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An Analogue of Pfister’s Local-Global Principle in the Burnside Ring

par MARTIN EPKENHANS

RÉSUMÉ. Soit $N/K$ une extension galoisienne de groupe de Galois $G$. On étudie l’ensemble $T(G)$ des combinaisons linéaires sur $\mathbb{Z}$ de caractères de l’anneau de Burnside $B(G)$, qui induisent des combinaisons $\mathbb{Z}$-linéaires des formes trace de sous-extensions de $N/K$ qui sont triviales dans l’anneau de Witt $W(K)$ de $K$. On montre que le sous-groupe de torsion de $B(G)/T(G)$ est le noyau de l’homomorphisme signature.

ABSTRACT. Let $N/K$ be a Galois extension with Galois group $G$. We study the set $T(G)$ of $\mathbb{Z}$-linear combinations of characters in the Burnside ring $B(G)$ which give rise to $\mathbb{Z}$-linear combinations of trace forms of subextensions of $N/K$ which are trivial in the Witt ring $W(K)$ of $K$. In particular, we prove that the torsion subgroup of $B(G)/T(G)$ coincides with the kernel of the total signature homomorphism.

1. INTRODUCTION

Let $L/K$ be a finite, separable extension of fields of characteristic $\neq 2$. With it we associate the ‘trace form’ which is defined by $\text{tr}_{L/K} : L \to K : x \mapsto \text{tr}_{L/K}x^2$. P.E. Conner started to investigate the connection of the trace form of $L/K$ and the trace form of a normal closure of $L/K$. His work yields some polynomial vanishing theorems for trace forms (see [1]). These identities come from identities in the Burnside ring of the Galois group $G = G(N/K)$ of $N/K$. We study the trace ideal $T(G)$ in $B(G)$, which is roughly speaking the set of $\mathbb{Z}$-linear combinations of trace forms of subextensions of $N/K$ which are trivial in the Witt ring $W(K)$ of $K$.

We first recall the definition of the Burnside ring $B(G)$ of a finite group $G$. A theorem of Springer [6] gives rise to a homomorphism $h_{N/K} : B(G) \to W(K)$. The trace ideal $T(G)$ is a finitely generated subgroup of the free abelian group $B(G)$. We introduce a signature homomorphism $\text{sign}_\sigma : B(G) \to \mathbb{Z}$ for each element $\sigma \in G$ of order $\leq 2$. These signature homomorphisms correspond to signatures of the Witt ring. We conclude that $T(G)$ is contained in the intersection $L(G)$ of all kernels of signatures. The
main theorem states that $T(G)$ and $L(G)$ are of equal rank. Hence the torsion subgroup of $B(G)/T(G)$ is given by the kernel of the total signature homomorphism. In section 7 we reduce our approach to 2-groups. The general case follows by induction via the Frattini subgroup of $G$.

2. Notation

We first fix our notations. Let $K$ be a field. Then $K^*$ denotes the multiplicative group of $K$, $K^{*2}$ is the group of squares in $K^*$. We write $K_s$ for a separable closure of $K$.

Let $N/K$ be a Galois extension, then $G(N/K)$ denotes the Galois group of $N/K$. If $H < G(N/K)$ then $N^H$ is the fixed field of $H$ in $N$. Let $Aut(K)$ be the group of field automorphisms of $K$.

Now let $K$ be a field of characteristic $\neq 2$. Let $\psi, \varphi$ be non-degenerate quadratic forms over $K$. Then $\det_K \psi$ is the determinant of $\psi$. If $p$ is a real place of $K$ then $\text{sign}_p \psi$ is the signature of $\psi$ with respect to $p$. $\psi \otimes \varphi$ is the product of $\psi$ and $\varphi$. For $m \in \mathbb{Z}$, $m \times \psi$ is the $m$-fold sum of $\psi$. $\psi \simeq \varphi$ indicates the isometry of $\psi$ and $\varphi$ over $K$. Let $L/K$ be a field extension. Then $\psi_L$ is the lifting of $\psi$ to a form over $L$ by scalar extension. $W(K)$ is the Witt ring of $K$. Let $K^1, \ldots, K^n \in K^*$. Then $<a_1, \ldots, a_n>$ is the diagonal form $a_1X_1^2 + \ldots + a_nX_n^2$ over $K$. $<a_1, \ldots, a_n>$ is the diagonal form over $K$. Then $\psi_L$ is the lifting of $\psi$ to a form over $L$ by scalar extension. $W(K)$ is the Witt ring of $K$. Let $a_1, \ldots, a_n \in K^*$. Then $<a_1, \ldots, a_n>$ is the diagonal form $a_1X_1^2 + \ldots + a_nX_n^2$ over $K$. $<a_1, \ldots, a_n> = \otimes_{i=1}^{n} a_i$ is the $n$-fold Pfister form defined by $a_1, \ldots, a_n$.

Let $L/K$ be a finite and separable field extension. The trace form of $L/K$ is the non-degenerate quadratic form $tr_{L/K}: L \rightarrow K : x \mapsto tr_{L/K}(x^2)$. We denote the trace form also by $<L/K>$, resp $<L>$ if no confusion can arise.

Let $M$ be a set. Then $\sharp M$ is the cardinality of $M$. $ord(G)$, $ord(\sigma)$ is the order of the finite group $G$, resp. of the element $\sigma \in G$.

