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Linear fractional transformations of continued fractions with bounded partial quotients


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Linear Fractional Transformations of Continued Fractions with Bounded Partial Quotients

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RéSUMÉ. Soit θ un nombre réel de développement en fraction continue

\[ \theta = [a_0, a_1, a_2, \ldots], \]

et soit

\[ M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \]

une matrice d'entiers tel que \( \det M \neq 0 \). Si θ est à quotients partiels bornés, alors \[ \frac{a_0 + b}{c_0 + d} = [a^*_0, a^*_1, a^*_2, \ldots] \] est aussi à quotients partiels bornés. Plus précisément, si \( a_j \leq K \) pour tout \( j \) suffisamment grand, alors \( a^*_j \leq \frac{1}{|\det(M)|(K + 2)} \) pour tout \( j \) suffisamment grand. Nous donnons aussi une borne plus faible qui est valable pour tout \( a^*_j \) avec \( j \geq 1 \). Les démonstrations utilisent la constante d’approximation diophantienne homogène \( L_\infty(\theta) = \limsup_{q \to \infty} (q||q\theta||)^{-1} \). Nous montrons que

\[ \frac{1}{|\det(M)|} L_\infty(\theta) \leq L_\infty \left( \frac{a\theta + b}{c\theta + d} \right) \leq |\det(M)| L_\infty(\theta). \]

ABSTRACT. Let θ be a real number with continued fraction expansion

\[ \theta = [a_0, a_1, a_2, \ldots], \]

and let

\[ M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \]

be a matrix with integer entries and nonzero determinant. If θ has bounded partial quotients, then \[ \frac{a_0 + b}{c_0 + d} = [a^*_0, a^*_1, a^*_2, \ldots] \] also has bounded partial quotients. More precisely, if \( a_j \leq K \) for all sufficiently large \( j \), then \( a^*_j \leq \frac{1}{|\det(M)|(K + 2)} \) for all sufficiently large \( j \). We also give a weaker bound valid for all \( a^*_j \) with \( j \geq 1 \). The proofs use the homogeneous Diophantine approximation constant \( L_\infty(\theta) = \limsup_{q \to \infty} (q||q\theta||)^{-1} \). We show that

\[ \frac{1}{|\det(M)|} L_\infty(\theta) \leq L_\infty \left( \frac{a\theta + b}{c\theta + d} \right) \leq |\det(M)| L_\infty(\theta). \]

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1. INTRODUCTION.

Let \( \theta \) be a real number whose expansion as a simple continued fraction is
\[
\theta = [a_0, a_1, a_2, \ldots],
\]
and set
\[
K(\theta) := \sup_{i \geq 1} a_i.
\]
where we adopt the convention that \( K(\theta) = +\infty \) if \( \theta \) is rational. We say that \( \theta \) has bounded partial quotients if \( K(\theta) \) is finite. We also set
\[
K_\infty(\theta) := \limsup_{i \geq 1} a_i,
\]
with the convention that \( K_\infty(\theta) = +\infty \) if \( \theta \) is rational. Certainly \( K_\infty(\theta) \leq K(\theta) \), and \( K_\infty(\theta) \) is finite if and only if \( K(\theta) \) is finite.

A survey of results about real numbers with bounded partial quotients is given in [17]. The property of having bounded partial quotients is equivalent to \( \theta \) being a badly approximable number, which is a number \( \theta \) such that
\[
\liminf_{q \to \infty} q ||q\theta|| > 0,
\]
in which \( ||x|| = \min(x - \lfloor x \rfloor, \lfloor x \rfloor - x) \) denotes the distance from \( x \) to the nearest integer and \( q \) runs through integers.

This note proves two quantitative versions of the theorem that if \( \theta \) has bounded partial quotients and \( M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) is an integer matrix with \( \det(M) \neq 0 \), then \( \psi = \frac{a\theta + b}{c\theta + d} \) also has bounded partial quotients.

