ARTŪRAS DUBICKAS

The mean values of logarithms of algebraic integers


<http://www.numdam.org/item?id=JTNB_1998__10_2_301_0>
The mean values of logarithms of algebraic integers

par ARTŪRAS DUBICKAS

Résumé. Soit $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_d$ l’ensemble des conjugués d’un entier algébrique $\alpha$ de degré $d$, n’étant pas une racine de l’unité. Dans cet article on propose de minorer

$$M_p(\alpha) = p \sqrt[1/p]{d \sum_{i=1}^{d} \left| \log |\alpha_i| \right|^p}$$

où $p > 1$.

Abstract. Let $\alpha$ be an algebraic integer of degree $d$ with conjugates $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_d$. In the paper we give a lower bound for the mean value

$$M_p(\alpha) = p \sqrt[1/p]{d \sum_{i=1}^{d} \left| \log |\alpha_i| \right|^p}$$

when $\alpha$ is not a root of unity and $p > 1$.

1. Introduction.

Let $\alpha$ be an algebraic number of degree $d \geq 2$ with

$$P(x) = a_dx^d + a_{d-1}x^{d-1} + \cdots + a_0 = a_d(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_d)$$

as its minimal polynomial over $\mathbb{Z}$ and $a_d$ positive. Following Mahler, the Mahler measure of $\alpha$ is defined by

$$M(\alpha) = a_d \prod_{i=1}^{d} \max \{1, |\alpha_i|\}.$$

The house of an algebraic number is the maximum of the modulus of its conjugates:

$$|\alpha| = \max \{|\alpha_1|, |\alpha_2|, \ldots, |\alpha_d|\}.$$

Put also

$$d(\alpha) = \max \left\{ \frac{|\alpha|}{|\alpha|}, \frac{|\alpha|}{|\alpha|} \right\} = \max \{ |\alpha_1|, \ldots, |\alpha_d|, 1/|\alpha_1|, \ldots, 1/|\alpha_d| \}.$$
for the "symmetric deviation" of conjugates from the unit circle. Denote for $p > 0$

$$M_p(\alpha) = \left( \sum_{i=1}^{d} |\log|\alpha_i||^p \right)^{1/p}.$$  

Our main concern here is the lower bound for this mean value when $\alpha$ is an algebraic integer ($a_d = 1$) which is not a root of unity.

In 1933, D.H. Lehmer [8] asked whether it is true that for every positive $\varepsilon$ there exists an algebraic number $\alpha$ for which $1 < M(\alpha) < 1 + \varepsilon$. In its strong form Lehmer's problem has been reformulated as whether it is true that if $\alpha$ is not a root unity then $M(\alpha) > \alpha_0 = 1.1762808...$ where $\alpha_0$ is the root of the polynomial

$$x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1.$$  

In 1971, C.J. Smyth [16] proved that if $\alpha$ is a non-reciprocal algebraic integer then $M(\alpha) \geq \theta = 1.32471...$ where $\theta$ is the real root of the polynomial $x^3 - x - 1$. This result reduces Lehmer's problem to the case of reciprocal algebraic integers (those with minimal polynomial satisfying the identity $P(x) \equiv x^dP(1/x)$). P.E. Blanksby and H.L. Montgomery [2] used Fourier analysis to prove that $M(\alpha) > 1 + 1/52d \log(6d)$. In 1978, C.L. Stewart [18] proved that $M(\alpha) > 1 + 1/104d \log d$. Although this result is weaker than the previous one, the method used has become very important and led to further improvements. Recently M. Mignotte and M. Waldschmidt [12] obtained Stewart's result via the interpolation determinant.

In 1979, E. Dobrowolski [4] obtained a remarkable improvement of these results showing that for each $\varepsilon > 0$, there exists an effective $d(\varepsilon)$ such that for $d > d(\varepsilon)$

$$M(\alpha) > 1 + (1 - \varepsilon) \left( \frac{\log \log d}{\log d} \right)^3.$$  

D.C. Cantor and E.G. Straus [3] in 1982 introduced the interpolation determinant to simplify Dobrowolski's proof and to replace the constant $1 - \varepsilon$ by $2 - \varepsilon$. Finally, R. Louboutin [9] was able to improve this constant to $9/4 - \varepsilon$. M. Meyer [11] obtained Louboutin's result using a version of Siegel's lemma due to Bombieri and Vaaler. Recently P. Voutier [19] showed that inequality (1) holds for all $d \geq 2$ with the weaker constant $1/4$ instead of $1 - \varepsilon$.