3. The Burnside ring $B(G)$

Let $G$ be a finite group and let $H < G$ be a subgroup of $G$. We denote the transitive action of $G$ on the set of left cosets $G/H = \{aH, a \in G\}$ by $(G, G/H)$. The transitive and faithful actions of $G$ on finite sets are in one-to-one correspondence with the set of conjugacy classes of subgroups of $G$. A subgroup $H$ of $G$ induces a transitive action of degree $[G:H]$, hence a representation of dimension $[G:H]$. Let $\chi_H$ denote the corresponding character. We sometimes write $\chi_H^G$ to indicate that the character is defined on $G$.

Definition 1. Let $G$ be a finite group. The Burnside ring $B(G)$ of $G$ is the free abelian group freely generated by the set 
$\{\chi_H | H \text{ runs over representatives of conjugacy classes of subgroups of } G\}$
and with multiplication given by
\[ \chi U_1 \cdot \chi U_2 = \bigoplus_{\sigma \in U_1 \backslash G / U_2} \chi U_1 \cap \sigma U_2 \sigma^{-1}, \]
where the sum runs over a set of representatives of the double cosets in \( U_1 \backslash G / U_2 \).

Remark 2. \( \chi_G \) is the multiplicative identity, \( \chi\{e\} =: \chi_1 \) is the regular character.

Another way of defining the multiplication is as follows. Let \( \rho_i : G \to GL(V_i), i = 1, 2 \) be representations of \( G \). Then \( \rho_1 \otimes \rho_2 : G \times G \to GL(V_1 \otimes V_2) \) is a representation of \( G \times G \) on \( V_1 \otimes V_2 \). According to the diagonal embedding \( G \to G \times G \) the representation \( \rho_1 \otimes \rho_2 \) restricts to a representation of \( G \) on \( V_1 \otimes V_2 \). For \( \rho_i = (\rho_i, G / U_i) \) we get \( \rho_1 \otimes \rho_2 |_G = \bigoplus_{\sigma \in U_1 \backslash G / U_2} (G, G / (U_1 \cap \sigma U_2 \sigma^{-1})). \)

4. THE HOMOMORPHISM \( h_{N/K} : \mathcal{B}(G(N/K)) \to W(K) \)

Proposition 3 (T.A. Springer). Let \( N/K \) be a finite Galois extension with Galois group \( G(N/K) = G \). Then there is a well-defined ring homomorphism
\[ h_{N/K} : \mathcal{B}(G) \to W(K) \]
with
\[ h_{N/K}(\chi_H) = \langle N^H \rangle \]
for all subgroups \( H \) of \( G \).

Proof. Let \( H < G \) be a subgroup of \( G \). Then \( h_{N/K} \) is well-defined as a group homomorphism since \( \langle N^H \sigma H^{-1} \rangle = \langle \sigma(N^H) \rangle = \langle N^H \rangle \). Now the assertion follows from the next lemma. \( \square \)

Lemma 4. Let \( N/K \) be a finite Galois extension with Galois group \( G = G(N/K) \). Let \( U_1, U_2 \) be subgroups of \( G(N/K) \). Then
\[ \langle N^{U_1} \rangle \otimes \langle N^{U_2} \rangle = \bigoplus_{\sigma \in U_1 \backslash G / U_2} \langle N^{U_1 \cap \sigma U_2 \sigma^{-1}} \rangle, \]
where the sum runs over a set of representatives of the double cosets \( U_1 \backslash G / U_2 \).

Proof. (see [2], I.6.2) Let \( \alpha \in N \) with \( N^{U_i} = K(\alpha) \) and let \( f \in K[X] \) be the minimal polynomial of \( \alpha \) over \( K \). Set \( L := N^{U_2} \). From Frobenius reciprocity [5], 2.5.6 we get
\[ \langle N^{U_i} \rangle \otimes \langle N^{U_2} \rangle = \langle K(\alpha) \rangle \otimes \langle L \rangle = tr_{L/K}((tr_{K(\alpha)/K}(1))L) \]
\[ = tr_{L/K}(L[X]/(f))/L \]
\[ = \bigoplus_{i=1, \ldots, r} tr_{L/K}(L[X]/(f_i))/L, \]
where \( f = f_1 \cdots f_r \) is the decomposition of \( f \) into monic irreducible polynomials in \( L[X] \). Now consider \( tr_{L/K}(L[X]/(g))/L \) for some monic prime
divisor $g \in L[X]$ of $f$. Then $g$ is the minimal polynomial of some conjugate $\sigma(\alpha)$ of $\alpha$ over $L$. Hence

$$\text{tr}_{L/K} <(L[X]/(g))/L> = \text{tr}_{L/K}(\text{tr}_{L(\sigma)/L}<1>) = <L(\sigma(\alpha))>.$$ 

Now

$$L(\sigma(\alpha)) = L \cdot K(\sigma(\alpha)) = L \cdot \sigma(K(\alpha)) = N^{U_2} \cdot \sigma(N^{U_1}) = N^{U_1 \cap \sigma U_2 \sigma^{-1}}.$$ 

The action of $G$ on the roots of $f$ induces an action of $U_2$ on the roots of $f$, which is equivalent to the action of $U_2$ on $G/U_1$. Each orbit of this action corresponds to a monic irreducible factor $g \in L[X]$ of $f$. \hfill \Box

5. The Trace Ideal in $B(G)$

**Definition 5.** Let $G$ be a finite group. Set

$$\mathcal{T}(G) := \cap \ker(h_{N/K}),$$

where the intersection is taken over all Galois extensions $N/K$ over all fields $K$ of characteristic $\neq 2$ with Galois group $G(N/K) \simeq G$. We call $\mathcal{T}(G)$ the **trace ideal of** $B(G)$.

