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Mean Square of the Remainder Term
in the Dirichlet Divisor Problem

par Yuk-Kam LAU and Kai-Man TSANG

1. Introduction and Main Results

Let $d(n)$ denote the divisor function. In this paper we shall consider a remainder term associated with the mean square of the error term $\triangle(x)$ in the Dirichlet divisor problem, which is defined as

$$\triangle(x) = \sum_{n \leq x} d(n) - x(\log x + 2\gamma - 1) \ .$$

Here γ is Euler's constant. The upper bound $\triangle(x) \ll x^{1/2}$ was first obtained by Dirichlet in 1838. This was gradually sharpened by many authors in the ensuing one and a half century. Iwaniec and Mozzochi [5] proved in 1988 that $\triangle(x) \ll x^{7/22+\varepsilon}$ for any $\varepsilon > 0$, by employing intricated techniques for the estimation of certain exponential sums. Such methods, however, do not seem capable of proving the conjectured best bound: $\triangle(x) \ll x^{1/4+\varepsilon}$.

Besides this problem, there are plenty of papers written on other interesting properties of $\triangle(x)$. For instance, Tong [9] showed that $\triangle(x)$ changes sign at least once in every interval of the form $[X, X + c_0\sqrt{X}]$ where c_0 is a certain positive constant. Recently Heath-Brown and Tsang [2] showed that this is essentially best possible: – the length of the intervals cannot be reduced to $o(\sqrt{X} \log^{-5} X)$. In contrast to this erratic behaviour, $\triangle(x)$, when considered in the mean, has very nice asymptotic formula. A classical result of Tong [10] says that

$$(1.1) \qquad \int_2^X \triangle(x)^2 dx = \left((6\pi^2)^{-1} \sum_{m=1}^\infty d(m)^2 m^{-3/2}\right) X^{3/2} + F(X)$$

with $F(X) \ll X \log^5 X$. The order of the remainder term $F(X)$ has significant connection with that of $\triangle(x)$. Indeed, Ivić's argument in Theorem

3.8 of [4] shows that $\triangle(x) \ll (U \log x)^{1/3}$ for any upper bound U of $F(X)$. Thus from the result $\triangle(x) = \Omega(x^{1/4})$ we infer that

$$(1.2) \qquad F(X) = \Omega(X^{3/4}/\log X) \ .$$

Ivić conjectured that $F(X) \ll X^{3/4+\varepsilon}$ is true for any $\varepsilon > 0$. This is a very strong bound since it implies $\triangle(x) \ll x^{1/4+\varepsilon}$. There are not many results on $F(X)$ in the literature. Tong's bound was slightly improved to $F(X) \ll X \log^4 X$ by Preissmann [7] in 1988. However, the gap between this and the Ω-result (1.2) is still very wide.

In this paper we shall prove the following.

THEOREM 1. *We have*

$$F(X) = \Omega_-(X \log^2 X) \ .$$

THEOREM 2. *For $X \geq 2$ we have*

$$\int_2^X F(x)dx = -(8\pi^2)^{-1}X^2 \log^2 X + c_1 X^2 \log X + \mathcal{O}(X^2)$$

for a certain constant c_1.

Theorem 1, which is a direct consequence of Theorem 2, disproves the above conjecture of Ivić. Unfortunately we are still unable to obtain a comparable Ω_+-result for $F(x)$. In fact we believe that there is an asymptotic formula for $F(x)$ of the form

$$(1.3) \qquad F(x) = -(4\pi^2)^{-1}x \log^2 x + c_2 x \log x + \mathcal{O}(x)$$

with a certain constant c_2. In a forthcoming paper, the second author [11] proves that

$$\int_X^{2X} \left(F(x + \sqrt{X}) - F(x)\right)^2 dx \asymp X^3 \ .$$

Using Preissmann's bound we see easily that

$$\int_X^{2X} \left(F(x + \sqrt{X}) - F(x)\right)dx = \int_{2X}^{2X+\sqrt{X}} - \int_X^{X+\sqrt{X}} F(x)dx$$
$$\ll X^{3/2} \log^4 X \ .$$

These two results together shows that $F(x + \sqrt{X}) - F(x)$ changes signs in $[X, 2X]$ and

$$F(x + \sqrt{X}) - F(x) = \Omega(X) .$$

Consequently, if (1.3) is true the $\mathcal{O}$-term on the right hand side is oscillatory and cannot be reduced.