The first result bounds \( K_\infty(\frac{a\theta + b}{c\theta + d}) \) in terms of \( K_\infty(\theta) \) and depends only on \( |\det(M)| \):

**THEOREM 1.1.** Let \( \theta \) have a bounded partial quotients. If \( M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) is an integer matrix with \( \det(M) \neq 0 \), then
\[
\frac{1}{|\det(M)|} K_\infty(\theta) - 2 \leq K_\infty \left( \frac{a\theta + b}{c\theta + d} \right) \leq |\det(M)|(K_\infty(\theta) + 2).
\]

The second result upper bounds \( K(\frac{a\theta + b}{c\theta + d}) \) in terms of \( K(\theta) \), and depends on the entries of \( M \):
THEOREM 1.2. Let $\theta$ have bounded partial quotients. If $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is an integer matrix with $\det(M) \neq 0$, then

$$K \left( \frac{a\theta + b}{c\theta + d} \right) \leq |\det(M)|(K(\theta) + 2) + |c(c\theta + d)|.$$  

(1.4)

The last term in (1.4) can be bounded in terms of the partial quotient $a_0$ of $\theta$, since

$$|c\theta + d| \leq |c||(a_0| + 1) + |d| \leq |ca_0| + |c| + |d|.$$  

Theorem 1.2 gives no bound for the partial quotient $a_0^* := \lfloor \frac{a\theta + b}{c\theta + d} \rfloor$ of $\frac{a\theta + b}{c\theta + d}$. Chowla [3] proved in 1931 that $K(\frac{a}{d}\theta) < 2ad(K(\theta) + 1)^3$, a result rather weaker than Theorem 1.2.

We obtain Theorem 1.1 and Theorem 1.2 from stronger bounds that relate Diophantine approximation constants of $\theta$ and $\frac{a\theta + b}{c\theta + d}$, which appear below as Theorem 3.2 and Theorem 4.1, respectively. Theorem 3.2 is a simple consequence of a result of Cusick and Mendès France [5] concerning the Lagrange constant of $\theta$ (defined in Section 2).

The continued fraction of $\frac{a\theta + b}{c\theta + d}$ can be directly computed from that for $\theta$, as was observed in 1894 by Hurwitz [9], who gave an explicit formula for the continued fraction of $2\theta$ in terms of that of $\theta$. In 1912 Châtelet [2] gave an algorithm for computing the continued fraction of $\frac{a\theta + b}{c\theta + d}$ from that of $\theta$, and in 1947 Hall [7] also gave a method. Let $M(n, \mathbb{Z})$ denote the set of $n \times n$ integer matrices. Raney [15] gave for each $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M(2, \mathbb{Z})$ with $\det(M) \neq 0$ an explicit finite automaton to compute the additive continued fraction of $\frac{a\theta + b}{c\theta + d}$ from the additive continued fraction of $\theta$.

In connection with the bound of Theorem 1.1, Davenport [6] observed that for each irrational $\theta$ and prime $p$ there exists some integer $0 \leq a < p$ such that $\theta' = \theta + \frac{a}{p}$ has infinitely many partial quotients $a_n(\theta') \geq p$. Mendès France [13] then showed that there exists some $\theta' = \theta + \frac{a}{p}$ having the property that a positive proportion of the partial quotients of $\theta'$ have $a_n(\theta') \geq p$.

Some other related results appear in Mendès France [11,12]. Basic facts on continued fractions appear in [1,8,10,18].

2. BADLY APPROXIMABLE NUMBERS

Recall that the continued fraction expansion of an irrational real number
\[ \theta = [a_0, a_1, \ldots] \text{ is determined by} \]
\[ \theta = a_0 + \theta_0 , \quad 0 < \theta_0 < 1 , \]

and for \( n \geq 1 \) by the recursion
\[ \frac{1}{\theta_{n-1}} = a_n + \theta_n , \quad 0 < \theta_n < 1 . \]

The \( n \)-th complete quotient \( \alpha_n \) of \( \theta \) is
\[ \alpha_n := \frac{1}{\theta_n} = [a_n, a_{n+1}, a_{n+2}, \ldots] . \]

The \( n \)-th convergent \( \frac{p_n}{q_n} \) of \( \theta \) is
\[ \frac{p_n}{q_n} = [a_0, a_1, \ldots, a_n] , \]
whose denominator is given by the recursion \( q_{-1} = 0, q_0 = 1, \) and \( q_{n+1} = a_{n+1}q_n + q_{n-1} \). It is well known (see [8, §10.7]) that
\[ ||q_n \theta|| = \left| q_n \theta - p_n \right| = \frac{1}{q_n \alpha_{n+1} + q_{n-1}} . \]

Since \( a_{n+1} \leq \alpha_{n+1} < a_{n+1} + 1 \) and \( q_{n-1} \leq q_n \), this implies that
\[ \frac{1}{a_{n+1} + 2} < q_n ||q_n \theta|| \leq \frac{1}{a_{n+1}} , \]
for \( n \geq 0 \).