6. The Main Results

**Theorem 6.** Let $G$ be a finite group. Then the trace ideal $\mathcal{T}(G)$ of $B(G)$ is a free abelian group of rank

$$\text{rank}(\mathcal{T}(G)) = \text{rank}(B(G)) - \#\{\text{conjugacy classes of elements } \sigma \in G \text{ of order } \leq 2\}.$$ 

The proof of theorem 6 will be organized as follows. We start by defining in a rather canonical way signatures for elements in the Burnside ring. By lemma 8, the trace ideal is contained in the kernel $L(G)$ of the total signature homomorphism. We compute the rank $L(G)$ in lemma 14. Now the assertion follows from the equality of the ranks of $\mathcal{T}(G)$ and $L(G)$, whose proof will be the subject of sections 7 and 8. In section 7 we reduce the proof of theorem 6 to 2-groups. Section 8 contains the proof of theorem 6 for 2-groups. It runs via induction over the Frattini subgroup of $G$.

If $G$ is a finite group then $RC(G)$ denotes a set of representatives of the conjugacy classes of subgroups of $G$. Further, $RC_2(G)$ denotes a set of representatives of the conjugacy classes of elements of order 1 or 2 in $G$. Let $G_2$ be a 2-Sylow subgroup of $G$. Then we can choose $RC_2(G) \subset G_2$.

In the sequel we will use the following proposition of Sylvester.
Proposition 7. Let $K$ be field, $\mathfrak{p}$ be an ordering of $K$. Then for any separable polynomial $f(X) \in K[X]$ the signature of the trace form of $K[X]/(f(X))$ over $K$ equals the number of real roots of $f(X)$ with respect to the ordering $\mathfrak{p}$.

For a proof see [7].

Lemma 8. Let $G$ be a finite group and let $\sigma \in G$ be an element of order $\leq 2$. Then there is a Galois extension $N/K$ of algebraic number fields and an isomorphism $\iota : G \xrightarrow{\sim} G(N/K)$ such that

1. $K \subset \mathbb{R}$ and $N \subset \mathbb{C}$.
2. $\iota(\sigma)$ is induced by the complex conjugation.

Proof. Set $n := \text{ord}(G)$.

1. $\text{ord}(\sigma) = 2$. If $n = 2$, set $K = \mathbb{Q}, N = \mathbb{Q}(\sqrt{-1})$.

Now let $n = 2m \geq 4$. Consider the quadratic form $\psi = (m - 1)x < 1, -1> \perp <1, -2>$ as a form over $\mathbb{Q}$. Then $\det_{\mathbb{Q}} \psi \notin \mathbb{Q}^2$ and $\text{sign}_{\mathbb{Q}} \psi = 0$. By theorems 1 and 3 of [4] there is a field extension $L/\mathbb{Q}$ with normal closure $N/\mathbb{Q}$ such that $N \subset \mathbb{C}, G(N/\mathbb{Q}) \cong S_n$ and $L/\mathbb{Q}$ has trace form $\psi$. Here $S_n$ denotes the symmetric group on $n$ elements.

Let $\alpha \in L$ be a primitive element of $L/\mathbb{Q}$. Since $\text{sign}_{\mathbb{Q}} <L>= 0$ no conjugate of $\alpha$ is real (see proposition 7). Let $M := \{\alpha_1, \bar{\alpha}_1, \ldots, \alpha_m, \bar{\alpha}_m\}$ be the set of conjugates of $\alpha$. $\bar{\alpha}$ is the complex conjugate of $\alpha \in \mathbb{C}$. Let $\varphi : G \to M$ be a bijection such that for each $a \in G$ the set $\varphi(\{a, \sigma(a)\})$ consists of a pair of complex conjugate elements of $M$. Now according to the identification given by $\varphi$ we get a monomorphism $\iota : G \hookrightarrow S(M) \xrightarrow{\sim} G(N/\mathbb{Q})$. Then $\iota(\sigma)$ is given by the complex conjugation on $N$. Set $K := N^{\iota(G)}$. Since $\iota(\sigma) \in \iota(G)$ the field $K$ is real.

2. $\sigma = \text{id}$. Set $\psi = (n - 1)x <1> \perp <2>$.

Then $\det_{\mathbb{Q}} \psi \notin \mathbb{Q}^2$ and $\text{sign}_{\mathbb{Q}} \psi = n$. Now choose $L, N$ and $\alpha \in L$ as above. Since $\text{sign}_{\mathbb{Q}} \psi = \text{sign}_{\mathbb{Q}} <L>= n$ all conjugates of $\alpha$ are real. Hence $L \subset N \subset \mathbb{R}$. Choose any injection $\iota : G \hookrightarrow G(N/\mathbb{Q})$ and set $K := N^{\iota(G)} \subset \mathbb{R}$. □

Set

$$X = \sum_{\mathcal{H} \in \text{RC}(G)} m_{\mathcal{H}} \cdot \chi_{\mathcal{H}}, \quad m_{\mathcal{H}} \in \mathbb{Z}.$$

Let $N/K$ be a Galois extension with Galois group $G(N/K) = G$. Let $\mathfrak{p}$ be a real place of $K$. Then

$$h_{N/K}(X) = \sum_{\mathcal{H} \in \text{RC}(G)} m_{\mathcal{H}} \cdot <N^\mathcal{H}> = 0.$$
gives
\begin{equation}
\text{sign}_p h_{N/K}(X) = 0 = \sum_{\mathcal{H} \in RC(\mathcal{G})} m_{\mathcal{H}} \cdot \text{sign}_p <N^{\mathcal{H}}>
\end{equation}

