Universal codes and unimodular lattices


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Universal Codes and Unimodular Lattices

par Robin Chapman et Patrick Solé

Résumé. Les codes résidus quadratiques binaires de longueur $p + 1$ produisent par construction $B$ et bourrage des réseaux de type II comme le réseau de Leech. Récemment, il a été prouvé que les codes résidus quadratiques quaternaires produisent les mêmes réseaux par construction $A$ modulo 4. Nous montrons de manière directe l’équivalence des deux constructions pour $p \leq 31$. En dimension 32 nous obtenons un réseau extrémal de type II qui n’est pas isomètre au réseau de Barnes-Wall $BW_{32}$. On considère également l’équivalence entre construction $B$ modulo 4 plus bourrage et construction $A$ modulo 8. En dimension 48 elles conduisent toutes deux à une nouvelle description du réseau extrémal de type II appelé $P_{48q}$.

Abstract. Binary quadratic residue codes of length $p + 1$ produce via construction $B$ and density doubling type II lattices like the Leech. Recently, quaternary quadratic residue codes have been shown to produce the same lattices by construction $A$ modulo 4. We prove in a direct way the equivalence of these two constructions for $p \leq 31$. In dimension 32, we obtain an extremal lattice of type II not isometric to the Barnes-Wall lattice $BW_{32}$. The equivalence between construction $B$ modulo 4 plus density doubling and construction $A$ modulo 8 is also considered. In dimension 48 they both led to a new description of the extremal type II lattice $P_{48q}$.

1. Introduction

In [2], Bonnecaze, Solé and Calderbank introduce for primes $p \equiv \pm 1 \pmod{8}$, codes $\widehat{Q}$ and $\widehat{N}$, the universal extended quadratic residue codes, of length $p + 1$ over the 2-adic integers $\mathbb{Z}_2$. For positive integers $s$ they consider their reductions $\widehat{Q}_2^s$ and $\widehat{N}_2^s$, modulo $2^s$; $\widehat{Q}_2$ and $\widehat{N}_2$ are just the standard binary extended quadratic residue codes, while $\widehat{Q}_4$ and $\widehat{N}_4$ are the quaternary quadratic residue codes. Given a code $C$ of length $n$ over $\mathbb{Z}_4$

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define $\Lambda(C)$ as the set of vectors in $Z^n$ which reduce modulo 4 to elements of $C$. If $p \equiv -1 \pmod{8}$ the lattice $\frac{1}{2}\Lambda(\bar{Q}_4)$ is even and unimodular ([2] Corollary 4.1); if $p = 7$ it is the $E_8$ lattice, while if $p = 23$ it is the Leech lattice.

Here we show, by means of an explicit isomorphism, that if $p \equiv -1 \pmod{8}$ and $p \leq 31$ then $\frac{1}{2}\Lambda(\bar{Q}_4)$ is isometric to a lattice $L(\bar{Q}_3)$ constructed from the binary quadratic residue code in a manner (construction $B$ plus density doubling) generalizing the original construction of the Leech lattice. If $p = 23$ this yields a short proof of what is perhaps the simplest construction of the Leech lattice [2]. If $p = 31$ this, combined with results of Koch and Venkov, shows that $BSBM_{32}$ introduced in [1] is not isometric to the Barnes-Wall lattice $BW_{32}$. In section §4 we consider a quaternary analogue of this situation, replacing construction $B$ by construction $B$ modulo 4, and construction $A$ modulo 4 by construction $A$ modulo 8. We show, inter alia, that $P_{48}$ can be obtained in the latter way from a quadratic residue code of length 48 over $Z_8$.

2. The main result

Throughout this section we assume that $p$ is a prime satisfying $p \equiv -1 \pmod{8}$. We also fix an integer $r$ such that $r \equiv 1 \pmod{4}$, and $r^2 + p \equiv 0 \pmod{32}$. In addition if $p \leq 31$ we will assume that $r^2 + p = 32$. (If $p = 7$, 23 or 31, then $r = 5$, -3 or 1 respectively.)