We consider the following Diophantine approximation constants. For an irrational number \( \theta \) define its type \( L(\theta) \) by
\[ L(\theta) = \sup_{q \geq 1} (q ||q \theta||)^{-1} , \]
and define the homogeneous Diophantine approximation constant or Lagrange constant \( L_\infty(\theta) \) of \( \theta \) by
\[ L_\infty(\theta) = \limsup_{q \geq 1} (q ||q \theta||)^{-1} . \]
We use the convention that if $\theta$ is rational, then $L(\theta) = L_\infty(\theta) = +\infty$. (N.B.: some authors study the reciprocal of what we have called the Lagrange constant.)

The best approximation properties of continued fraction convergents give

\begin{equation}
L(\theta) = \sup_{n \geq 0} (q_n||q_n\theta||)^{-1}
\end{equation}

and

\begin{equation}
L_\infty(\theta) = \limsup_{n \geq 0} (q_n||q_n\theta||)^{-1}.
\end{equation}

The set of values taken by $L_\infty(\theta)$ over all $\theta$ is called the Lagrange spectrum [4]. It is well known that $L_\infty(\theta) \geq \sqrt{5}$ for all $\theta$. If $\theta = [a_0, a_1, a_2, \ldots]$, then another formula for $L_\infty(\theta)$ is

\begin{equation}
L_\infty(\theta) = \limsup_{j \to \infty} ([a_j, a_{j+1}, \ldots] + [0, a_{j-1}, a_{j-2}, \ldots, a_1]);
\end{equation}

see [4, p. 1].

There are simple relations between these quantities and the partial quotient bounds $K(\theta)$ and $K_\infty(\theta)$, cf. [16, pp. 22--23].

**Lemma 2.1.** For any irrational $\theta$ with bounded partial quotients, we have

\begin{equation}
K(\theta) \leq L(\theta) \leq K(\theta) + 2.
\end{equation}

**Proof.** This is immediate from (2.2) and (2.3). \qed

**Lemma 2.2.** For any irrational $\theta$ with bounded partial quotients

\begin{equation}
K_\infty(\theta) \leq L_\infty(\theta) \leq K_\infty(\theta) + 2.
\end{equation}

**Proof.** This is immediate from (2.2) and (2.4). \qed

Although we do not use it in the sequel, we note that both inequalities in (2.7) can be slightly improved. Since $q_n \leq (a_n + 1)q_{n-1}$, (2.1) yields

\[ q_n||q_n\theta|| \leq \frac{1}{a_{n+1} + \frac{q_{n-1}}{q_n}} \leq \frac{1}{a_{n+1} + 1/(a_n + 1)} . \]
Since $a_n \leq K_\infty(\theta)$ from some point on, this and (2.4) yield

$$L_\infty(\theta) \geq K_\infty(\theta) + \frac{1}{K_\infty(\theta) + 1}.$$ 

Next, from (2.1) we have

$$q_n ||q_n\theta|| = \frac{q_n}{a_{n+1}q_n + q_{n-1}} = \frac{1}{a_{n+1} + \frac{1}{a_{n+2}} + \frac{q_{n-1}}{q_n}}.$$ 

Hence

$$(q_n ||q_n\theta||)^{-1} = a_{n+1} + \frac{1}{a_{n+2}} + \frac{q_{n-1}}{q_n}.$$ 

Let $K = K_\infty(\theta)$. Then for all $n$ sufficiently large we have

$$a_{n+2} \geq 1 + \frac{1}{K + 1} = \frac{K + 2}{K + 1},$$

so

$$(q_n ||q_n\theta||)^{-1} \leq K + \frac{K + 1}{K + 2} + 1 = K + 2 - \frac{1}{K + 2}.$$ 