Let \( \mathcal{H} < \mathcal{G} \) and \( N^{\mathcal{H}} = K(\alpha) \). By proposition 7, \( \text{sign}_p <N^{\mathcal{H}}> \) equals the number of real conjugates of \( \alpha \) with respect to the ordering \( p \). Let \( \sigma \in G(N/K) \) be the automorphism which is induced by the complex conjugation. Then \( \text{sign}_p <N^{\mathcal{H}}> \) is the number of fixed points of the action of \( <\sigma> \) on the set of conjugates of \( \alpha \), which equals the number of fixed points of the action of \( <\sigma> \) on \( \mathcal{G}/\mathcal{H} \). Therefore the equation (I) is already determined by \( \mathcal{G} \) and the conjugacy class of the complex conjugation in \( \mathcal{G} \). This leads to the following definition.

**Definition 9.** Let \( \sigma \in \mathcal{G} \) be an element of order \( \leq 2 \). Let \( \mathcal{H} \) be a subgroup of \( \mathcal{G} \) and let \( \chi_{\mathcal{H}} \in \mathcal{B}(\mathcal{G}) \) be the corresponding character. Set

\[
\text{sign}_\sigma \chi_{\mathcal{H}} = \#\{\text{fixed points of } <\sigma>, \mathcal{G}/\mathcal{H}\}.
\]

Of course, \( \text{sign}_\sigma \chi_{\mathcal{H}} = \chi_{\mathcal{H}}(\sigma) \). Since our approach is motivated by quadratic form considerations we feel it is more convenient to talk about signatures.

As usual \( C_\mathcal{G}(\sigma) \) denotes the centralizer of \( \sigma \) in \( \mathcal{G} \). Let \( \mathcal{G}\sigma = \{\rho^{-1}\sigma\rho \mid \rho \in \mathcal{G}\} \) be the set of conjugates of \( \sigma \) in \( \mathcal{G} \).

**Proposition 10.** Let \( \mathcal{G} \) be a finite group, \( \mathcal{H} < \mathcal{G} \) a subgroup of \( \mathcal{G} \). Let \( \sigma \in \mathcal{G} \) be an element of order \( \leq 2 \). Then

\[
\text{sign}_\sigma \chi_{\mathcal{H}} = \frac{\text{ord}(C_\mathcal{G}(\sigma))\#(\mathcal{G}\sigma \cap \mathcal{H})}{\text{ord}(\mathcal{H})} = \frac{[\mathcal{G}:\mathcal{H}]\#(\mathcal{G}\sigma \cap \mathcal{H})}{\#\mathcal{G}\sigma}
\]

**Proof.** Consider the action of \( <\sigma> \) on \( \mathcal{G}/\mathcal{H} \). Let \( \rho \in \mathcal{G} \). Then \( \rho\mathcal{H} \) is a fixed point if and only if \( \rho^{-1}\sigma\rho \in \mathcal{H} \). Hence we can assume that

\[
\mathcal{G}\sigma \cap \mathcal{H} = \{\sigma_1, \ldots, \sigma_r\}
\]

is a set of \( r > 0 \) elements. Let

\[
M = \{(\rho, \sigma_i) \mid \rho^{-1}\sigma\rho = \sigma_i\} \subset \mathcal{G} \times \{\sigma_1, \ldots, \sigma_r\}.
\]

Obviously the cardinality of \( M \) is the product of \( \text{ord}(\mathcal{H}) \) and the number of fixed points. Further, for \( i = 1, \ldots, r \) we get

\[
\#\{\rho \in \mathcal{G} \mid (\rho, \sigma_i) \in M\} = \text{ord}(C_\mathcal{G}(\sigma)).
\]

Hence \( \#M = \text{ord}(C_\mathcal{G}(\sigma)) \cdot \#\mathcal{G}\sigma \cap \mathcal{H} \). \( \square \)

We abbreviate \( \chi_{<\tau>} \) to \( \chi_\tau \).

**Corollary 11.** In the situation of proposition 10 we get

1. \( \text{sign}_\sigma \chi_{\mathcal{H}} \equiv [\mathcal{G}:\mathcal{H}] \mod 2 \).
2. \( \text{sign}_{id} \chi_{\mathcal{H}} = [\mathcal{G}:\mathcal{H}] \).
3. $\text{sign}_\sigma \chi_\mathcal{H} \neq 0$ if and only if $\mathcal{H}$ contains some conjugate of $\sigma$.

4. Let $\tau \in \mathcal{G}$ be an element of order $\leq 2$. Then $\text{sign}_\tau \chi \neq 0$ if and only if $\sigma$ and $\tau$ are conjugate or $\sigma = \text{id}$.

5. Let $\tau$ and $\sigma$ be two conjugate involutions. Then

$$2 \cdot \# \mathcal{G} \cdot \text{sign}_\sigma \chi_\tau = \text{ord}(\mathcal{G}).$$

6. If $\mathcal{H}$ is a normal subgroup of $\mathcal{G}$, then $\text{sign}_\sigma \chi_\mathcal{H} = 0$ or $= [\mathcal{G} : \mathcal{H}]$.

$\text{sign}_\sigma$ extends to a homomorphism on $\mathcal{B}(\mathcal{G})$.

