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On the discrepancy of Markov-normal sequences


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On the discrepancy of Markov-normal sequences

par M.B. LEVIN

RÉSUMÉ. On construit une suite normale de Markov dont la dis-
crépance est $O(N^{-1/2} \log^2 N)$, améliorant en cela un résultat don-
nant l’estimation $O(e^{-c(\log N)^{1/2}})$.

ABSTRACT. We construct a Markov normal sequence with a dis-
crepancy of $O(N^{-1/2} \log^2 N)$. The estimation of the discrepancy
was previously known to be $O(e^{-c(\log N)^{1/2}})$.

A number $\alpha \in (0, 1)$ is said to be normal to the base $q$, if in a $q$-ary
expansion of $\alpha$,

$$\alpha = .d_1d_2\cdots = \sum_{i=1}^{\infty} d_i/q^i, \quad d_i \in \{0, 1, \ldots, q-1\}$$

each fixed finite block of digits of length $k$ appears with an asymptotic
frequency of $q^{-k}$ along the sequence $(d_i)_{i \geq 1}$. Normal numbers were intro-
duced by Borel (1909). Borel proved that almost every number (in the
sense of Lebesgue measure) is normal to the base $q$. But only in 1935 did
Champernowne give the explicit construction of such a number, namely

$$\theta = .123456789101112\ldots.$$ 

obtained by successively concatenating all the natural numbers.

Let $P = (p_{i,j})_{0 \leq i, j \leq q-1}$ be an irreducible Markov transition matrix,
$(\mu_i)_{0 \leq i \leq q-1}$ the stationary probability vector of $P$ and $\mu$ its probability
measure.

A number $\alpha$ (sequence $(d_i)_{i \geq 1}$) is said to be Markov-normal if in a $q$-ary
expansion of $\alpha$ each fixed finite block of digits $b_0 b_1 \ldots b_k$ appears with an
asymptotic frequency of $p_{b_0} p_{b_1} \ldots p_{b_k-1}$. 

According to the individual ergodic theorem $\mu$-almost all sequences (num-
bbers) are normals.

Markov normal numbers were introduced by Postnikov and Piatecki-
Shapiro [1]. They also obtained, by generalizing Champernowne’s method,
the explicit construction of these numbers. Another Champernowne con-
struction of Markov normal numbers was obtained in Smorodinsky-Weiss

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[2] and in Bertrand-Mathis [3]. In [4] Chentsov gave the construction of Markov normal numbers using \textit{completely uniformly distributed sequences} (for the definition, see [5]) and the standard method of modelling Markov chains. In [6] Shahov proposed using a \textit{normal periodic systems of digits} (for the definition, see [5]) to construct Markov normal numbers. In [7] he obtained the estimate of discrepancy of the sequence \(\{\alpha q^n\}_{n=1}^N\) to be \(O(e^{-c(\log N)^{1/3}})\). In this article we construct a Markov normal sequence with the discrepancy of sequence \(\{\alpha q^n\}_{n=1}^N\) equal to \(O(N^{-1/2}\log^2 N)\).

Let \((x_n)_{n \geq 1}\) be a sequence of real numbers, \(\mu\) - measure on \([0,1)\). The quantity

\[
D(\mu, N) = \sup_{\gamma \in [0,1)} \left| \frac{1}{N} \# \{ n \in [1, N] \mid 0 \leq \{x_n\} < \gamma \} - \mu[0, \gamma] \right|
\]

is called the \textit{discrepancy} of \((x_n)_{n=1}^N\).

The sequence \(\{\{x_n\}\}_{n \geq 1}\) is said to be \(\mu\)-\textit{distributed} in \([0,1)\) if \(D(\mu, N) \to 0\).

Let the measure \(\mu\) be such that

\[
\mu((\gamma_n, \gamma_n + 1/2q^n)) = p_{e_1}p_{e_2}...p_{e_{n-1}e_n}, \quad \gamma_n = .c_1...c_n, \quad n = 1, 2, ...
\]

where \(c_k \in \{0, 1,...,q-1\}, \quad k = 1, 2,...\). It is known that if and only if \(\alpha\) is Markov normal number, the sequence \(\{\alpha q^n\}_{n=1}^\infty\) is \(\mu\)-distributed.

The discrepancy \(D(\mu, N)\) satisfies \(D(\mu, N) = O(N^{-1/2}(\log \log N)^{1/3})\) for almost all \(\alpha\).

The following facts are known from the theory of finite Markov chains [8,9]:

Let a Markov chain have \(d\) cyclic class \(C_1,...,C_d\). We enumerate the states \(e_1,...,e_q\) of the Markov chain in such a way, that if \(e_i \in C_m, \quad e_j \in C_n\) and \(i > j,\) then \(m \geq n\). Here matrix \(P\) has \(d^2\) blocks \((P_{i,j})_{0 \leq i,j \leq d-1}\), where \(P_{i,j} = 0\) except for \(P_{1,2}, P_{2,3}, P_{d-1,d}, P_{d,d}\). Matrix \(P^d\) has a block-diagonal structure. Let \(P_1,...,P_d\) be the block diagonal of matrix \(P^d\). There exists a number \(k_0\) such that all the elements of matrices \(P_i^{k_0}\) \((i = 1, ..., d)\) are greater than zero [9, ch. 4]. Let \(\theta\) be the minimal element of these matrices, and \(p_{ij}^{(k)}\) the \(ij\) element of matrix \(P^k\), \(k = 1, 2, ....\)

It is evident that

\[
\theta = \min_{i,j} p_{ij}^{(d)k_0},
\]

where we choose minimum values for \(i,j\) so that \(e_i, e_j\) are included in the same cyclic class.