We conclude that

$$L_\infty(\theta) \leq K_\infty(\theta) + 2 - \frac{1}{K_\infty(\theta) + 2}.$$ 

3. LAGRANGE CONSTANTS AND PROOF OF THEOREM 1.1.

An integer matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $\det(M) \neq 0$, acts as a linear fractional transformation on a real number $\theta$ by

$$M(\theta) := \frac{a\theta + b}{c\theta + d}.$$ 

Note that $M_1(M_2(\theta)) = M_1M_2(\theta)$. 
**Lemma 3.1.** If $M$ is an integer matrix with $\det(M) = \pm 1$, then the Lagrange constants of $\theta$ and $M(\theta)$ are related by

$$L_\infty(M(\theta)) = L_\infty(\theta).$$

**Proof.** This is well-known, cf. [14] and [5, Lemma 1], and is deducible from (2.5).

The main result of Cusick and Mendès France [5] yields:

**Theorem 3.2.** For any integer $m \geq 1$, let

$$G_m = \{M \in M(2, \mathbb{Z}) : |\det(M)| = m\}.$$ 

Then for any irrational number $\theta$,

$$\sup_{M \in G_m} (L_\infty(M(\theta))) = mL_\infty(\theta).$$

and

$$\inf_{M \in G_m} (L_\infty(M(\theta))) \geq \frac{1}{m}L_\infty(\theta).$$

**Proof.** Theorem 1 of [5] states that

$$\max_{\frac{a}{d} = m, 0 \leq b < d} \left( L_\infty\left( \frac{a\theta + b}{d} \right) \right) = mL_\infty(\theta).$$

Let $GL(2, \mathbb{Z})$ denote the group of $2 \times 2$ integer matrices with determinant $\pm 1$. We need only observe that for any $M$ in $G_m$ there exists some $\tilde{M} \in GL(2, \mathbb{Z})$ such that $\tilde{M}M = \begin{bmatrix} a' & b' \\ 0 & d' \end{bmatrix}$ with $a'd' = m$ and $0 \leq b' < d'$. For if so, and $\psi = \frac{a\theta + b}{c\theta + d}$, then Lemma 3.1 gives

$$L_\infty(\psi) = L_\infty(\tilde{M}(\psi)) = L_\infty(\tilde{M}M(\theta)) = L_\infty\left( \frac{a'\theta + b'}{d'} \right),$$

whence (3.4) implies (3.2). To construct $\tilde{M} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, we must have

$$Ca + Dc = 0.$$
Take $C = \frac{\text{lcm}(a,c)}{a}$ and $D = -\frac{\text{lcm}(a,c)}{c}$. Then $\gcd(C, D) = 1$, so we may complete this row to a matrix $\tilde{M} \in GL(2, \mathbb{Z})$. Multiplying this by a suitable matrix $\begin{pmatrix} \pm 1 & c \\ 0 & \pm 1 \end{pmatrix}$ yields the desired $\tilde{M}$.

The lower bound (3.3) follows from the upper bound (3.2). We use the adjoint matrix

$$M' = \text{adj}(M) = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix},$$

which has $M'M = \text{det}(M)I = mI$ and $\text{det}(M') = \text{det}(M)$. If $\theta' = M(\theta)$, then

$$M'(\theta') = M'(M(\theta)) = M'M(\theta) = \theta.$$

We prove by contradiction. Suppose (3.3) were false, so that for some $M \in G_m$ and some $\theta$ we have

$$L_\infty(M(\theta)) < \frac{1}{m} L_\infty(\theta).$$

This states that

$$mL_\infty(\theta') < L_\infty(M'(\theta')),$$

which contradicts (3.2) for $\theta'$, since $\text{det}(M') = \text{det}(M) = m$. \hfill \Box

**Remark.** The lower bound (3.3) holds with equality for some values of $\theta$ and not for other values. If for given $\theta$ we choose an $M \in G_m$ which gives equality in (3.2), so that $L_\infty(M(\theta)) = mL_\infty(\theta)$, then equality holds in (3.3) for $\theta' = \text{adj}(M)(\theta)$. However, if $L_\infty(\theta) = \sqrt{5}$, as occurs for $\theta = \frac{1 + \sqrt{5}}{2}$, then $L_\infty(M(\theta)) > L_\infty(\theta)$ for all $M$; hence (3.3) does not hold with equality when $m \geq 2$.

