chi)$$

for all $y \in Y$ and all $\chi \in \text{Hom}(O^*_\ell, F^*_\ell) = \text{Hom}(X, F^*_\ell)$, by reduction modulo $\ell$. They are not zero for any $(y, \chi)$.

**Corollary.** $\pi, \pi' \in \text{Cusp}_{F_\ell} G$ are isomorphic if and only if they have the same central character and the same epsilon factors

$$\varepsilon(\pi \otimes \chi, y) = \varepsilon(\pi' \otimes \chi, y)$$

for all $y \in Y$, and for all character $\chi \in \text{Hom}(O^*_\ell, F^*_\ell)$. 

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*[Proof and further details are omitted for brevity.]*
Final remarks. a) When \( n > 2 \), the groups \( GL(m, O_F)^* \) for \( m \leq n - 1 \) replace \( O_F^* \).

b) Using the explicit description for the irreducible representations of dimension \( n \) of \( W_F \) [V3], one could try to prove a similar theorem for the irreducible integral \( \overline{Q}_F \)-representations of \( W_F \) of dimension \( n \). To my knowledge this is a known and harder problem, which is not solved in the complex case.

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