We first outline a construction of lattices from binary codes of length $p + 1$. Consider a self-orthogonal linear subcode $C$ of $Z_2^{p+1}$, containing the all-ones word. Define $L(C)$ to be the sublattice of $Z_2^{p+1}$ generated by the following types of vectors:

1. all vectors of shape $(8 \ 0^p)$,
2. all vectors of shape $(4^2 \ 0^{p-1})$,
3. all vectors of shape $(2^a \ 0^{p+1-a})$ whose support coincides with the support of an element of $C$,
4. any vector of shape $(1^p)$.

This can be recast as the union of two cosets

$$2B(C) \cup ((r \ 1^p) + 2B(C)),$$

of the lattice $2B(C)$ obtained, up to scaling, by construction $B$ applied to $C$ namely

$$B(C) := C + 2P_{p+1} + 4Z^{p+1},$$

with $P_{p+1}$ denoting the parity-check code of length $p + 1$. It is clear that $L(C)$ is a lattice, of index $4^{p+1}/|C|$ in $Z^{p+1}$. If the code $C$ is doubly even, then the norm of each vector in $L(C)$ is divisible by 16. It follows that if $C$ is
self-dual and doubly even then the lattice $\frac{1}{\sqrt{8}} L(C)$ is even and unimodular. If $C$ is $\hat{Q}_2$ or $\hat{N}_2$ then it has these properties. We give four examples of this construction.

<table>
<thead>
<tr>
<th>$p$</th>
<th>lattice</th>
<th>reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>Gosset</td>
<td>[7]</td>
</tr>
<tr>
<td>23</td>
<td>Leech</td>
<td>[6, p.131]</td>
</tr>
<tr>
<td>31</td>
<td>$BW_{32}$</td>
<td>[7, 9]</td>
</tr>
<tr>
<td>31</td>
<td>$BSBM_{32}$</td>
<td>[1]</td>
</tr>
</tbody>
</table>

By the norm of an element in Euclidean space we mean the square of its length, and the minimal norm of a lattice is the least norm of a non-zero element of the lattice. It is easy to see that the minimum norm of $\frac{1}{\sqrt{8}} L(\hat{Q}_2)$ is $\min(4, 2\lfloor \frac{p+1}{16} \rfloor, \frac{1}{2} \text{mw}(\hat{Q}_2))$ where $\text{mw}(C)$ is the minimum (Hamming) weight of the code $C$.

**Theorem 1.** The lattices $\frac{1}{2} \Lambda(\hat{Q}_4)$ and $\frac{1}{\sqrt{8}} L(\hat{Q}_2)$ are isometric for $p \leq 31$.

**Proof.** Assume $p \leq 31$. We recall the definition of $\hat{Q}$ from §III of [2]. Let $\delta$ be the square root of $-p$ in $\mathbb{Z}_2$ with $\delta \equiv -1 \pmod{4}$. Note then that $\delta \equiv -r \pmod{16}$. The vectors $m_\alpha$ ($\alpha \in F_p \cup \{\infty\}$) are defined as the rows of the matrix

$$M = \begin{pmatrix} \delta & 1 & \cdots & 1 \\ -1 & & & \\ \vdots & & W + \delta I & \\ -1 & & & \end{pmatrix}$$

where

$$W_{ij} = \left( \frac{j - i}{p} \right).$$

(The rows and columns of this matrix are labelled in the order $\infty, 0, 1, \ldots, p-1$.) The matrix $W$ is called a Jacobsthal matrix, and is instrumental in building Hadamard matrices of Paley type [10, Chap. II]. We collect here the properties that we need