Let \(f(j)\) be the number of cyclic class states \(e_j\) \((e_j \in C_{f(j)}, j = 0,...,q-1)\).
According to [9, ch.4] we obtain

\[ |p_{ij}^{(kd+f(j)-f(i))} - dp_{j}| \leq (1 - 2\theta)^{-1+k/k_0}, \]

\[ p_{ij}^{(kd+f(j)-f(i)+l)} = 0, \quad l = 1, 2, \ldots, d - 1, \quad k = 1, 2, \ldots. \]

Let

\[ p = \max_{0 \leq i, j \leq q-1} (p_{ij}, p_{ij}), \quad A_n = [p^{-n}], \quad n = 1, 2, \ldots. \]

We have, from the irreducibility of matrix \( P \), that

\[ p < 1 \quad \text{and} \quad A_n \to \infty. \]

We use matrices \( P_n = (p_{ij}(n))_{0 \leq i, j \leq q-1} \) with the rational elements

\[ p_{ij}(n) = v_{ij}(n)/A_n, \]

and we choose \( v_{ij}(n) \) as follows:

Let \( i \) be fixed and \( p_{ij} \) be greater than zero. Then we denote

\[ v_{ij}(n) = [A_n p_{ij}], \quad \text{if} \quad j \neq j_0, \quad \text{and} \quad v_{ij}(n) = A_n - \sum_{j \neq j_0} v_{ij}(n). \]

It is evident that

\[ \sum_{j=0}^{q-1} p_{ij}(n) = 1, \quad |v_{ij}(n) - A_n p_{ij}| \leq q, \quad i, j = 0, \ldots, q - 1, \quad n = 1, 2, \ldots. \]

If \( k_1 \) is sufficiently large, then using (3) and (6)-(8), we obtain

\[ \min_{ij} p_{ij}^{(dko)}(n) \geq \theta/2, \quad n > k_1, \]

where we choose minimum values for \( i, j \) so that \( e_i, e_j \) belong to the same cyclic class.

It is evident that \( P_n \) \((n > k_1)\) is an irreducible matrix with a \( d \)-cyclic class.

Applying (3), (4) and (9) we obtain

\[ |p_{ij}^{(kd+f(j)-f(i))}(n) - dp_{j}(n)| \leq (1 - \theta)^{-1+k/k_0}, \quad k = 1, 2, \ldots. \]

\[ p_{ij}^{(kd+f(j)-f(i)+l)}(n) = 0, \quad l = 1, 2, \ldots, d - 1, \quad i, j = 0, \ldots, q - 1, \]

where \( n > k_1 \), and \( (p_j(n))_{j<q} \) is the stationary probability vector of \( P_n \).

According to [7, 10] there exist integers \( v_0(n), \ldots, v_{q-1}(n), L_n > 0 \), such that

\[ p_j(n) = v_j(n)/L_n \quad v_0(n) + \ldots + v_{q-1}(n) = L_n \]
If \(k_1\) is sufficiently large, then applying (5)-(8) and (11), we obtain
\[
\max_{i,j} (p_i(n), p_{ij}(n))) \leq (p+1)/2 < 1, \quad n > k_1,
\]
\[
\min_{0 \leq i \leq q-1} p_i(n) \geq \bar{p} = 1/2 \min_{0 \leq i \leq q-1} p_i > 0.
\]

Let the measure \(\mu_n\) on \([0,1)\) be such that
\[
\mu_n([\gamma_r, \gamma_r + 1/q^r]) = p_{c_1}(n)p_{c_1c_2}(n)\ldots p_{c_{r-1}c_r}(n),
\]
\[
\gamma_r = c_1\ldots c_r, \quad n, r = 1, 2, \ldots
\]
where \(c_r \in \{0,1,\ldots,q-1\}, \quad r = 1, 2, \ldots\).

**Lemma 1.** Let \(\gamma = c_1\ldots c_n, \ldots\). Then
\[
\mu[0, \gamma] = \mu_n[0, \gamma_n] + O(np^n),
\]
\[
\mu[0, \gamma] = \mu_n[0, \gamma_n + 1/q^n] + O(np^n),
\]
where the \(O\)-constant depends only on \(P\).