In 1965, A. Schinzel and H. Zassenhaus [13] conjectured that there exists an absolute positive constant $\gamma$ such that $|\alpha| > 1 + \gamma/d$ whenever $\alpha$ is not a root of unity. The best known result on this problem is due to the author [5]:
we have

\begin{equation}
|\alpha| > 1 + \left( \frac{64}{\pi^2} - \varepsilon \right) \frac{1}{d} \left( \frac{\log \log d}{\log d} \right)^3.
\end{equation}

where \( d > d_1(\varepsilon) \). In fact, both inequalities (1), (2) and the respective conjectures can be considered in terms of the lower bound for \( M_p(\alpha) \). Indeed, notice that

\[
M_1(\alpha) = \frac{2\log M(\alpha) - \log |a_0|}{d}.
\]

Therefore, for \( |a_0| \geq 2 \),

\[
M_1(\alpha) = \frac{2\log |a_0| - \log |a_0|}{d} \geq \frac{\log 2}{d}.
\]

If \( |a_0| = 1 \), then

\[
M_1(\alpha) = \frac{2\log M(\alpha)}{d}.
\]

Louboutin’s result can be written as follows

\begin{equation}
dM_1(\alpha) > \left( \frac{9}{2} - \varepsilon \right) \left( \frac{\log \log d}{\log d} \right)^3.
\end{equation}

Taking \( p = \infty \), we can write the inequality (2) in the following form

\begin{equation}
dM_\infty(\alpha) = d \log d(\alpha) \geq d \log |\alpha| > \left( \frac{64}{\pi^2} - \varepsilon \right) \left( \frac{\log \log d}{\log d} \right)^3.
\end{equation}

The function \( p \to M_p(\alpha) \) is nondecreasing. Hence the inequality \( dM_p(\alpha) \geq c_p \) where \( 1 < p < \infty \) and \( c_p > 0 \) lies between the conjecture of Lehmer \( p = 1 \) and the ”symmetric” form of the conjecture of Schinzel and Zassenhaus \( p = \infty \) (see also [1] for a problem which lies between these two conjectures). We have noticed above that the conjectural value for \( c_1 \) is \( 2\log \alpha_0 \). It would be of interest to find out whether it is true that \( d(\alpha) \geq \sqrt{2} \). The equality holds for the polynomial \( x^d - 2 \). We conjecture that the answer to the above question is affirmative, so that \( c_\infty = \log 2 \). In this paper, we take up the interpolation determinant again (see [3],[5],[9],[10], [19]) and fill the gap between inequalities (3) and (4) (Theorem 2). One can also consider the mean value of conjugates of an algebraic integer

\[
m_p(\alpha) = \sqrt[p]{\frac{1}{d} \sum_{i=1}^{d} |\alpha_i|^p}
\]

and the mean value of the differences

\[
t_p(\alpha) = \sqrt[p]{\frac{2}{d(d-1)} \sum_{i \leq j} |\alpha_i - \alpha_j|^p}.
\]
The lower bound for $m_1(\alpha)$ where $\alpha$ is a totally positive integer was considered by I. Schur [14], C.L. Siegel [15], C.J. Smyth [17]. In 1988, M. Langevin [7] solved Favard’s problem proving that $t(\alpha) := \max_{i,j} |\alpha_i - \alpha_j| > 2 - \varepsilon$ for an algebraic integer of a sufficiently large degree. The author [6] proved that $t_2(\alpha) > \sqrt{\varepsilon} - \varepsilon$. The problem of finding an upper bound for $t_\infty(\alpha) := 1/ \min_{i \neq j} |\alpha_i - \alpha_j|$ is known as a separation problem. In this article, we apply the lower bound for $M_2(\alpha)$ to estimate $m_p(\alpha)$ from below (Theorem 3).