(J1) $JW = WJ = 0$

(J2) $WW^T = pI - J$

(J3) $A := \sum_{i=0}^{p-1} W_{-i,1} = -1$

(J4) $B := \sum_{i=0}^{p-1} W_{i,1} = 0$

where $J$ stands for the all-one matrix. See [10, Chap. II, Lemma 7] for proofs of (J1) and (J2). To prove (J3), (J4) observe firstly that by (J1) we have, knowing that $-1$ is not a quadratic residue, that $A + B = -1$. 
Secondly, writing $\chi$ for the Jacobi symbol we have

$$B = \frac{1}{2} \sum_{x \in F_p, x \neq 0} \chi(1 - x^2)$$

and by the character property of $\chi$

$$B = \frac{1}{2} \sum_{x \in F_p, x \neq 0} \chi(1 - x)\chi(1 + x) = 0,$$

the last equality coming from $(J2)$.

The coordinate positions in the code are labelled $\infty, 0, 1, \ldots, p - 1$, regarded as elements of the projective line over $F_p$. The universal extended quadratic residue code is now defined as

$$\widehat{Q} = \left( \sum_{\alpha \in F_p \cup \{\infty\}} Q_{2\alpha} m_{\alpha} \right) \cap \mathbb{Z}_{2\infty}^{p+1},$$

where $Q_{2\infty}$ is the field of 2-adic numbers. A similar definition holds for $\widehat{N}$ with $W_{i,j}$ replaced by $W_{i,-j}$.

We can now describe $\Lambda(\widehat{Q}_4)$ as the set of vectors in $\mathbb{Z}^{p+1}$ congruent modulo 4 to elements of $\widehat{Q}$. Let $n_{\alpha} \in \mathbb{Z}^{p+1}$ be the rows of the matrix

$$N = \left( \begin{array}{cccc} -r & 1 & \cdots & 1 \\ -1 & & & \\ \vdots & & W - rI \\ -1 & & & \end{array} \right)$$

so that for $\alpha \in F_p \cup \{\infty\}$ we have $n_{\alpha} \equiv m_{\alpha} \pmod{16}$. Since by $(J2)$ we have $NN^t = 32I$ the matrix $\frac{1}{\sqrt{32}}N$ is orthogonal. We claim that this matrix maps $\frac{1}{\sqrt{8}}L(\widehat{N}_2)$ to $\frac{1}{2} \Lambda(\widehat{Q}_4)$. This is equivalent to saying that $N$ maps $\frac{1}{8}L(\widehat{N}_2)$ to $\Lambda(\widehat{Q}_4)$. Note that these codes and lattices are preserved by the automorphism $\sigma$ coming from the permutation $(0 1 2 \cdots p - 1)$ on $F_p \cup \{\infty\}$, and this automorphism maps $m_{\alpha}$ to $m_{\alpha+1}$ and $n_{\alpha}$ to $n_{\alpha+1}$. This automorphism is the shift in the cyclic construction of the QR codes. We proceed to show that the images by $N$ of the four types of vectors in construction $L$ above lie in $\Lambda(\widehat{Q}_4)$.

Since the $n_{\alpha} \in \Lambda(\widehat{Q}_4)$, the matrix $N$ takes the coordinate vectors, which lie in $\frac{1}{8}L(\widehat{N}_2)$, into $\Lambda(\widehat{Q}_4)$. For convenience let $(a, b; c; d)$ denote the vector with $\infty$-coordinate $a$, 0-coordinate $b$, and generic $\alpha$-coordinate $c$, and
generic \( \beta \)-coordinate \( d \) where \( \alpha \) and \( \beta \) are any quadratic residue, and quadratic non-residue respectively. Now

\[
\frac{1}{2}(n_\infty + n_0) = \left( \frac{r - 1}{2}, \frac{-r - 1}{2}; 0; -1 \right)
\]