**Proof.** It follows from (2), (5) and (6) that
\[
\mu[0, \gamma_n] + \sum_{r \geq n+1} \sum_{k=0}^{c_r-1} p_{c_1}p_{c_1c_2}\ldots p_{c_{r-1}b} = \mu[0, \gamma_n] + O(p^n).
\]

We apply (2), (14) and obtain
\[
\mu[0, \gamma_n] = \mu_n[0, \gamma_n] + \sum_{r=1}^{n} \sum_{k=0}^{c_r-1} \sigma_r(b),
\]
\[
\sigma_r(b) = p_{c_1}p_{c_1c_2}\ldots p_{c_{r-1}b} - p_{c_1}(n)p_{c_1c_2}(n)\ldots p_{c_{r-1}b}(n).
\]
If \(p_{c_1}p_{c_1c_2}\ldots p_{c_{r-1}b} = 0\), then \(p_{c_1c_j} = 0\) and according to (5), (7), (8), (11) we have \(p_{c_1c_j}(n) = O(p^n)\) and
\[
\sigma_r(b) = O(p^n).
\]

Let \(p_{c_1}p_{c_1c_2}\ldots p_{c_{r-1}b} \neq 0\). Then
\[
\sigma_r(b) = p_{c_1}p_{c_1c_2}\ldots p_{c_{r-1}b} \Delta_r,
\]
where
\[
\Delta_r = 1 - (1 + \frac{a_{c_1}(n) - L_np_{c_1}}{L_np_{c_1}}) \prod_{k=1}^{r-1} (1 + \frac{a_{c_kv_k}(n) - A_np_{c_kv_k}}{A_np_{c_kv_k}}),
\]
and \( v_k = c_{k+1} \) or \( b \).

On the basis of (5), (7), (8) and (11) we deduce that

\[
|\Delta_r| \leq \left(1 + \frac{B}{p \cdot A_n}\right)(1 + \frac{q}{p' \cdot A_n})^{r-1} - 1 \leq (1 + \varepsilon p^n)^r - 1, \quad p' = \min_{i,j \neq 0} p_{ij},
\]

where \(|\varepsilon| < 2qB/p'\).

It is easy to compute that

\[
\Delta_r = O(rp^n), \quad r \leq n.
\]

Hence and from (17) - (20) we obtain

\[
\mu(0, \gamma) - \mu_n[0, \gamma_n] = O(np^n + np^n \sum_{r=1}^{n} \sum_{k=0}^{c_r-1} p_{c_1c_2 \cdots p_{c_r-1}b}) = O(np^n)
\]

and formula (15) is proved. Statement (16) is proved analogously.

We obtain the Markov normal number \( \alpha = .d_1d_2 \ldots \) by concatenating blocks \( \alpha_n = (a_1, \ldots, a_{A_2n}) \), where \( a_i \in \{0, 1, \ldots, q - 1\}, \ i = 1, 2, \ldots \)

(21)

\[
\alpha = .\alpha_1' \ldots \alpha_n', \ldots
\]

We choose the numbers \( a_i \) as follows:

Let

(22)

\[
\Omega_n = \{\omega_n = (b_0, \ldots, b_{A_2n+n}) \mid b_0 \in \{0, \ldots, L_n - 1\}, \ b_1, b_2, \ldots \in \{0, \ldots, A_n - 1\}\}
\]

\[
S_0 = [0, v_0(n)), \ S_j = [v_0(n) + \ldots + v_{j-1}(n), v_0(n) + \ldots + v_j(n)),
\]

\[
S_{i,0} = [0, v_{i,0}(n)), \quad S_{i,j} = [v_{i,0}(n) + \ldots + v_{i,j-1}(n), v_{i,0}(n) + \ldots + v_{i,j}(n))
\]

(\(i = 0, \ldots, q - 1, \ j = 1, \ldots, q - 1\)).

We set \( a_0 = i, \) if \( b_0 \in S_i, \ i = 0, \ldots, q - 1. \) If we choose the numbers \( a_0, \ldots, a_{k-1}, \) then we set

(23)

\[
a_k = i, \ \text{if} \ b_k \in S_{a_{k-1},i}, \ i = 0, \ldots, q - 1.
\]

Let

(24)

\[
\alpha_n = \alpha_n(\omega_n) = .a_1, \ldots, a_{A_2n+n}, \quad n = 1, 2, \ldots
\]

(25)

\[
R(\beta, \gamma)(\mu_n, \alpha, M) = \#\{n \in [1, M] \mid \beta \leq \{\alpha q^n\} < \gamma\} - M \mu_n[\beta, \gamma],
\]

(26)

\[
E_n(\omega_n) = \max_{1 \leq M \leq A_2n} \max_{\gamma_n} |R(0, \gamma_n)(\mu_n, \alpha_n(\omega_n), M)|,
\]
We choose $\omega_n$ (and consequently $\alpha_n(\omega_n)$) such that

$$E_n = \min_{\omega_n \in \Omega_n} E_n(\omega_n).$$

Proof. (To follow later.)