One of the key ingredients in our argument is an asymptotic formula for the sum

$$\sum_{m \leq x} d(m)d(m + h) .$$

Such a sum has been investigated by several authors in connection with other problems in analytic number theory. In our proof we use a result of Heath-Brown [1] which is quite sufficient for our purpose. (see (2.12)-(2.15) below)

2. Notations and some Preparation

Throughout the paper, ε denotes an arbitrary small positive number which need not be the same at each occurrence. The symbols $c_0, c_1, c_2, \ldots$ etc. denote certain constants. We shall also use the well-known inequality $d(n) \ll n^{\varepsilon}$ from time to time without explicit reference. The constants implicit in the symbols $\mathcal{O}$ and $\ll$ depend at most on ε.

A useful formula for studying problems concerning $\triangle(x)$ was obtained by Voronoi [12] at the beginning of this century. The formula expresses $\triangle(x)$ as an infinite series involving the Bessel functions. In practice, the following truncated form of the formula

$$\triangle(x) = (\pi\sqrt{2})^{-1} x^{1/4} \sum_{n \leq N} d(n) n^{-3/4} \cos(4\pi\sqrt{nx} - \pi/4)$$
$$+ \mathcal{O}(x^{\varepsilon} + x^{1/2+\varepsilon} N^{-1/2})$$

for $1 \leq N \ll x$ is quite sufficient. However, for our present problem, the above $\mathcal{O}$-term is far too large and we shall use instead the following approximation to $\triangle(x)$ given by Meurman [6, Lemma 3].

LEMMA 1. *For $x \geq 1$ and $M \gg x$, let*

$$\delta_M(x) = (\pi\sqrt{2})^{-1} x^{1/4} \sum_{n \leq M} d(n) n^{-3/4} \cos(4\pi\sqrt{nx} - \pi/4) .$$

Then $\triangle(x) = \delta_M(x) + R(x)$ where $R(x) \ll x^{-1/4}$ if $\|x\| \gg x^{5/2}M^{-1/2}$ and $R(x) \ll x^\varepsilon$ otherwise.

Using this we obtain

LEMMA 2. *Let $x \geq 2$ and $x^7 \ll M \ll x^{100}$. Then*

$$\int_2^x \triangle(u)^2 du = \int_2^x \delta_M(u)^2 du + \mathcal{O}(x) .$$

Proof. Firstly,

$$\int_2^x \triangle(u)^2 du = \int_2^x \delta_M(u)^2 du + 2\int_2^x \delta_M(u)R(u)du + \int_2^x R(u)^2 du .$$

Next, by Lemma 1, we have

$$(2.1) \qquad \int_2^x R(u)^2 du \ll \sum_{n=2}^{[x]+1} n^\varepsilon n^{5/2}M^{-1/2} + \int_2^x (u^{-1/4})^2 du \ll \sqrt{x} .$$

Moreover, following the argument of [3, Theorem 13.5] we show that

$$\int_2^x \delta_M(u)^2 du \asymp x^{3/2}$$

for $M \ll x^{100}$. Thus, by Cauchy-Schwarz's inequality and (2.1) we have

$$\int_2^x \delta_M(u)R(u)du \ll x$$

and hence our lemma.

Square out $\delta_M(u)$ and then integrate term by term, we get

$$\int_2^x \delta_M(u)^2 du$$

$$= (4\pi^2)^{-1} \sum_{m,n \leq M} d(m)d(n)(mn)^{-3/4} \int_2^x \sqrt{u} \cos\left(4\pi(\sqrt{n} - \sqrt{m})\sqrt{u}\right)du$$

$$+ (4\pi^2)^{-1} \sum_{m,n \leq M} d(m)d(n)(mn)^{-3/4} \int_2^x \sqrt{u} \sin\left(4\pi(\sqrt{n} + \sqrt{m})\sqrt{u}\right)du.$$

In the first double sum the diagonal terms yield a total contribution of

$$(4\pi^2)^{-1} \sum_{m \le M} d(m)^2 m^{-3/2} \frac{2}{3}(x^{3/2} - 2^{3/2})$$

$$= (6\pi^2)^{-1} \sum_{m=1}^{\infty} d(m)^2 m^{-3/2} x^{3/2} + \mathcal{O}(x^{3/2} M^{\varepsilon - 1/2} + 1) \, .$$