2. STATEMENT OF THE RESULTS.

The notations are the following. Let $G(x)$ be a real valued function in $[0; 1]$ such that $G(0) = 1, G(1) = 0$. Let also the derivative of $G(x)$ be continuous and negative in the interval $(0; 1)$. Put

\begin{equation}
I = \int_0^1 G(x) dx,
\end{equation}

(2.5)

\begin{equation}
J = \int_0^1 \left( G(x) \right)^2 dx,
\end{equation}

(2.6)

\begin{equation}
L = \int_0^1 \left( G'(x) \right)^2 dx.
\end{equation}

(2.7)

Put also for brevity

$$
\delta(d) = \left( \frac{\log \log d}{\log d} \right)^3.
$$

Let $\alpha$ be a reciprocal algebraic integer, i.e. $d = 2m, m \in \mathbb{N}, \alpha_{2m} = 1/\alpha_1, \alpha_{2m-1} = 1/\alpha_2, \ldots, \alpha_{m+1} = 1/\alpha_m$ where $|\alpha_1| \geq |\alpha_2| \geq \cdots \geq |\alpha_m| \geq 1$. Suppose also that $\alpha$ is not a root of unity. With these hypotheses, our main result is the following:

**Theorem 1.** For every $\varepsilon > 0$ there exists $d_0(\varepsilon)$ such that we have

\begin{equation}
\sum_{j=1}^{d/2} \left( I - \frac{2j}{d} J \right) \log |\alpha_j| > \frac{1 - \varepsilon}{L} \delta(d)
\end{equation}

(2.8)

whenever $d > d_0(\varepsilon)$.

The constant $d_0(\varepsilon)$ and the constants $d_1(\varepsilon), d_2(\varepsilon), d_3(\varepsilon), d_4, d_5(p)$ used below are effective. Taking $G(x) = (1 - x)^2$, we get $I = 1/3, J = 1/5, L = 4/3$. Hence the following inequality holds:
Corollary 1. For every $\varepsilon > 0$ there exists $d_1(\varepsilon)$ such that

$$\sum_{j=1}^{d/2} \left(1 - \frac{6j}{5d}\right) \log |\alpha_j| > \left(\frac{9}{4} - \varepsilon\right) \delta(d)$$

whenever $d > d_1(\varepsilon)$.

This inequality obviously implies Louboutin's result. On the other hand, taking $G(x) = 1 - \sin(\pi x/2)$, we have $I = 1 - 2/\pi$, $J = 3/2 - 4/\pi$, $L = \pi^2/8$. Hence

$$\sum_{j=1}^{d/2} \left(1 - \frac{2}{\pi} - \left(3 - \frac{8}{\pi}\right) \frac{j}{d}\right) \log |\alpha_j| > \left(\frac{8}{\pi^2} - \varepsilon\right) \delta(d).$$

We can replace in the inequality above $\log |\alpha_j|$ by $|\alpha_j| - 1$, and so Theorem 1 yields the following Corollary.

Corollary 2. For every $\varepsilon > 0$ there exists $d_2(\varepsilon)$ such that for $d > d_2(\varepsilon)$ we have

$$\sum_{j=1}^{d/2} \tau_j |\alpha_j| > 1 + \left(\frac{64}{\pi^2} - \varepsilon\right) \frac{\delta(d)}{d},$$

where

$$\tau_j = \left(1 - \frac{2}{\pi}\right) \frac{8}{d} - \left(3 - \frac{8}{\pi}\right) \frac{8j - 4}{d^2}.$$

Corollary 2 implies the inequality (2), since $\sum_{j=1}^{d/2} \tau_j = 1$. The following theorem fills the gap between (3) and (4).

Theorem 2. Let $1 < p < \infty$ and $\varepsilon > 0$. Then there is $d_3(\varepsilon)$ such that for $d > d_3(\varepsilon)$ we have

$$dM_p(\alpha) > \left(b_p - \varepsilon\right) \delta(d),$$

where the constant $b_p$ is given by

$$b_p = 2 \frac{(2p - 1)J}{L \left(\frac{p}{p-1}(I(2p-1)/(p-1) - (I - J)(2p-1)/(p-1))\right)^{1-1/p}}.$$

We are not solving the problem of computing the maximum in (9) for a fixed $p$ from the interval $(1; \infty)$. However, notice that if $G(x) = (1 - x)^{1.7}$ and $p = 2$ then by (5)-(7) and (9) we get $b_2 > 6.2679$.