Let

$$n_1 = 0, \ldots, n_{k+1} = n_k + A_{2k}, \quad k = 1, 2, \ldots.$$

Every natural $N$ can be represented uniquely in the following form with integers $k$

$$N = n_k + M_1, \quad 0 \leq M_1 < A_{2k}, \quad k = 1, 2, \ldots.$$

Let

$$T_\gamma(\alpha, Q, M) = \# \{ n \in (Q, Q + M) \mid \{\alpha q^n\} < \gamma \},$$

$$R_\gamma(\mu, \alpha, Q, M) = T_\gamma(\alpha, Q, M) - M \mu[0, \gamma).$$

For $Q = 0$ we use the symbols $T_\gamma(\alpha, M)$ and $R_\gamma(\mu, \alpha, M)$.

Theorem 1. Let the number $\alpha$ be defined by (21), (23), (24) and (27). Then $\alpha$ is Markov-normal and the following estimate is true:

$$D(\mu, N) = O(N^{-1/2} \log^2 N),$$

where the $O$-constant depends only on $P$.

Proof. Using (29), (30) and (31), we obtain

$$R_\gamma(\mu, \alpha, N) = \sum_{r=1}^{k-1} R_\gamma(\mu, \alpha, n_r, A_{2r}) + R_\gamma(\mu, \alpha, n_k, M_1).$$

According to (21), (24) and (25) we have

$$R_\gamma(\mu, \alpha, n_r, M) = R_\gamma(\mu, \alpha_r, M), \quad M < A_{2r} - 2r.$$

It follows from (31) that

$$R_\gamma(\mu, \alpha_r, M) = T_\gamma(\alpha_r, M) - M \mu[0, \gamma),$$

and

$$T_{\gamma_r}(\alpha_r, M) \leq T_\gamma(\alpha_r, M) \leq T_{\gamma_{r+1/q}}(\alpha_r, M).$$
It is evident that

\[ |R_\gamma(\mu, \alpha_r, M)| \leq |T_{\gamma_r}(\alpha_r, M) - M\mu[0, \gamma)| + |T_{\gamma_r+1/q^*}(\alpha_r, M) - M\mu[0, \gamma)|. \]

We apply (31) and obtain

\[ |R_\gamma(\mu, \alpha_r, M)| \leq |R_{\gamma_r}(\mu, \alpha_r, M)| + |R_{\gamma_r+1/q^*}(\mu, \alpha_r, M)| + M(\mu[0, \gamma) - \mu[0, \gamma_r)| + |\mu[0, \gamma) - \mu[0, \gamma_r + 1/q^*)|. \]

On the basis of (26)-(28), Lemma 1 and Lemma 2 we deduce that

\[ R_\gamma(\mu, \alpha_r, M) = O(p^{-r}r^2). \]

According to (34) we have for \( M < A_{2r - r} \)

(35) \[ R_\gamma(\mu, \alpha, n_r, M) = O(p^{-r}r^2). \]

It follows from (31) that

\[ R_\gamma(\mu, \alpha_r, M) = R_\gamma(\mu, \alpha_r, M - 2r) + O(r), \]

It is evident from this that statement (35) is valid both for \( M < A_{2r - 2r} \) as well as for \( M \in [A_{2r - 2r}, A_{2r}] \).

Substituting (35) into (33) and bearing in mind (30) we deduce

\[ R_\gamma(\mu, \alpha, N) = \sum_{r=1}^{k-1} O(p^{-r}r^2) + O(p^{-k}k^2) = O(p^{-k}k^2). \]

Using (29), (30) and (5) we obtain

\[ R_\gamma(\mu, \alpha, N) = O(N^{1/2} \log^2 N). \]

Hence and from (1), (31) the statement of the theorem follows.

We denote

(36) \[ \delta(a) = \begin{cases} 1, & \text{if } a = 0; \\ 0, & \text{otherwise}. \end{cases} \]

It is easy to see that

(37) \[ \delta(a) = \frac{1}{N} \sum_{m=1}^{N} e^{2\pi i am}, \quad 0 \leq a \leq N - 1. \]

\textbf{Lemma 3.} \textit{Let} \( 1 \leq M \leq A_{2n} \) \textit{and}

\[ G_M = \sum_{z=1}^{M} g_z. \]
Then

\begin{equation}
|G_M| \leq \sum_{m=0}^{A_{2n}-1} \frac{1}{m+1} \sum_{x=1}^{A_{2n}} g_x e^{2\pi i \frac{m(x-y)}{A_{2n}}}.
\end{equation}

Proof. According to (36) we have

\[G_M = \sum_{y=1}^{M} \sum_{z=1}^{A_{2n}} g_z \delta(x - y).\]

Using (37), we obtain

\begin{equation}
|G_M| = \left| \sum_{m=0}^{A_{2n}-1} \frac{1}{A_{2n}} \sum_{y=1}^{M} \sum_{z=1}^{A_{2n}} g_z e^{2\pi i \frac{m(x-y)}{A_{2n}}} \right| \leq
\end{equation}

\[\leq \sum_{m=0}^{A_{2n}-1} \frac{1}{A_{2n}} \left| \sum_{y=1}^{M} \sum_{z=1}^{A_{2n}} e^{2\pi i \frac{m(y-z)}{A_{2n}}} \right| \left| \sum_{x=1}^{A_{2n}} g_x e^{2\pi i \frac{mx}{A_{2n}}} \right|.
\]