**Proof of Theorem 1.1.** Theorem 3.2 gives $L_\infty(M(\theta)) \leq \text{det}(M)L_\infty(\theta)$. Now apply Lemma 2.2 twice to get

$$K_\infty(M(\theta)) \leq L_\infty(M(\theta))$$

$$\leq |\text{det}(M)|L_\infty(\theta)$$

$$\leq |\text{det}(M)|(K_\infty(\theta) + 2).$$

(3.5)

To obtain the lower bound, we use the adjoint $M' = \text{adj}(M) = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$, and apply (3.5) with $M'$ and $\theta' = M(\theta)$ to obtain

$$K_\infty(\theta) = K_\infty(M'(M(\theta))) \leq |\text{det}(M')|(K_\infty(M(\theta)) + 2).$$
Since $|\det(M)| = |\det(M')|$, this yields

$$K_{\infty}(M(\theta)) \geq \frac{1}{|\det(M)|}K_{\infty}(\theta) - 2. \quad \Box$$

4. NUMBERS OF BOUNDED TYPE AND PROOF OF THEOREM 1.2

Recall that the type $L(\theta)$ of $\theta$ is the smallest real number such that $q||q\theta|| \geq \frac{1}{L(\theta)}$ for all $q \geq 1$.

**Theorem 4.1.** Let $\theta$ have bounded partial quotients. If $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is an integer matrix with $\det(M) \neq 0$, then

$$L\left( \frac{a\theta + b}{c\theta + d} \right) \leq |\det(M)|L(\theta) + |c(c\theta + d)| .$$

**Proof.** Set $\psi = \frac{a\theta + b}{c\theta + d}$. Suppose first that $c = 0$ so that $\det(M) = |ad| > 0$. Then $L(\psi) \geq \frac{1}{x}$, where

$$x := q||q\psi|| = q||q\left( \frac{a\theta + b}{d} \right)|| = q||q\left( \frac{a\theta + b}{d} \right) - p|| .$$

We have

$$|ad|x = |aq| |aq\theta + (bq - dp)|$$

$$\geq |aq|||aq\theta|| \geq \frac{1}{L(\theta)} .$$

For any $\epsilon > 0$ we may choose $q$ in (4.2) so that $\frac{1}{x} \geq L(\psi) - \epsilon$. Then

$$|\det(M)|L(\theta) = |ad||L(\theta)| \geq \frac{1}{x} \geq L(\psi) - \epsilon .$$

Letting $\epsilon \to 0$ yields (4.1) when $c = 0$.

Suppose now that $c \neq 0$. Again $L(\psi) \geq \frac{1}{x}$ where

$$x := q||q\psi|| = q||q\left( \frac{a\theta + b}{c\theta + d} \right) - p|| .$$
We have

\[(4.5) \quad |c\theta + d| x = q |(qa - pc)\theta - (pd - qb)| ,\]

so that

\[|c\theta + d| \begin{vmatrix} \frac{qa - pc}{q} \end{vmatrix} x = |qa - pc| |(qa - pc)\theta - (pd - qb)| \]

\[(4.6) \quad \geq |qa - pc| \| (qa - pc)\theta \| .\]

We first treat the case \(qa - pc = 0\). Now

\[
\begin{bmatrix} a & -c \\ -b & d \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} = \begin{bmatrix} qa - pc \\ pd - qb \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} ,
\]

since \(\text{det} \begin{bmatrix} a & -c \\ -b & d \end{bmatrix} = \text{det}(M) \neq 0\). Thus if \(qa - pc = 0\) then \(|pd - qb| \geq 1\), hence (4.5) gives

\[(4.7) \quad |c\theta + d| x = q|pd - qb| \geq 1 .\]

It follows that \(qa - pc \neq 0\) provided that

\[(4.8) \quad \frac{1}{x} > |c\theta + d| .\]

We next treat the case when \(qa - pc \neq 0\). Now from the definition of \(L(\theta)\) we see

\[(4.9) \quad |qa - pc| \| (qa - pc)\theta \| \geq \frac{1}{L(\theta)} .\]

Given \(\epsilon > 0\), we may choose \(q\) so that \(\frac{1}{x} \geq L(\psi) - \epsilon\), and we obtain from (4.6) and (4.9) that

\[(4.10) \quad |c\theta + d| \begin{vmatrix} \frac{qa - pc}{q} \end{vmatrix} L(\theta) \geq \frac{1}{x} \geq L(\psi) - \epsilon .\]