Corollary 3. There is $d_4 > 0$ such that for $d > d_4$ we have

$$dM_2(\alpha) > 6.2679 \delta(d).$$
Theorem 3. If \( \alpha \) is an algebraic integer which is not a root of unity, then for every \( p > 0 \) there exists \( d_5(p) \) such that for \( d > d_5(p) \) we have

\[
\left( m_p(\alpha) \right)^p > 1 + 19.64 \left( p \delta(d)/d \right)^2.
\]

In particular,

\[
m_1(\alpha) = \frac{|\alpha_1| + \cdots + |\alpha_d|}{d} > 1 + 19.64 \left( \frac{\delta(d)}{d} \right)^2.
\]

Proof of Theorem 1. Let \( f(x) \) be a continuous non-negative function in \([0; 1]\) such that \( \int_0^1 f(x)dx = 1 \), and let \( G(x) = \int f(y)dy \). Put

\[
s = \left[ \frac{L}{2} \left( \frac{\log d}{\log \log d} \right)^2 \right],
\]

where \( L \) is the Euler’s constant.

\[
k_0 = \left[ \frac{s^2 \log s}{\log d} \right],
\]

\[
k_r = \left[ \frac{s f \left( \frac{r}{s} \right)}{d} \right], \quad 1 \leq r \leq s.
\]

Define

\[
h_0(z) = h(z) = \left( 1, z, z^2, \ldots, z^{N-1} \right)^t,
\]

\[
h_k(z) = \frac{z^k d^k h(z)}{k!} = \left( 0, \ldots, \frac{(N-2)}{k} z^{N-2}, \frac{(N-1)}{k} z^{N-1} \right)^t.
\]

Consider the determinant

\[
D = \det \left| \left| h_{u_r}(\alpha_j^{p_r}) \right| \right|,
\]

where the matrix consists of \( N = (k_0 + k_1 + \cdots + k_s)d \) columns, \( u_r = 0, 1, \ldots, k_r - 1, j = 1, 2, \ldots, d \). Here \( p_r \) is the \( r \)-th prime number \((p_0 = 1, p_1 = 2, p_2 = 3, \ldots)\). Recall that \( \alpha \) is reciprocal and \( \alpha_{2m} = 1/\alpha_1, \ldots, \alpha_{m+1} = 1/\alpha_m \). Then see [(3), (5), (9), (10), (19)] the determinant \( D \) is given by

\[
D = \pm \prod \left( \alpha_i^{p_u} - \alpha_j^{p_v} \right)^{k_u k_v} \left( \alpha_i^{-p_u} - \alpha_j^{-p_v} \right)^{k_u k_v} \prod \left( \alpha_i^{p_u} - \alpha_j^{-p_v} \right)^{k_u k_v}
\]

where the first product is taken over \( i, j = 1, 2, \ldots, m \) and \( 0 \leq u \leq v \leq s \) (if \( u = v \), then \( i < j \)). The second product is taken over all \( i, j = 1, 2, \ldots, m; u, v = 0, 1, 2, \ldots, s \). Let us denote these products by \( P_1 \) and \( P_2 \) respectively.
We first consider $P_1$. We have:

$$P_1 = \pm \prod \left( \alpha_i^{p_u} - \alpha_j^{p_v} \right) 2^{k_u k_v} \prod \alpha_i^{-p_u} \alpha_j^{-p_v} \alpha_k k_v$$

$$= \pm \prod \left( \alpha_i^{p_u} - \alpha_j^{p_v} \right) 2^{k_u k_v} \prod \alpha_i^{-p_u} \alpha_j^{-p_v} \prod \alpha_k^{-p_v} k_v$$

$$\times \prod_{i<j; u<v} (\alpha_i \alpha_j)^{-p_u k_v^2}$$

$$= \pm M(\alpha)^{-m \left( \sum_{u<v} p_u k_k k_v + \sum_{u>v} p_u k_k k_v \right)} (m-1) \sum p_u k_v^2$$

Next, we have for the product $P_2$

$$P_2 = \prod \left( 1 - \alpha_i^{-p_u} \alpha_j^{-p_v} \right) k_u k_v \prod \alpha_i^{-p_u} \alpha_j^{-p_v}$$

$$= \prod \left( 1 - \alpha_i^{-p_u} \alpha_j^{-p_v} \right) k_u k_v \times M(\alpha) \sum p_u k_k k_v$$

Combining these results we find

$$D = \pm \prod \left( \alpha_i^{p_u} - \alpha_j^{p_v} \right) 2^{k_u k_v} \prod \left( 1 - \alpha_i^{-p_u} \alpha_j^{-p_v} \right) k_u k_v \times M(\alpha) \sum p_u k_v^2$$

Now from each term $\alpha_i^{p_u} - \alpha_j^{p_v}$ in the first product we take

1. $\alpha_i^{p_u}$, if $u = v, i < j$;
2. $\alpha_j^{p_v}$, if $u < v, j \leq i$;
3. $\alpha_i^{p_u} \alpha_j^{p_v-p_u}$, if $u < v, i < j$.