Let $0 < N_2 - N_1 < A_{2n}$. It is known [5, p. 1] that

\begin{equation}
\frac{1}{A_{2n}} \left| \sum_{y=N_1}^{N_2} e^{2\pi i \frac{m(y-z)}{A_{2n}}} \right| \leq \min(1, \frac{1}{A_{2n}|\sin \frac{xm}{A_{2n}}|}) \leq \frac{1}{m+1}.
\end{equation}

From (39) and (40) we give the assertion of the lemma. ■

**Lemma 4.** Let $0 \leq u_1 \leq u_2 < A_{2n}$, $m \geq 0$ $i, j = 0, ..., q - 1$, $n > k_1$. Then

\[S = \sum_{x=u_1}^{u_2} e^{2\pi i \frac{mx}{A_{2n}}} (p_{ij}^{(x)}(n)/p_j(n) - 1) = O(1),
\]

where the constant in symbol $O$ depends only on $P$.

Proof. Let $N_1 = [u_1/d]$, $N_2 = [u_2/d]$. We change the variable $x = dy + z$ and obtain according to (13)

\[S = \frac{2d}{p} + \sum_{y=N_1}^{N_2} \sum_{z=1}^{d} e^{2\pi i \frac{m(y+z)}{A_{2n}}} (p_{ij}^{(dy+z)}(n)/p_j(n) - 1), \text{ where } |\epsilon| < 1
\]

Let

\[\sigma_y = \sum_{z=1}^{d} e^{2\pi i \frac{mz}{A_{2n}}} (p_{ij}^{(dy+z)}(n)/p_j(n) - 1).
\]

It follows that

\begin{equation}
S = \frac{2d}{p} + \sum_{y=N_1}^{N_2} e^{2\pi i \frac{mx+y}{A_{2n}}} \sigma_y.
\end{equation}
Applying (13), (10), we obtain

\[ \sigma_y = d e^{2\pi i \frac{m_1}{A_{2n}}} + \epsilon_1 \frac{d}{p_j(n)} (1 - \theta)^{-1+y/k_0} - \sum_{z=1}^{d} e^{2\pi i \frac{m_1}{A_{2n}}}, \]

where \(|\epsilon_1| < 1, \ z_1 = f(j) - f(i)\).

Substituting this formula into (41), we obtain according to (13), that

\[ (42) \quad S = S_1 S_2 + \epsilon_1 \sum_{y=N_1}^{N_2} \frac{d}{p} (1 - \theta)^{-1+y/k_0}, \quad |\epsilon_1| \leq 1, \]

where

\[ (43) \quad S_1 = \sum_{y=N_1}^{N_2} e^{2\pi i \frac{m_1}{A_{2n}}}, \quad S_2 = \sum_{z=1}^{d} (e^{2\pi i \frac{m_1}{A_{2n}}} - e^{2\pi i \frac{m_1}{A_{2n}}}). \]

It is known that

\[ (44) \quad |e^{2\pi i \frac{m_1(z_1-z)}{A_{2n}}} - 1| = 2|\sin \pi m(z_1-z)/A_{2n}| \leq 2\pi m d/A_{2n}. \]

Using (40) we get

\[ S_1 \leq A_{2n}/(md + 1). \]

Hence and from (42-44) the assertion of the lemma follows. \( \square \)

We consider further that \( a_i, \ i = 1, 2, ... \) is the sign of the number \( \alpha_n(\omega_n) \).

It follows from (25), that

\[ (45) \quad R_{[0,\gamma_\nu]}(\mu_n, \alpha_n, M) = \sum_{r=1}^{n} \sum_{b=0}^{c_r-1} R_{[\gamma_{r-1}+b/q^r, \gamma_{r-1}+(b+1)/q^r]}(\mu_n, \alpha_n, M), \]

and

\[ R_{[\gamma_{r-1}+b/q^r, \gamma_{r-1}+(b+1)/q^r]}(\mu_n, \alpha_n, M) = \]

\[ \sum_{z=1}^{M} \delta(a_{z+1} - c_1) ... \delta(a_{z+r} - b) - M \mu_n[\gamma_{r-1} + b/q^r, \gamma_{r-1} + (b+1)/q^r]. \]

Hence and from (45) we get

\[ (46) \quad R_{[0,\gamma_\nu]}(\mu_n, \alpha_n, M) = \]

\[ = \sum_{r=1}^{n} \sum_{b=0}^{c_r-1} M \sum_{z=1}^{M} (\delta(a_{z+1} - c_1) ... \delta(a_{z+r} - b) - \mu_n[\gamma_{r-1} + b/q^r, \gamma_{r-1} + (b+1)/q^r]). \]
LEMMA 5. Let \( n > k_1 \),

\[
B(r, c) = \sum_{z, y=1}^{A_2n} e^{2\pi iz\frac{r}{A_2n}} (\mu_n^2[\gamma_r, \gamma_r + \frac{1}{q^r}) + \sigma_1(x, y) - \mu_n[\gamma_r, \gamma_r + \frac{1}{q^r}](\sigma_2(x) + \sigma_2(y))),
\]

where

\[
\sigma_1(x, y) = \frac{1}{|\Omega_n|} \sum_{\omega_n \in \Omega_n} \delta(a_{x+1} - c_1)\delta(a_{x+r} - c_r)\delta(a_{y+1} - c_1)\delta(a_{y+r} - c_r),
\]

\[
\sigma_2(x) = \frac{1}{|\Omega_n|} \sum_{\omega_n \in \Omega_n} \delta(a_{x+1} - c_1)\delta(a_{x+r} - c_r).
\]