**Proposition 12.** Let $\mathcal{G}$ be a finite group and let $\sigma \in \mathcal{G}$ be an element of order $\leq 2$. Then there is a unique homomorphism

$$\text{sign}_\sigma : \mathcal{B}(\mathcal{G}) \to \mathbb{Z}$$

with $\text{sign}_\sigma \chi_\mathcal{U} = \#\{\text{fixed points of } (\langle \sigma \rangle, \mathcal{G}/\mathcal{U})\}$ for all subgroups $\mathcal{U}$ of $\mathcal{G}$.

**Proof.** We consider the representations and characters over fields of characteristic 0. Let $\rho : \mathcal{G} \to GL(V)$ be the underlying representation of $\chi_\mathcal{U}$. Hence we get $\text{sign}_\sigma \chi_\mathcal{U} = \text{trace}(\rho(\sigma)) = \chi_\mathcal{U}(\sigma)$. Since $\text{trace}(A \otimes B) = \text{trace}(A) \cdot \text{trace}(B)$, $\text{sign}_\sigma$ is a ring homomorphism.

We conclude that $\mathcal{T}(\mathcal{G})$ is contained in the intersection of all kernels of signature homomorphisms.

**Definition 13.** Let $\mathcal{G}$ be a finite group. Set

$$L(\mathcal{G}) := \left\{ \sum_{\mathcal{H} \in \mathcal{RC}(\mathcal{G})} m_\mathcal{H} \chi_\mathcal{H} \mid \sum_{\mathcal{H} \in \mathcal{RC}(\mathcal{G})} m_\mathcal{H} \cdot \text{sign}_\sigma \chi_\mathcal{H} = 0 \right\} \subset \mathcal{B}(\mathcal{G}).$$

for all $\sigma \in \mathcal{RC}_2(\mathcal{G})$.

**Lemma 14.** Let $\mathcal{G}$ be a finite group of order $n$. The system of linear equations given by

$$\sum_{\mathcal{H} \in \mathcal{RC}(\mathcal{G})} \text{sign}_\sigma \chi_\mathcal{H} \cdot x_\mathcal{H} = 0, \ \sigma \in \mathcal{RC}_2(\mathcal{G})$$

has rank $\# \mathcal{RC}_2(\mathcal{G})$.

**Proof.** Let $\sigma_1 = \text{id}, \sigma_2, \ldots, \sigma_r$ be the $r$ distinct elements of $\mathcal{RC}_2(\mathcal{G})$. Consider the coefficients $\text{sign}_{\sigma_i} \chi_{\langle \sigma_i \rangle}$ for $i, j = 1, \ldots, r$. We get $\text{sign}_{\sigma_i} \chi_{\langle \sigma_i \rangle} = \text{ord}(\mathcal{G})/\text{ord}(\sigma_i) \in \{n, n/2\}$ for $i = 1, \ldots, r$. For $j = 2, \ldots, r$ we have $\text{sign}_{\sigma_j} \chi_{\langle \sigma_i \rangle} \neq 0$ if and only if $i = j$. 

$\square$
Remark 15. By lemma 14, $L(G)$ is a free abelian group of rank
$$\text{rank}(B(G)) - \# \text{RC}_2(G).$$
Further, $T(G) \subset L(G)$ by lemma 8 and the remarks following it. We get
$$\text{rank}(T(G)) = \text{rank}(L(G))$$
if and only if there exists a positive integer $a \in \mathbb{Z}$
with $a \cdot L(G) \subset T(G)$.

By Pfister's local-global principle, $L(G)$ is the set of all $X \in B(G)$ such that
$h_{N/K}(X)$ is a torsion form for any Galois extension $N/K$ with $G(N/K) \simeq G$.
Hence the rank formula of theorem 6 is equivalent to the existence of an
integer $l \in \mathbb{Z}$, $l \geq 0$ depending only on $G$ such that $2^l$ annihilates $h_{N/K}(L(G))$
for any Galois extension $N/K$ with Galois group $G$.

Since $T(G) \subset L(G)$ each signature homomorphism $\text{sign}_\sigma$ induces a unique
signature homomorphism $\text{sign} : B(G)/T(G) \to \mathbb{Z}$. Hence we easily get from
Theorem 6:

Theorem 16 (Local-Global Principle). An element $X \in B(G)$ is a tor-
sion element in $B(G)/T(G)$ if and only if $\text{sign}_\sigma(X) = 0$ for every $\sigma \in G$ of
order $\leq 2$. Every torsion element of $B(G)/T(G)$ has 2-power order.

7. REDUCTION TO 2-GROUPS

Proposition 17. Let $G$ be a group of odd order. Then
$$T(G) = L(G).$$
Hence $\text{rank}(T(G)) = \text{rank}(B(G)) - 1$.

Proof. Let $N/K$ be a Galois extension with Galois group $G(N/K) \simeq G$.
Let $L$ be an intermediate field of $N/K$. Then $[L : K] = 1$ (see [2],
cor. I.6.5). Let $X = \sum_{H \in \text{RC}(G)} m_H \cdot \chi_H$. Then $h_{N/K}(X) = \sum_{H \in \text{RC}(G)} m_H \cdot [G : H] < 1>$.
Since ord($G$) is odd, $L(G)$ is defined by the equation
$$\sum_{H \in \text{RC}(G)} m_H \cdot [G : H] = 0$$
(see corollary 11). Now the statement about
the ranks follows from remark 15. \hfill \Box

Let $H, U$ be subgroups of $G$. Then the representation defined by the action
of $G$ on $G/U$ restricts to a representation of $H$ on $G/U$. This defines a ring
homomorphism
$$\text{res}^G_H : B(G) \to B(H),$$
the ‘restriction map’. We get
$$\text{res}^G_H \chi^G_{\sigma U} = \oplus_{\sigma \in H \cap G/U} \chi^H_{H \cap \sigma U \sigma^{-1}} \in B(H),$$
where $\chi^H_{H \cap \sigma U \sigma^{-1}} \in B(H)$ is a character of $H$.