This is the key point of our argument. Write the determinant $D$ as follows

$$D = \pm \prod \alpha_i^{2 p_u k_u^2} \left( 1 - (\alpha_j/\alpha_i)^{p_u} \right) 2^{k_u^2} \prod \alpha_j^{2 p_v k_v k_v} \left( \alpha_i^{p_u} \alpha_j^{-p_v} - 1 \right) 2^{k_u k_v}$$

$$\times \prod \alpha_i^{2 p_u k_u k_v} \alpha_j^{2(p_v-p_u) k_v k_v} \left( \alpha_j^{p_u-p_v} - (\alpha_j/\alpha_i)^{p_v} \right) 2^{k_u k_v}$$

$$\times \prod \left( 1 - \alpha_i^{-p_u} \alpha_j^{-p_v} \right) k_u k_v \times M(\alpha) \sum p_u k_v^2$$
Denote $y_1 = \alpha_2/\alpha_1$, $y_2 = \alpha_3/\alpha_2, \ldots, y_{m-1} = \alpha_m/\alpha_{m-1}$, $y_m = 1/\alpha_m$. Then $D$ can be expressed in the form

$$D = \pm M(\alpha) \prod_{i} \alpha_i^{2p_i k_i^2} \prod_{j} \alpha_j^{2p_j k_j k_v} \prod_{u,v} \alpha_i^{2p_i k_u k_v} \alpha_j^{2(p_v - p_u) k_u k_v} \times p(y_1, y_2, \ldots, y_m)$$

$$= \prod_{j=1}^m \alpha_j^{s_j} \times p(y_1, y_2, \ldots, y_m)$$

where $p(y_1, \ldots, y_m)$ is a polynomial in $y_1, y_2, \ldots, y_m$. The power $s_j$ is given by

$$s_j = \left( \sum_{u<v} p_u k_u^2 + 2(m - j) \sum_{u<v} p_u k_u k_v + 2(m - j + 1) \sum_{u<v} p_v k_u k_v \right)$$

$$+ 2(m - j) \sum_{u<v} p_u k_u k_v + 2(j - 1) \sum_{u<v} (p_v - p_u) k_u k_v$$

$$= (2m - 2j + 1) \sum_{u<v} p_u k_u^2 + 2m \sum_{u<v} p_v k_u k_v + (2m - 4j + 2) \sum_{u<v} p_u k_u k_v$$

$$= (2m - 2j + 1) \sum_{u<v} p_u k_u^2 + 2m \sum_{u<v} p_v k_u k_v$$

$$+ (2m - 4j + 2) \left( \sum_{u<v} p_u k_u \sum_{v<u} k_v - \sum_{v<u} p_u k^2_u - \sum_{v<u} p_u k_v k_u \right)$$

$$= (d - 4j + 2) \sum_{u<v} p_u k_u \sum_{v<u} k_v + (4j - 2) \sum_{v<u} p_u k_v k_u + (2j - 1) \sum_{v<u} p_v k_u^2$$

$$= (d - 4j + 2) \sum_{u<v} p_u k_u \sum_{v<u} k_v + (4j - 2) \sum_{v<u} p_u k_v k_u - (2j - 1) \sum_{v<u} p_v k_u^2$$

Using the maximum modulus principle and the inequalities $|y_j| \leq 1$, $j = 1, 2, \ldots, m$, we have

$$|p(y_1, y_2, \ldots, y_m)| \leq |p(y_1^0, y_2^0, \ldots, y_m^0)|,$$

where $|y_1^0| = |y_2^0| = \cdots = |y_m^0| = 1$. Now by Hadamard’s inequality we find (see [5])

$$\log |D| \leq \frac{1}{2} d \log \left( d \sum_{v=0}^s k_v \right) \sum_{v=0}^s k_v^2 + \sum_{j=1}^{d/2} s_j \log |\alpha_j|.$$
\[
\sum_{v=1}^{s} k_v \log p_v \sim \sum_{v} s f \left( \frac{u}{s} \right) \log v \sim s^2 \log s \int_{0}^{1} f(x) dx \sim s^2 \log s \sim \frac{L^2}{2} \frac{(\log d)^4}{(\log \log d)^3},
\]

\[
k_0 \sim \frac{L^2}{2} \left( \frac{\log d}{\log \log d} \right)^3,
\]

\[
\sum_{v=0}^{s} k_v^2 \sim k_0^2 + \sum_{v=1}^{s} s^2 f^2 \left( \frac{v}{s} \right) \sim k_0^2 + s^3 \int_{0}^{1} f^2(x) dx \sim \frac{3}{8} L^4 \left( \frac{\log d}{\log \log d} \right)^6.
\]