Then

\[
E_n \leq \frac{A_2n-1}{m+1} (\frac{Cn}{m})^{1/2} \left( \sum_{r=1}^{c_r-1} \sum_{m=0}^{e_r-1} \sum_{c_r=0}^{q-1} B(r, c) \right)^{1/2}.
\]

Proof. It follows from (46) and Lemma 3 that

\[
|R_{(0, \gamma_n)}(\mu_n, \alpha_n, M)| \leq \frac{1}{m+1} \left( \sum_{r=1}^{c_r-1} \sum_{m=0}^{e_r-1} \sum_{c_r=0}^{q-1} e^{2\pi iz\frac{r}{A_2n}} (\delta(a_{x+1} - c_1)\delta(a_{x+r} - b) - \mu_n[\gamma_{r-1} + b/q^r, \gamma_{r-1} + (b+1)/q^r]) \right).
\]

Changing the order of summation and applying the Cauchy inequality

\[
\left| \frac{1}{N} \sum_{n=1}^{N} g_n \right| \leq \left( \frac{1}{N} \sum_{n=1}^{N} |g_n|^2 \right)^{1/2},
\]

we obtain that

\[
|R_{(0, \gamma_n)}(\mu_n, \alpha_n, M)| \leq \frac{A_2n-1}{m+1} (\frac{Cn}{m})^{1/2} \left( \sum_{r=1}^{c_r-1} \sum_{m=0}^{e_r-1} \sum_{c_r=0}^{q-1} e^{2\pi iz\frac{r}{A_2n}} (\delta(a_{x+1} - c_1)\delta(a_{x+r} - b) - \\
- \mu_n[\gamma_{r-1} + b/q^r, \gamma_{r-1} + (b+1)/q^r]) \right)^{1/2}.
\]

We change the variable \( b \) to \( c_i \) and assume, on the right-hand side, the summation on \( c_i, i = 1, ..., r - 1 \).

It is evident that

\[
|R_{(0, \gamma_n)}(\mu_n, \alpha_n, M)| \leq \frac{A_2n-1}{m+1} (\frac{Cn}{m})^{1/2} \left( \sum_{r=1}^{c_r-1} \sum_{m=0}^{e_r-1} \sum_{c_r=0}^{q-1} \sum_{c_r=0}^{q-1} \sum_{c_r=0}^{q-1} \right).
\]
We denote by $S(cvn)$ the right-hand side of formula (52).
It is evident that $S(wn)$ does not depend on $M$ and $\gamma_n$.
Applying (26), we obtain
\[ E_n(\omega_n) \leq S(\omega_n) \]
and
\[ E_n \leq \frac{1}{|\Omega_n|} \sum_{\omega_n \in \Omega_n} E_n(\omega_n) \leq \frac{1}{|\Omega_n|} \sum_{\omega_n \in \Omega_n} S(\omega_n). \]
Changing the order of summation and using (51), we obtain
\[ E_n \leq \sum_{m=0}^{A_{2n}^{-1}} \frac{(q_n)^{1/2}}{m + 1} \left( \sum_{r=1}^{n} \sum_{c_1=0}^{q-1} \sum_{c_r=0}^{q-1} \frac{1}{|\Omega_n|} \sum_{\omega_n \in \Omega_n} \sum_{x=1}^{A_{2n}^{-1}} e^{2\pi i \frac{mx}{A_{2n}}} (\delta(a_{x+1} - c_1) \ldots \delta(a_{x+r} - c_r) - \mu_n[\gamma_r, \gamma_r + 1/q^r]) \right)^{1/2}. \]
Hence and from (47)-(49) we deduce formula (50). 

**Lemma 6.** Let $n > k_1$. Then
\[ \sigma_2(x) = \mu_n[\gamma_r, \gamma_r + 1/q^r]. \]

**Proof.** Applying (49) and (22), we get
\[ \sigma_2(x) = \frac{1}{L_n A_{2n}^{x+r}} \sum_{b_0=0}^{L_n-1} \sum_{b_1=0}^{A_n-1} \ldots \sum_{b_{x+r}=0}^{A_n-1} \delta(a_{x+1} - c_1) \ldots \delta(a_{x+r} - c_r). \]
According (23), we obtain
\[ a_{x+i} = c_i \quad \text{if and only if} \quad b_{x+i} \in S_{c_{i-1} c_i}, i = 2, 3, \ldots. \]
It follows that
\[ \sigma_2(x) = \frac{1}{L_n A_{2n}^{x+r}} \sum_{b_0=0}^{L_n-1} \sum_{b_1=0}^{A_n-1} \ldots \sum_{b_{x+1}=0}^{A_n-1} \delta(a_{x+1} - c_1) \sum_{b_{x+2} \in S_{c_1 c_2}} \ldots \sum_{b_{x+r} \in S_{c_{r-1} c_r}} 1 = \]
\[ = \frac{1}{L_n A_{2n}^{x+r}} \sum_{b_0=0}^{L_n-1} \sum_{b_1=0}^{A_n-1} \ldots \sum_{b_{x+1}=0}^{A_n-1} \delta(a_{x+1} - c_1) u_{c_1 c_2}(n) \ldots u_{c_{r-1} c_r}(n). \]
Using (7) we get
\[ \sigma(x) = \sigma p_{c_1 c_2}(n) \ldots p_{c_{r-1} c_r}(n), \]
where