Proposition 18. Let $G$ be a finite group and let $H < G$. Let $\sigma \in H$ be an
element of order $\leq 2$. Then
commutes.

Proof. Let $U < G$. We compute the signature of $\text{res}_H^G \chi_U^G$ as follows: Restrict the action of $G$ on $G/U$ to $H$. Then count the number of fixed points of $<\sigma>$ according to this action. Of course, this number equals $\text{sign}_\sigma \chi_U^G$. □

There is an additive but not multiplicative corestriction map $\text{cor}_H^G : B(H) \to B(G)$ defined by $\text{cor}_H^G \chi_H^U = \chi_U^G$.

**Proposition 19.** Let $G$ be a finite group, $H < G$. Let $N/K$ be a Galois extension with $G(N/K) = G$. Let $s^* : W(K) \to W(N^H)$ be the lifting homomorphism. Then

\[
\begin{array}{ccc}
B(G) & \xrightarrow{h_{N/K}} & W(K) \\
\text{res}_H^G & \downarrow & \downarrow s^* \\
B(H) & \xrightarrow{h_{N/N^H}} & W(N^H)
\end{array}
\]

and

\[
\begin{array}{ccc}
B(G) & \xrightarrow{h_{N/K}} & W(K) \\
\text{cor}_H^G & \downarrow & \downarrow \text{tr}_{N^H/K} \\
B(H) & \xrightarrow{h_{N/N^H}} & W(N^H)
\end{array}
\]

commute.

Proof. We use the notation of lemma 4 and its proof. Set $L := N^H$. Then

\[
\begin{aligned}
\text{h}_{N/L}(\text{res}_H^G(\chi_U^G)) &= \prod_{\sigma \in U \backslash G/H} h_{N/L}(\chi_{H^G \sigma U \sigma^{-1}}^H) \\
&= \prod_{\sigma \in U \backslash G/H} \langle N^H \cap \sigma U \sigma^{-1} \rangle / L = \prod_{i=1, \ldots, r} \langle L[X]/(f_i) \rangle / L \\
&= \langle (L[X]/(f_1 \cdots f_r)) \rangle / L = \langle (K[X]/(f)) \otimes L \rangle \\
&= s^*(N^U/K) = s^* \circ h_{N/K}(\chi_U^G).
\end{aligned}
\]
Lemma 20. Let $\mathcal{H} < G$ be finite groups.

1. Then $\text{res}_G^\mathcal{H}(L(G)) \subset L(\mathcal{H})$.

2. Let $[G : \mathcal{H}]$ be odd.
   (a) Then $\text{res}_G^\mathcal{H}(X) \in L(\mathcal{H})$ if and only if $X \in L(G)$.
   (b) $\text{res}_G^\mathcal{H}(X) \in T(\mathcal{H})$ implies $X \in T(G)$.

Proof. 1. follows from proposition 18.
2. Choose $RC_2(G) \subset RC_2(\mathcal{H})$ and apply proposition 18.
(b) Let $N/K$ be a Galois extension with $G(N/K) = G$ and let $X \in B(G)$ with $\text{res}_G^\mathcal{H}(X) \in T(\mathcal{H})$. Now $h_{N/N1} \circ \text{res}_G^\mathcal{H}(X) = 0 = s^* \circ h_{N/K}(X)$ by proposition 19. By a theorem of Springer $s^*$ is injective (see [5], 2.5.3). Thus $h_{N/K}(X) = 0$.

From $X \in T(G)$ we get $X \in \ker(h_{N/K})$, hence $\text{res}_G^\mathcal{H}(X) \in \ker(h_{N/N1})$. But we do not get $\text{res}_G^\mathcal{H}(X) \in T(\mathcal{H})$. We only get $\text{res}_G^\mathcal{H}(X) \in \bigcap \ker(h_{N/K})$, where the intersection runs over all Galois extensions $N/K$ with Galois group $G$ and such that $G < \text{Aut}(N)$.

Let $\text{exp}(G)$ denote the exponent of $G$.

Proposition 21. Let $G$ be a finite group and let $G_2$ be a 2-Sylow subgroup of $G$.

1. Then the rank formula of theorem 6 holds for $G$ if it holds for any 2-Sylow subgroup of $G$, in which case $\text{exp}(L(G)/T(G))$ divides the exponent of $L(G_2)/T(G_2)$.

2. Suppose there is a set $X$ of fields such that $G \subset \text{Aut}(N)$ for any $N \in X$ and such that

$$T(G_2) = \bigcap_{N \in X} \bigcap_{U \text{Aut}(N), U \cong G_2} \ker(h_{N/NU}).$$

Then $X \in T(G)$ if and only if $\text{res}_{G_2}^G(X) \in T(G_2)$. Hence $L(G)/T(G)$ is isomorphic to a subgroup of $L(G_2)/T(G_2)$.