However, the bound

\[\left| q \left( \frac{a\theta + b}{c\theta + d} \right) - p \right| \leq \frac{1}{2} \]
implies that
\[
\left| \frac{qa - pc}{c} \right| = \left| q \left( \frac{a}{c} \right) - p \right| \leq \left| q \left( \frac{a\theta + b}{c\theta + d} \right) - q \left( \frac{a}{c} \right) \right| + \left| q \left( \frac{a}{c} \right) - p \right|
\leq q |\det(M)| \left| \frac{1}{c(c\theta + d)} \right| + \frac{1}{2}.
\]

Multiplying this by $\frac{c}{q}$ and applying it to the left side of (4.10) yields
\[
(4.11) \quad L \left( \frac{a\theta + b}{c\theta + d} \right) - \epsilon \leq |\det(M)|L(\theta) + \frac{1}{2} \frac{|c(c\theta + d)|}{q}.
\]

Letting $\epsilon \to 0$ and using $q \geq 1$ yields
\[
(4.12) \quad L \left( \frac{a\theta + b}{c\theta + d} \right) \leq |\det(M)|L(\theta) + \frac{1}{2} |c(c\theta + d)|,
\]
provided that (4.8) holds. Now (4.8) fails to hold only if
\[
(4.13) \quad L \left( \frac{a\theta + b}{c\theta + d} \right) \leq |c\theta + d|.
\]

The last two inequalities imply (4.1) when $c \neq 0$. \(\square\)

**Proof of Theorem 1.2.** Applying Theorem 4.1 and Lemma 2.1 gives
\[
K \left( \frac{a\theta + b}{c\theta + d} \right) \leq L \left( \frac{a\theta + b}{c\theta + d} \right) \leq |\det(M)|L(\theta) + |c(c\theta + d)| \leq |\det(M)|(K(\theta) + 2) + |c(c\theta + d)|,
\]
which is the desired bound. \(\square\)

**Remarks.** (1). The proof method of Theorem 4.1 can also be used to directly prove the bounds
\[
(4.14) \quad \frac{1}{|\det(M)|}L_\infty(\theta) \leq L_\infty(M(\theta)) \leq |\det(M)|L_\infty(\theta),
\]
of Theorem 3.2, from which Theorem 1.1 can be easily deduced. The lower bound in (4.14) follows from the upper bound as in the proof of Theorem 3.2. We sketch a proof of the upper bound in (4.14) for the case
\( \psi = \frac{a\theta + b}{c\theta + d} \) with \( c \neq 0 \). For any \( \epsilon^* > 0 \) and all sufficiently large \( q^* \geq q^*(\epsilon^*) \), we have

\[
q^* ||q^*\theta|| \geq \frac{1}{L_\infty(\theta) + \epsilon^*}.
\]

We choose \( q = q_n(\psi) \) for sufficiently large \( n \), and note that

\[
q^* = |q_n(\psi)a - p_n(\psi)c| \to \infty
\]
as \( n \to \infty \), since \( \psi \) is irrational. We can then replace (4.9) by (4.15), and then deduce (4.11) with \( L(\theta) \) replaced by \( L_\infty(\theta) + \epsilon^* \). Letting \( q \to \infty \), \( \epsilon \to 0 \) and \( \epsilon^* \to 0 \) in that order yields the upper bound in (4.14).

(2). For a given matrix \( M \) consider the set of attainable ratios

\[
\mathcal{V}(M) := \left\{ \frac{L_\infty(M(\theta))}{L_\infty(\theta)} : \theta \text{ has bounded partial quotients} \right\}.
\]

By Lemma 3.1 the set \( \mathcal{V}(M) \) depends only on its \( SL(2, \mathbb{Z}) \)-double coset

\[
[M] = \{ N_1MN_2 : N_1, N_2 \in SL(2, \mathbb{Z}) \}.
\]

Theorem 3.2 shows that

\[
\mathcal{V}(M) \subseteq \left[ \frac{1}{|\det(M)|} , \frac{1}{|\det(M)|} \right].
\]

It is an interesting open problem to determine the set \( \mathcal{V}(M) \). Both \( |\det(M)| \) and \( \frac{1}{|\det(M)|} \) lie in \( \mathcal{V}(M) \), as follows from Theorem 3.2 and the remark following it.

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