\begin{equation}
\sigma = \frac{1}{L_n A_n^{x+1}} \sum_{b_0=0}^{L_n-1} \sum_{b_1=0}^{A_n-1} \cdots \sum_{b_{z+1}=0}^{A_n-1} \delta(a_{z+1} - c_1).
\end{equation}

It is obvious that

\begin{equation}
\sum_{d_0, \ldots, d_z=0}^{q-1} \prod_{i=0}^{z} \delta(a_i - d_i) = 1.
\end{equation}

Hence and from (55) we obtain, changing the order of summation

\begin{equation}
\sigma = \sum_{d_0, \ldots, d_z=0}^{q-1} \frac{1}{L_n A_n^{x+1}} \sum_{b_0=0}^{L_n-1} \sum_{b_1=0}^{A_n-1} \cdots \sum_{b_{z+1}=0}^{A_n-1} \prod_{i=0}^{z} \delta(a_i - d_i) \delta(a_{z+1} - c_1).
\end{equation}

According to (53), (36) and (22), we have

\begin{equation}
\sigma = \sum_{d_0, \ldots, d_z=0}^{q-1} \frac{1}{L_n A_n^{x+1}} \sum_{b_0 \in S_{d_0}} \sum_{b_1 \in S_{d_0 d_1}} \cdots \sum_{b_{z+1} \in S_{d_z c_1}} 1 =
\end{equation}

\begin{equation}
= \sum_{d_0, \ldots, d_z=0}^{q-1} \frac{1}{L_n A_n^{x+1}} v_{d_0}(n) v_{d_0 d_1}(n) v_{d_z c_1}(n).
\end{equation}

Applying (7) and (11), we obtain

\begin{equation}
\sigma = p_{c_1}(n).
\end{equation}

On the basis of (54) and (14) the lemma is proved. \hfill \blacksquare

**Lemma 7.** Let \( n > k_1, \ |y - x| > r. \) Then

\begin{equation}
\sigma_1(x, y) = \mu_n^2 [\gamma_r, \gamma_r + 1/q^r] p_{c_1 c_1}(n) / p_{c_1}(n).
\end{equation}

**Proof.** Let \( y > x. \)

Applying (48) and (22), we obtain \( \sigma_1(x, y) =
\end{equation}

\begin{equation}
= \frac{1}{L_n A_n^{x+r}} \sum_{b_0=0}^{L_n-1} \sum_{b_1=0}^{A_n-1} \cdots \sum_{b_{x+r+1}=0}^{A_n-1} \delta(a_{x+1} - c_1) \cdots \delta(a_{x+r} - c_r) \delta(a_{y+1} - c_1) \cdots \delta(a_{y+r} - c_r).
\end{equation}

As in the proof of Lemma 6, we get

\begin{equation}
\sigma_1(x, y) = p_{c_1}(n) (p_{c_1 c_1}(n) \cdots p_{c_r c_r}(n))^2 \sigma,
\end{equation}

where

\begin{equation}
\sigma = \frac{1}{A_n^{x+r}} \sum_{b_{x+r+1}=0}^{A_n-1} \cdots \sum_{b_{y+1}=0}^{A_n-1} \delta(a_{x+r} - c_r) \delta(a_{y+1} - c_1).
\end{equation}
As in (56), we have
\[ \sum_{d_1, \ldots, d_{y-z-r}=0}^{q-1} \prod_{i=1}^{y-z-r} \delta(a_{x+r+i}-d_i) = 1. \]
Hence and from (59), changing the order of summation, we obtain
\[ \sigma = \sum_{d_1, \ldots, d_{y-z-r}=0}^{q-1} \frac{1}{A_n^{y-z-r}} \sum_{b_{x+r+1}+1=0}^{A_n-1} \cdots \sum_{b_{y+1}+1=0}^{A_n-1} \delta(a_{x+r} - c_r) \times \]
\[ \times \prod_{i=1}^{y-z-r} \delta(a_{x+r+i} - d_i) \delta(a_{y+1} - c_1). \]
Using (53), (36) and (22), we get
\[ \sigma = \sum_{d_1, \ldots, d_{y-z-r}=0}^{q-1} \frac{1}{A_n^{y-z-r}} \sum_{b_{x+r+1+1} \in S_{c_r, d_1}} \cdots \sum_{b_{y+1+1} \in S_{d_{y-z-r}, c_1}} 1. \]
Applying (7) and (11), we obtain
\[ \sigma = \sum_{d_1, \ldots, d_{y-z-r}=0}^{q-1} p_{c, d_1}(n) p_{d_1, d_2}(n) \cdots p_{d_{y-z-r}, c_1}(n) = p_{c, c_1}^{(y-z-r)}(n). \]
It follows from (58), that
\[ \sigma_1(x, y) = (p_{c_1}(n)p_{c_1, c_2}(n) \cdots p_{c_{r-1}, c_r}(n))^2 p_{c, c_1}^{(y-z-r)}(n)/p_{c_1}(n). \]
Similarly for \( x < y \). According to (14) the lemma is proved. ■