Proof. 1. If the rank formula holds for $G_2$, then by remark 15 there is a positive integer $a$ with $a \cdot L(G_2) \subset T(G_2)$. Let $X \in L(G)$. Then $\text{res}_{G_2}^G(X) \in L(G_2)$ and $\text{res}_{G_2}^G(aX) = a \cdot \text{res}_{G_2}^G(X) \in a \cdot L(G_2) \subset T(G_2)$. Hence $aX \in T(G)$ by
lemma 20(2)(b). The proof of (2) is left to the reader.

8. PROOF OF THEOREM 6

Let $J_2(G)$ be the set of involutions of the 2-group $G$. For a subgroup $H$ of $G$ define

$$X_H^G := X_H := \text{ord}(H) \cdot \chi_H^G - \chi_1^G + \sum_{\tau \in R_C^2(G), \tau \neq 1} \#(\mathcal{C} \cap H) \cdot (\chi_1^G - 2 \cdot \chi_1^G)$$

and let

$$M_G := \{ X_H | H \in R_C(G) - R_C^2(G) \}.$$

By proposition 10 and corollary 11, $M$ is a free subset of $L(G)$ which consists of $\text{rank}(L(G))$ elements. We will prove by induction that $M_G$ is contained in $\mathcal{B}(G)$.

**Lemma 22.** Let $G$ be a 2-group. Then $M_G$ is a free subset of $\mathcal{B}(G)$ consisting of $\text{rank}(L(G))$ elements.

**Proof.** Observe that $\mathcal{B}(\mathbb{Z}_2) = 0$. Let $G$ be a group of order $2^i \geq 4$ and let $N/K$ be a Galois extension with Galois group $G$. Now we proceed by induction.

1. Let $H$ be a subgroup with $H \neq G$. Let $\tau, \tau' \in G$ be involutions. Then $\chi_\tau = \chi_{\tau'}$ if and only if $\tau' \in \mathcal{G}\tau$. Since $J_2(G)$ is the disjoint union of the conjugacy classes of the involutions of $G$ we get

$$J_2(H) = J_2(G) \cap H = \bigcup_{\tau \in R_C^2(G), \tau \neq 1} \mathcal{G}\tau \cap H.$$

Let $U < G$ be a maximal subgroup of $G$ which contains $H$. Then

$$X_H^U = \text{ord}(H) \cdot \chi_H^U - \chi_1^U + \sum_{\tau \in R_C^2(U), \tau \neq 1} \#(U \cap H) \cdot (\chi_1^U - 2 \cdot \chi_1^U)$$

$$= \text{ord}(H) \cdot \chi_H^U - \chi_1^U + \sum_{\tau \in J_2(H)} (\chi_1^U - 2 \cdot \chi_1^U).$$

Now $X_H^U \in \mathcal{B}(U)$ by induction hypothesis. Hence $h_{N/K}(X_H^U) = 0$, which gives $h_{N/K}(X_H^G) = \text{tr}_{N/K}(h_{N/K}(X_H^U)) = 0$ (see proposition 19). Hence $X_H^G \in \mathcal{B}(G)$ if $H \neq G$.

2. Next we have to prove $X_G^G \in \mathcal{B}(G)$. First we consider an elementary abelian group. Then

$$X_G^G = 2^i \cdot \chi_G^G + (2^i - 2) \cdot \chi_1^G - 2 \cdot \sum_{\tau \in G, \tau \neq 1} \chi_\tau^G.$$
Let $N = K(\sqrt{a_1}, \ldots, \sqrt{a_l})$. We know that $< N > = < 2^l > \otimes < -a_1, \ldots, -a_l >$ (see [3], prop. 1).

Now expand the Pfister form $< -a_1, \ldots, -a_l > = < 1, b_2, \ldots, b_{2l} >$. Then the entries $b_2, \ldots, b_{2l}$ are in one-to-one correspondence with the quadratic subextensions of $N/K$. There are exactly $2^{l-1} - 1$ elements $\tau \in G, \tau \neq id$ such that $K(\sqrt{b_i}) \subset N^{\tau}$. Hence

$$h_{N/K}(X_G^G) = 2^l \times 1 \cdot (2^l - 2) \times < N > - 2 \sum_{\tau \in G, \tau \neq id} < N^{\tau} >$$

$$= 0.$$

Now we can assume that $G$ is not an elementary abelian group. Let $U_1, \ldots, U_m$ be the maximal subgroups of $G$. Since $G$ is not a group of order 2, we get $J_2(G) \subset \bigcup_{i=1}^m U_i$. This gives

$$\sum_{\tau \in J_2(G)} (\chi_1 - 2 \cdot \chi_\tau) = \sum_{U = U_{i_1} \cap \ldots \cap U_{i_r}} (-1)^{r+1} \sum_{\tau \in J_2(U)} (\chi_1 - 2 \cdot \chi_\tau),$$

where the sum runs over the set of all non-empty subsets of $\{1, \ldots, m\}$. Let $\Phi(G)$ denote the Frattini subgroup of $G$. Let $2^k$ be its order and set $V = G/\Phi(G)$. Let $F$ be the fixed field of $\Phi(G)$. Then $F/K$ is an elementary abelian extension. Let $\{i_1, \ldots, i_r\} \subset \{1, \ldots, m\}$ be a set of $r$ different indices. Set $H = U_{i_1} \cap \ldots \cap U_{i_r}$. Then $X_H^H \in T(H)$ by induction hypothesis. We get $h_{N/N_H}(X_H^H) = 0$, which implies

$$\sum_{\tau \in J_2(H)} (< N/N_H > - 2 \times < N^{\tau}/N_H >) = < N/N_H > - \text{ord}(H) \times < 1 >.$$