which lies in \( \mathbb{Z}^{r+1} \) and is congruent to \( \frac{1}{2}(m_0 - m_\infty) \) modulo 8. Hence \( \frac{1}{2}(m_\infty + m_0) \in \Lambda(\hat{Q}_4) \). Applying \( \sigma \) it follows that \( \frac{1}{2}(m_\infty + m_\alpha) \in \Lambda(\hat{Q}_4) \) for all \( \alpha \in F_p \), and so \( \frac{1}{2}(m_\alpha + m_\beta) \in \Lambda(\hat{Q}_4) \) for all \( \alpha, \beta \in F_p \cup \{\infty\} \). Hence \( \frac{1}{8}vN \in \Lambda(\hat{Q}_4) \) for all \( v \) of the shape \((4^2 0^{p-1})\). We next compute

\[
\frac{1}{4} \left( n_\infty + \sum_{j \in Q'} n_j \right) = \left( \frac{-2r + p - 1}{8}, \frac{-p + 1}{8}; 0; -\frac{r - 1}{4} \right)
\]

where \( Q' \) is the set of quadratic non-residues modulo \( p \). The last coordinates come from (J3), (J4). Again this has integer coordinates, and is congruent modulo 4 to \( \frac{1}{4}(m_\infty + \sum_{j=0} m_j) \), so this vector lies in \( \Lambda(\hat{Q}_4) \). It follows that

\[
\frac{1}{4} \left( n_\infty + \sum_{j \in Q'} n_{j+k} \right) \in \Lambda(\hat{Q}_4)
\]

for each \( k \in F_p \). But \( \hat{N}_2 \) is generated by the vectors whose supports are the sets \( \{\infty\} \cup (k + Q') \). \((2, p.370, \text{III. A.})\). It follows that if \( v \) has the shape \((2^a 0^{p+1-a})\) and whose support is the same as that of an element of \( \hat{N}_2 \), then \( \frac{1}{8}vN \in \Lambda(\hat{Q}_4) \). Finally

\[
\frac{1}{8} \left( r n_\infty + \sum_{j \in F_p} n_j \right) = (-4, 0; 0; 0) \in \Lambda(\hat{Q}_4)
\]

and so \( \frac{1}{8}(r, 1; 1; 1)N \in \Lambda(\hat{Q}_4) \). Hence \( \frac{1}{8}L(\hat{N}_2)N \subseteq \Lambda(\hat{Q}_4) \), and comparing determinants we see that \( \frac{1}{8}L(\hat{N}_2)N = \Lambda(\hat{Q}_4) \). Since \( L(\hat{Q}_2) \) and \( L(\hat{N}_2) \) are isometric the Theorem follows. \( \square \)

3. Application to the cases of \( p = 23, 31 \).

If \((a_1, \ldots, a_n)\) is an element of a code over \( \mathbb{Z}_4 \), then its Euclidean weight is \( w(a_1) + \cdots + w(a_n) \) where

\[
w(a) = \begin{cases} 
0 & \text{if } a = 0, \\
1 & \text{if } a = \pm1, \\
4 & \text{if } a = 2.
\end{cases}
\]

The minimum Euclidean weight \( \text{mew}(C) \) of a code \( C \) over \( \mathbb{Z}_4 \) is the least Euclidean weight of its non-zero elements. If \( C \) is a linear code then the
minimum norm of $\Lambda(C)$ is $\min(16, \text{mew}(C))$. For $p = 23$ and $p = 31$, the minimum norm of $L(\tilde{N}_2)$ is 32, and so the minimum norm of $\Lambda(\tilde{Q}_4)$ is 16. Hence $\text{mew}(\tilde{Q}_4) \geq 16$. In [2] this is proved in a more elaborate way for $p = 23$.