Similarly,

\[
s_j \sim (d - 4j) s^5 \log s \int_{0}^{1} f(x)x dx + 4j s^5 \log s \int_{0}^{1} f(x) \left( \int_{0}^{x} f(y) dy \right) dx.
\]

Since

\[
\int_{0}^{1} f(x) x dx = - \int_{0}^{1} G'(x) x dx = \int_{0}^{1} G(x) dx = I
\]

and

\[
\int_{0}^{1} f(x) \left( \int_{0}^{x} f(y) dy \right) dx = \int_{0}^{1} f(x) x (1 - G(x)) dx
\]

\[
= I - \int_{0}^{1} f(x) x G(x) dx = I + \int_{0}^{1} G'(x) G(x) x dx
\]

\[
= I + \frac{1}{2} \int_{0}^{1} \left( G^2(x) \right)' x dx
\]

\[
= I - \frac{1}{2} \int_{0}^{1} G^2(x) dx = I - \frac{1}{2} J,
\]

we have

\[
s_j \sim s^5 \log s \left( (d - 4j)I + 4j(I - \frac{12}{J}) \right)
\]

\[
\sim (dI - 2jJ) \frac{L^5}{16} \frac{(\log d)^{10}}{(\log \log d)^9}.
\]
For a sufficiently large $d$ we have
\[ \sum_{j=1}^{d/2} (dI - 2jJ) \log |\alpha_j| \]
This inequality implies (8). \qed

Proof of Theorem 2. If $a$ is not reciprocal, then by Smyth’s result [16] $dM_1(\alpha) \geq 2 \log \theta$, and the theorem follows from $M_p(\alpha) \geq M_1(\alpha)$. Let $a$ be reciprocal. Then by (8) and by Hölder’s inequality we have
\[ 1 - \frac{\varepsilon}{L} \delta(d) < \sum_{j=1}^{d/2} (I - \frac{2j}{d}J) \log |\alpha_j| \]
\[ \leq \left( \sum_{j=1}^{d/2} (\log |\alpha_j|)^p \right)^{1/p} \left( \sum_{j=1}^{d/2} (I - \frac{2j}{d}J)^q \right)^{1/q} \]
where $1/p + 1/q = 1$.

Note first that for a reciprocal $a$
\[ \left( \sum_{j=1}^{d/2} (\log |\alpha_j|)^p \right)^{1/p} = (d/2)^{1/p} M_p(\alpha). \]

For $d$ tending to infinity we have
\[ \sum_{j=1}^{d/2} (I - \frac{2j}{d}J)^q \sim \frac{d}{2} \int_0^1 (I - Jx)^q \, dx \]
\[ = \frac{d(I^{q+1} - (I-J)^{q+1})}{2J(q+1)}. \]
Hence
\[ 1 - \frac{\varepsilon}{L} \delta(d) < \frac{d}{2} M_p(\alpha) \left( \frac{I^{q+1} - (I-J)^{q+1}}{J(q+1)} \right)^{1/q}, \]
and Theorem 2, where the constant $b_p$ is given by (9), follows. \qed
Proof of Theorem 3. We have

\[
(m_p(\alpha))^p = \frac{1}{d} \sum_{i=1}^{d} |\alpha_i|^p
\]

\[
= \frac{1}{d} \sum_{i=1}^{d} \exp(p \log |\alpha_i|)
\]

\[
= \frac{1}{d} \sum_{i=1}^{d} \sum_{j=0}^{\infty} \frac{(p \log |\alpha_i|)^j}{j!}
\]

\[
= \frac{1}{d} \sum_{j=0}^{\infty} \frac{p^j}{j!} \sum_{i=1}^{d} (\log |\alpha_i|)^j.
\]

If \(\alpha\) is reciprocal, then the inner sum equals \(d (M_j(\alpha))^j\) for even \(j\) and zero for odd \(j\). Hence

\[
(m_p(\alpha))^p = 1 + \sum_{k=1}^{\infty} \frac{p^{2k}}{(2k)!} (M_{2k}(\alpha))^{2k} > 1 + \frac{p^2}{2} (M_2(\alpha))^2.
\]

Utilizing Corollary 3 we have

\[
(M_2(\alpha))^2 > 39.28 \left( \frac{\delta(d)}{d} \right)^2,
\]

if \(d\) is large enough and the statement of Theorem 3 follows.