Here the main term is the same as that in (1.1). Hence by Lemma 2, we can write

$$(2.2) \qquad\qquad F(x) = S_1(x) + S_2(x) + \mathcal{O}(x) \, ,$$

where for any $y \ge 2$,

$$(2.3) \qquad S_1(y) = (2\pi^2)^{-1} \sum_{m < n \le M} d(m)d(n)(mn)^{-3/4} \times$$

$$\int_2^y \sqrt{u} \cos\left(4\pi(\sqrt{n} - \sqrt{m})\sqrt{u}\right) du \, ,$$

and

$$(2.4) \qquad S_2(y) = (4\pi^2)^{-1} \sum_{m,n \le M} d(m)d(n)(mn)^{-3/4} \times$$

$$\int_2^y \sqrt{u} \sin\left(4\pi(\sqrt{n} + \sqrt{m})\sqrt{u}\right) du \, .$$

From now on, we let X to be a sufficiently large number, $M = X^7$ and $L = \log X$. For any $\nu \ge 0$, let

$$(2.5) \qquad\qquad g(\nu) = \nu^{-3/2} J_{3/2}(\nu) - 4\nu^{-5/2} J_{5/2}(\nu) \, ,$$

where J_k denotes the Bessel function of order k. It is well-known that [13, §§3.3, 3.4]

$$J_k(z) \ll \min(|z|^k, |z|^{-1/2})$$

for any real z. Hence,

$$(2.6) \qquad\qquad g(\nu) \ll \min(1, \nu^{-2})$$

for any $\nu \ge 0$.

LEMMA 3. *We have*

(2.7)
$$\int_0^X F(x)dx = \sqrt{2}\pi^{-3/2}X^{5/2}\sum_{m<n\leq M}d(m)d(n)(mn)^{-3/4}g(\theta_{m,n}) + \mathcal{O}(X^2)$$

where $\theta_{m,n} = 4\pi(\sqrt{n}-\sqrt{m})\sqrt{X}$.

Proof. By [8, Lemma 4.2], for any real α and y, $y \geq 2$ we have

(2.8)
$$\int_2^y \sqrt{u}e^{i\alpha\sqrt{u}}du \ll y|\alpha|^{-1} .$$

We first obtain some preliminary bounds for $S_1(y)$ and $S_2(y)$. According to (2.3) and on applying (2.8), we have

(2.9) $$S_1(y) \ll y\sum_{m<n\leq M}d(m)d(n)(mn)^{-3/4}(\sqrt{n}-\sqrt{m})^{-1}$$

(2.9) $$S_1(y) \ll y\sum_{m<n\leq M}d(m)d(n)(mn)^{-3/4}(\sqrt{n}-\sqrt{m})^{-1}$$

$$\ll yM^{\varepsilon}\Big\{\sum_{m<n\leq 2m\leq M}(mn)^{-3/4}(\sqrt{n}+\sqrt{m})(n-m)^{-1}$$

$$+ \sum_{2m<n\leq M}m^{-3/4}n^{-5/4}\Big\}$$

$$\ll yM^{\varepsilon}\Big\{\sum_{m\leq M/2}m^{-1}\sum_{m<n\leq M}(n-m)^{-1} + \log M\Big\} \ll yM^{\varepsilon} .$$

Similarly,

(2.10) $$S_2(y) \ll y\sum_{m,n\leq M}d(m)d(n)(mn)^{-3/4}(\sqrt{n}+\sqrt{m})^{-1} \ll yM^{\varepsilon} .$$

Next, for $x \in [\sqrt{X}, X]$ we have $x^7 \ll M \ll x^{14}$ so that, by (2.2)

(2.11)
$$\int_{\sqrt{X}}^X F(x)dx = \int_{\sqrt{X}}^X S_1(x)dx + \int_{\sqrt{X}}^X S_2(x)dx + \mathcal{O}(X^2)$$

$$= \int_2^X S_1(x)dx + \int_2^X S_2(x)dx + \mathcal{O}(XM^{\varepsilon} + X^2) ,$$

since, by (2.9) and (2.10), $\int_2^{\sqrt{X}} S_i(x)dx \ll XM^\varepsilon$. The main term on the right hand side of (2.7) arises from $\int_2^X S_1(x)dx$. Indeed, by (2.3),

$$\int_2^X S_1(x)dx = (2\pi^2)^{-1} \sum_{m<n\leq M} d(m)d(n)(mn)^{-3/4} \times$$
$$\int_2^X \int_2^x \sqrt{u} \cos\left(4\pi(\sqrt{n}-\sqrt{m})\sqrt{u}\right)dudx \ .$$