In [8] Koch and Venkov show that for the five non-isomorphic doubly even self-dual binary codes $C_1, \ldots, C_5$ of length 32, the lattices $L(C_1), \ldots, L(C_5)$ are all non-isometric. We can take $C_1 = \tilde{Q}_2$, and $C_2$ to be the Reed-Muller code $RM(2, 5)$. Since $L(RM(2, 5))$ is isometric to the Barnes-Wall lattice $BW_{32}$ [9], it follows that $\frac{1}{2}\Lambda(\tilde{Q}_4)$ for $p = 31$ is not isometric to $BW_{32}$, confirming a conjecture of [1]. It is known that there are only two unimodular lattices in dimension 32 with minimal norm 4 and an automorphism of order 31 [12]. From the results of [1] and of the current paper we can infer than both can be constructed by construction $A$ mod 4 applied to an extended quaternary cyclic code: the quaternary Reed-Muller code $QRM(2, 5)$ in the case of $BW_{32}$ and the extended quadratic residue code $\tilde{Q}_4$ in the case of $BSBM_{32} := \frac{1}{2}\Lambda(\tilde{Q}_4)$. Both lattices also appear in [11, 4].

4. Quaternary Analogue

We assume in this § that $p \geq 47$ is a prime $\equiv -1 \pmod{8}$, and that the integer $r \equiv 1 \pmod{4}$ satisfies

$$r^2 + p = 96 = 16.6,$$

if $p = 47, 71$ and

$$r^2 + p = 128 = 16.8.$$

if $p = 79, 103, 127$. The corresponding values of $r$ are $r = -7, 5$ in first case and $r = -7, 5, 1$ in the second. For a quaternary code $C$ of length $p + 1$ we define

$$B_4(C) := C + 4P_{p+1} + 8Z^{p+1},$$

and

$$L_4(C) := 2B_4(C) \cup (\langle r^p \rangle + 2B_4(C)).$$

For an octonary code $C_8$ of length $p + 1$, we define

$$\Lambda_4(C_8) = C_8 + 8Z^{p+1}.$$

We have the following analogue of Theorem 1:

**Theorem 2.** The lattices $\frac{1}{4}L_4(\tilde{Q}_4)$ and $\frac{1}{\sqrt{8}}\Lambda_4(\tilde{Q}_8)$ are isometric for $p = 47, 71, 79, 103, 127$.

The proof is analogous to the proof of Theorem 1 and is omitted.

**Corollary 1.** For $p = 47$ the lattice $\frac{1}{\sqrt{8}}\Lambda(\tilde{Q}_8)$ has norm 6, and the code $\tilde{Q}_8$ has euclidean minimum weight 48.
Proof. Follows from the preceding theorem by noticing that $\hat{Q}_4$ has euclidean minimum weight 24 [1, 11, 5]. □

The lattice $L_4(\hat{Q}_4)$ was considered in [3] and is isometric to $P_{48q}$. Adopting the definition of $P_{48q}$ in §7.7 of [6], the orthogonal matrix

$$\frac{1}{\sqrt{96}} \begin{pmatrix} -7 & 1 & \cdots & 1 \\ -1 \\ \vdots & W - 7I \\ -1 \end{pmatrix}$$

takes $P_{48q}$ to $L_4(\widehat{N}_4)$ (which is isometric to $L_4(\hat{Q}_4)$) by a similar argument to Theorem 1. Similarly it is tantamount to conjecture that the conjectural extremal type II lattice of dimension 80 of example 3 of [13] is taken by

$$\frac{1}{\sqrt{128}} \begin{pmatrix} -7 & 1 & \cdots & 1 \\ -1 \\ \vdots & W - 7I \\ -1 \end{pmatrix}$$

into $L_4(\widehat{N}_4)$.

5. Conclusion

It would be interesting to lift the remaining three Conway-Pless codes over $\mathbb{Z}_4$ and obtain by construction $A_4$ the three remaining zero-defect lattices of the Koch-Venkov classification. Similarly the construction of $P_{48q}$ by construction $B_3$ applied to ternary QR codes and density doubling [6, p.149] suggests a construction modulo 6. Eventually, quaternary double circulant codes which produce an even extremal unimodular lattice in dimension 40 [5] should be amenable to a similar analysis.

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