Set $V' = H/\Phi(G)$ and suppose $H \neq \Phi(G)$. By (1) we know that $X_{V'}^{V'} \in T(V)$ for all subgroups $V'$ of $V$ with $V' \neq 1$. This gives

$$\text{ord}(H/\Phi(G)) \times < 1 > = < F/N_H > - \sum_{\tau \in J_2(V')} ( < F/N_H > - 2 \times < F^{\tau}/N_H >).$$
We further get
\[ h_{N/K}(\sum_{\tau \in J_2(\mathcal{H})} (\chi_1^G - 2 \cdot \chi_T^G)) = \sum_{\tau \in J_2(\mathcal{H})} (^{<N> - 2 \times <N^\tau>}) \]
\[ = \text{tr}_{N^H/K}[\sum_{\tau \in J_2(\mathcal{H})} (^{<N/N^H> - 2 \times <N^\tau/N^H>})] \]
\[ = \text{tr}_{N^H/K}(^{<N>N^H> - \text{ord}(\mathcal{H}) \times <1>}) \]
\[ = ^{<N>- \text{ord}(\mathcal{H}) \times <N^H>)} \]
\[ = ^{<N>- 2^k \times \text{tr}_{N^H/K}(\text{ord}(\mathcal{H}/\Phi(\mathcal{G})) \times <1>)} \]
\[ = ^{<N>- 2^k \times \text{tr}_{N^H/K}(<F/N^H>)} \]
\[ - \sum_{\tau \in J_2(\mathcal{H}/\Phi(\mathcal{G}))} (^{<F/N^H>- 2 \times <F^\tau/N^H>}) \]
\[ = ^{<N>- 2^k \times <F> \sum_{\tau \in J_2(\mathcal{H}/\Phi(\mathcal{G}))} (^{<F>- 2 \times <F^\tau>})} \]

If \( \mathcal{H} = \Phi(\mathcal{G}) \), then \( J_2(\mathcal{H}/\Phi(\mathcal{G})) \) is empty and \( N^H = F \). Hence the formula also holds in this situation.

Now \( \sum_{r=0}^{n}(-1)^r \binom{n}{r} = 0 \) implies
\[ h_{N/K}(X^G_y) = 2^l \times <1> - <N> + \sum_{\mathcal{H}} (-1)^{r+1} \sum_{\tau \in J_2(\mathcal{H})} (^{<N>- 2 \times <N^\tau>}) \]
\[ = 2^l \times <1>- 2^k \times <F> \]
\[ + 2^k \times \sum_{\mathcal{H}} (-1)^{r+1} \sum_{\tau \in J_2(\mathcal{H}/\Phi(\mathcal{G}))} (^{<F>- 2 \times <F^\tau>}) \]
\[ = 2^k \times [\text{ord}(\mathcal{V}) \times <1> - <F> + h_{F/K}(\sum_{\tau \in J_2(\mathcal{V})} (\chi_1^\mathcal{V} - 2 \cdot \chi_T^\mathcal{V}))] \]
\[ = 2^k h_{F/K}(X^\mathcal{V}_y) = 0 \]

by the above. \qed

9. OPEN QUESTIONS

We conclude with some open questions. How does the exponent of \( B(\mathcal{G})/T(\mathcal{G}) \) depend on \( \mathcal{G} \)?

Proposition 23. Let \( \mathcal{G} \) be a finite group. If a 2-Sylow subgroup \( \mathcal{G}_2 \) of \( \mathcal{G} \) is a normal subgroup of \( \mathcal{G} \), then the restriction homomorphism induces an epimorphism
\[ \text{res} : L(\mathcal{G}) \rightarrow L(\mathcal{G}_2)/T(\mathcal{G}_2) \]
Proof. Let \( \text{cor} : B(\mathcal{G}_2) \rightarrow B(\mathcal{G}) \) be the corestriction. This is an additive homomorphism. Since \( \mathcal{G}_2 \) is normal in \( \mathcal{G} \) we get \( \text{res} \circ \text{cor} = [\mathcal{G} : \mathcal{G}_2] \cdot \text{id} \).

By Theorem 6 there is an integer \( l \in \mathbb{N} \) such that \( 2^l \cdot L(\mathcal{G}_2) \subseteq T(\mathcal{G}_2) \). Let \( k, t \in \mathbb{Z}, k > 0 \) with \( k \cdot [\mathcal{G} : \mathcal{G}_2] = 1 + t \cdot 2^l \). Then \( \text{res} \circ \text{cor}(kX) = X + t \cdot 2^l X \equiv X \mod T(\mathcal{G}_2) \).

This leads to the following question: Does the restriction homomorphism induces an isomorphism

\[
\text{res} : L(\mathcal{G})/T(\mathcal{G}) \rightarrow L(\mathcal{G}_2)/T(\mathcal{G}_2)
\]

We know that the answer is affirmative if \( \mathcal{G} \) is an abelian group whose 2-Sylow subgroup is cyclic or elementary abelian. In these cases \( L(\mathcal{G})/T(\mathcal{G}) \) has exponent 2. If \( \mathcal{G} \) is the dihedral group of order 8, then the exponent is 2. In the case of the quaternion group \( Q_8 \) of order 8 we get \( \exp(L(Q_8)/T(Q_8)) = 4 \).

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