Write $\theta = 4\pi(\sqrt{n}-\sqrt{m})\sqrt{X}$ for short. Then the above double integral is equal to

$$\int_2^X (X-u)\sqrt{u} \cos\left(4\pi(\sqrt{n}-\sqrt{m})\sqrt{u}\right)du$$
$$= 2X^{5/2} \int_{\sqrt{2/X}}^1 (1-v^2)v^2 \cos(\theta v)dv$$
$$= 2X^{5/2}\Bigg\{ \int_0^1 (1-v^2) \cos(\theta v)dv - \int_0^1 (1-v^2)^2 \cos(\theta v)dv$$
$$- \int_0^{\sqrt{2/X}} (1-v^2)v^2 \cos(\theta v)dv \Bigg\} \ .$$

By the well-known integral representation

$$J_{k+\frac{1}{2}}(z) = \frac{2}{\sqrt{\pi}}\left(\frac{z}{2}\right)^{k+\frac{1}{2}}\frac{1}{k!}\int_0^1 (1-v^2)^k \cos(zv)dv \ , \quad k = 0,1,2,\dots$$

for the Bessel functions [13, §3.3], the first two integrals on the right hand side is equal to

$$\sqrt{2\pi}\left(\theta^{-3/2}J_{3/2}(\theta) - 4\theta^{-5/2}J_{5/2}(\theta)\right) = \sqrt{2\pi}g(\theta) \ ,$$

by (2.5). Moreover, using integration by parts we find that

$$\int_0^{\sqrt{2/X}} (1-v^2)v^2 \cos(\theta v)dv \ll X^{-1}\theta^{-1} \ .$$

Hence

$$\int_2^X (X-u)\sqrt{u} \cos\left(4\pi(\sqrt{n}-\sqrt{m})\sqrt{u}\right)du = 2\sqrt{2\pi}X^{5/2}g(\theta) + \mathcal{O}(X^{3/2}\theta^{-1}),$$

and then

$$\int_2^X S_1(x)dx = \sqrt{2}\pi^{-3/2}X^{5/2} \sum_{m<n\leq M} d(m)d(n)(mn)^{-3/4}g(\theta)$$
$$+ \mathcal{O}\Big(X \sum_{m<n\leq M} d(m)d(n)(mn)^{-3/4}(\sqrt{n} - \sqrt{m})^{-1}\Big) .$$

The sum inside the $\mathcal{O}$-term can be treated by the argument in (2.9), and we then find that the $\mathcal{O}$-term is bounded by XM^ε, which is smaller than that on the right hand side of (2.7).

In view of (2.11) and (2.7), it remains to bound the two integrals $\int_0^{\sqrt{X}} F(x)dx$ and $\int_2^X S_2(x)dx$ by X^2. By Preissmann's bound, we have

$$\int_0^{\sqrt{X}} F(x)dx \ll X\log^4 X$$

which is acceptable. Next, by (2.4),

$$\int_2^X S_2(x)dx = (4\pi^2)^{-1} \sum_{m,n\leq M} d(m)d(n)(mn)^{-3/4}\times$$
$$\int_2^X (X - u)\sqrt{u}\sin\big(4\pi(\sqrt{n} + \sqrt{m})\sqrt{u}\big)du$$
$$= (2\pi^2)^{-1}X^{5/2} \sum_{m,n\leq M} d(m)d(n)(mn)^{-3/4}\times$$
$$\int_{\sqrt{2/X}}^1 (1 - v^2)v^2\sin\big(4\pi(\sqrt{n} + \sqrt{m})\sqrt{X}v\big)dv .$$

The inner integral, on applying integration by parts twice, is found to be

$$\ll X^{-3/2}(\sqrt{n} + \sqrt{m})^{-1} + X^{-1}(\sqrt{n} + \sqrt{m})^{-2}.$$

Thus,

$$\int_2^X S_2(x)dx \ll X \sum_{m\leq n\leq M} d(m)d(n)(mn)^{-3/4}n^{-1/2}+$$
$$X^{3/2} \sum_{m\leq n\leq M} d(m)d(n)(mn)^{-3/4}n^{-1}$$
$$\ll XM^\varepsilon + X^{3/2} \ll X^{3/2} .$$

This completes the proof of Lemma 3.