**Lemma 8.** Let \( n > k_1 \), \( |y - x| \leq r \). Then
\[ \sigma_1(x, y) \leq \mu_n[\gamma_r, \gamma_r + 1/q^r](\frac{1+p}{2})|y-x|-1. \]

**Proof.** Let \( y \geq x \).
As in the proof of Lemma 6 and Lemma 7, we get
\[ \sigma_1(x, y) \leq p_{c_1}(n)p_{c_1, c_2}(n) \cdots p_{c_{y-x-1}, c_{y-x}}(n)p_{c_{y-x}, c_1}(n)p_{c_1, c_2}(n) \cdots p_{c_{r-1}, c_r}(n). \]
It follows from (12), that
\[ \sigma_1(x, y) \leq p_{c_1}(n)p_{c_1, c_2}(n) \cdots p_{c_{r-1}, c_r}(n)(\frac{1+p}{2})^{y-x-1}. \]
Similarly for \( x < y \). According to (14) the lemma is proved. ■

**Lemma 9.** Let \( n > k_1 \). Then
\[ B(r, c) = O(A_{2n} \mu_n[\gamma_r, \gamma_r + 1/q^r]). \]

\[ (60) \]
Proof. Applying (47) and Lemma 6, we obtain
\[ B(r, c) = \sum_{x,y=1}^{A_{2n}} \sigma(x,y), \]
where
\[ \sigma(x,y) = e^{2\pi i \frac{m(x-y)}{A_{2n}}} (\sigma_1(x,y) - \mu_n^2(\gamma_r, \gamma_r + \frac{1}{q^r})). \]
Let
\[ B_1 = \sum_{1 \leq x, y \leq A_{2n}, |y-x| \leq r} \sigma(x,y), \quad B_2 = \sum_{1 \leq x, y \leq A_{2n}, y-x \geq r} \sigma(x,y), \quad B_3 = \sum_{1 \leq x, y \leq A_{2n}, z-y \geq r} \sigma(x,y). \]
According to Lemma 8, (12) and (14) we obtain
\[ |B_1| \leq \mu_n(\gamma_r, \gamma_r + \frac{1}{q^r}) \sum_{1 \leq x, y \leq A_{2n}, |y-x| \leq r} \frac{1}{q^r} = O(A_{2n} \mu_n(\gamma_r, \gamma_r + \frac{1}{q^r})). \]

It follows from Lemma 7 that
\[ B_2 = \mu_n^2(\gamma_r, \gamma_r + 1/q^r) \sum_{x=1}^{A_{2n}} \sum_{y=x+r}^{A_{2n}} e^{2\pi i \frac{m(x-y)}{A_{2n}}} (p_{c, c_1}(y-x-r)(n)/p_{c_1}(n) - 1). \]
Changing the variable \( y \) to \( y_1 = y-x-r \) and applying Lemma 3, we obtain
\[ B_2 = O(A_{2n} \mu_n^2(\gamma_r, \gamma_r + 1/q^r)). \]
Similarly estimate is valid for \( B_3 \).
Hence and from (61)-(62) we obtain the assertion of the lemma. \( \square \)

Proof of Lemma 2. Substituting (60) into (50) and bearing in mind (5), we deduce
\[ E_n = O\left( \sum_{m=0}^{A_{2n}-1} \frac{(nq)^{1/2}}{m+1} \left( \sum_{r=1}^{n} \sum_{c_1=0}^{q-1} \cdots \sum_{c_r=0}^{q-1} A_{2n} \mu_n(\gamma_r, \gamma_r + \frac{1}{q^r}) \right)^{1/2} \right) = O(\sqrt{A_{2n}n} \sum_{m=0}^{A_{2n}-1} \frac{1}{m+1}) = O(p^{-n}n^2). \]
Lemma 2 is proved. \( \square \)

Remark. By a similar method and the method in [12] a Markov normal vector for the multidimensional case can be be constructed. By the method
in [12] one can reduce the logarithmic multiplier in (32) to \(O(\log N^{3/2})\). To reduce the logarithmic multiplier further see [15].

**Problem.** According to [12-14] the Borel and Bernoulli normal numbers exist with discrepancy \(O(N^{-2/3+\epsilon})\). It would be interesting to know whether Markov normal numbers exist with discrepancy \(O(N^{-\epsilon})\) where \(\epsilon > 1/2\).

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**REFERENCES**


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