Suppose now that \(\alpha\) is not reciprocal. If

\[
|\alpha_0| = \prod_{i=1}^{d} |\alpha_i| \geq 2
\]

then

\[
(m_p(\alpha))^p = \frac{|\alpha_1|^p + \cdots + |\alpha_d|^p}{d} \geq \prod_{i=1}^{d} |\alpha_i|^{p/d} \geq 2^{p/d} > 1 + \frac{p \log 2}{d}
\]

\[
> 1 + 19.64 \left( \frac{p \delta(d)}{d} \right)^2
\]

for \(d > d_5(p)\). Hence it is sufficient to consider the case when \(|\alpha_0| = 1\). Let \(\alpha_1, \alpha_2, \ldots, \alpha_r\) be the conjugates of \(\alpha\) lying strictly outside the unit circle. Put

\[
\Lambda = \prod_{i=1}^{r} |\alpha_i|.
\]
Then
\[
(m_p(\alpha))^p = \left(\frac{|\alpha_1|^p + \cdots + |\alpha_d|^p}{d}\right) \geq \frac{r}{d} (|\alpha_1| \cdots |\alpha_r|)^{p/r} + \frac{d-r}{d} (|\alpha_{r+1}| \cdots |\alpha_d|)^{p/(d-r)}
\]
\[
= \frac{r}{d} \Lambda^{p/r} + \frac{d-r}{d} \Lambda^{-p/(d-r)}.
\]
We shall show now that the last expression is greater than
\[
1 + \frac{(\log \theta)^2}{2} \left(\frac{p}{d}\right)^2
\]
where \(\theta = 1.32471\ldots\). Indeed, if
\[
\frac{r}{d} \Lambda^{p/r} + \frac{d-r}{d} \Lambda^{-p/(d-r)}
\]
then
\[
h'(\Lambda) = \frac{p}{d} \Lambda^{p/r-1} - \frac{p}{d} \Lambda^{-p/(d-r)-1}
\]
\[
= \frac{p}{\Lambda d} \left(\Lambda^{p/r} - \Lambda^{-p/(d-r)}\right).
\]
Therefore, the function \(h(\Lambda)\) is increasing in the interval \((1; \infty)\) and by Smyth's theorem
\[
h(\Lambda) \geq h(\theta) = \frac{r}{d} \theta^{p/r} + \frac{d-r}{d} \theta^{-p/(d-r)}.
\]
Put for brevity \(p = zd\) and \(r = yd\). We are going to prove that
\[
g(z) = y^{z/y} + (1 - y)\theta^{-z/(1-y)} - 1 - \frac{\log \theta}{2} z^2 > 0
\]
for \(z > 0\) and \(0 < y < 1\). Indeed, \(g(0) = 0\) and
\[
g'(z) = \theta^{z/y} \log \theta - \theta^{-z/(1-y)} \log \theta - (\log \theta)^2 z
\]
\[
> \theta^{z/y} \log \theta - \log \theta - (\log \theta)^2 z
\]
\[
> (1 + \frac{z \log \theta}{y}) \log \theta - \log \theta - (\log \theta)^2 z
\]
\[
= z \left(1 - \frac{1}{y} - 1\right) (\log \theta)^2 > 0.
\]
Therefore, with our hypotheses
\[
(m_p(\alpha))^p > 1 + \frac{(\log \theta)^2}{2} \left(\frac{p}{d}\right)^2 > 1 + 19.64 \left(\frac{p\delta(d)}{d}\right)^2
\]
for \(d > d_5(p)\). This completes the proof of Theorem 3. \(\square\)
REFERENCES


Arturas Dubickas
Department of Mathematics,
Vilnius University,
Naugarduko 24,
Vilnius 2006, Lithuania
E-mail: arturas.dubickas@maf.vu.lt