For any $y > 0$, let

$$(2.12) \qquad \psi_h(y) = \sum_{m \leq y} d(m)d(m+h) \ .$$

In his work on the fourth power moment of the Riemann zeta-function on the critical line, Heath-Brown [1] proved that

$$(2.13) \qquad \psi_h(y) = I_h(y) + E_h(y),$$

where the main term $I_h(y)$ is of the form

$$(2.14) \qquad I_h(y) = y \sum_{i=0}^{2} \log^i y \sum_{d|h} d^{-1}(\alpha_{i0} + \alpha_{i1} \log d + \alpha_{i2} \log^2 d)$$

for certain constants α_{ij}, and the remainder $E_h(y)$ satisfies

$$(2.15) \qquad E_h(y) \ll y^{5/6+\varepsilon}$$

uniformly for $1 \leq h \leq y^{5/6}$. In particular $\alpha_{20} = 6\pi^{-2}$, $\alpha_{21} = \alpha_{22} = 0$. We note that $I_h(y)$ is roughly of order $y \log^2 y$. In our proof of Theorem 2 in §3 we shall need $I_h'(y)$, the derivative of $I_h(y)$. By (2.14)

$$(2.16) \qquad I_h'(y) = a_2(h) \log^2 y + a_1(h) \log y + a_0(h)$$

where

$$a_2(h) = 6\pi^{-2} \sum_{d|h} d^{-1} \ ,$$

$$a_1(h) = \sum_{d|h} d^{-1}(12\pi^{-2} + \alpha_{10} + \alpha_{11} \log d + \alpha_{12} \log^2 d) \ ,$$

(2.17)

$$a_0(h) = \sum_{d|h} d^{-1} \sum_{j=0}^{2}(\alpha_{0j} + \alpha_{1j}) \log^j d \ .$$

For any $y > 0$, $Q > 3$ let

$$(2.18) \qquad \xi(y, Q) = \sum_{h \leq y} h^{-1}\big(4a_2(h) \log^2 Qh + 2a_1(h) \log Qh + a_0(h)\big) \ .$$

LEMMA 4. *We have*

$$\xi(y, Q) = \frac{4}{3} \log^3 Qy + c_3 \log^2 Qy - \frac{4}{3} \log^3 Q + c_4 \log^2 Q + c_5 \log Q$$
$$+ c_6 \log y + c_7 + \mathcal{O}(y^{-1} \log^3 y \log^2 Qy) \ .$$

Proof. In the argument below we use the symbol c to denote a certain constant which may not be the same at each occurrence.

Firstly, for $j = 0, 1, 2$ there are constants $\beta_0, \beta_1, \beta_2$ such that

$$(2.19) \qquad \sum_{h \leq y} a_j(h) = \beta_j y + B_j(y)$$

with $B_j(y) \ll \log^3 y$. (Note $B_j(1^-) = -\beta_j$). Indeed, by (2.17),

$$\sum_{h \leq y} a_0(h) = \sum_{d \leq y} d^{-1} \sum_{j=0}^{2} (\alpha_{0j} + \alpha_{1j})(\log^j d)(yd^{-1} + \mathcal{O}(1))$$

$$= y \sum_{d \leq y} d^{-2} \sum_{j=0}^{2} (\alpha_{0j} + \alpha_{1j}) \log^j d + \mathcal{O}\left(\sum_{d \leq y} d^{-1} \log^2 d \right)$$

$$= y \sum_{d=1}^{\infty} d^{-2} \sum_{j=0}^{2} (\alpha_{0j} + \alpha_{1j}) \log^j d + \mathcal{O}\left(y \sum_{d > y} d^{-2} \log^2 d \right) +$$

$$+ \mathcal{O}(\log^3 y)$$

$$= \beta_0 y + \mathcal{O}(\log^3 y)$$

with

$$\beta_0 = \sum_{d=1}^{\infty} d^{-2} \sum_{j=0}^{2} (\alpha_{0j} + \alpha_{1j}) \log^j d \ .$$

Similar argument establishes (2.19) for $j = 1$ and 2. Further we find that $\beta_2 = 1$.

Next by Riemann Stieltjes integration and (2.19), we have

$$\sum_{h \leq y} a_2(h) h^{-1} \log^2 Qh$$

$$= \int_1^y t^{-1} \log^2 Qt \, dt + \left[t^{-1} \log^2 Qt \, B_2(t) \right]_{1-}^y$$

$$- \int_1^y B_2(t) t^{-2} (2 \log Qt - \log^2 Qt) \, dt$$

$$= \frac{1}{3} (\log^3 Qy - \log^3 Q) + \log^2 Q + \mathcal{O}(y^{-1} \log^3 y \log^2 Qy)$$

$$- \int_1^y B_2(t) t^{-2} (-\log^2 Q + 2(1 - \log t) \log Q + 2 \log t - \log^2 t) \, dt$$

$$= \frac{1}{3} (\log^3 Qy - \log^3 Q) + \log^2 Q + c \log^2 Q + c \log Q + c$$

$$+ \mathcal{O}\left(\int_y^\infty (\log^3 t) t^{-2} (\log^2 Q + \log^2 t) \, dt \right) + \mathcal{O}(y^{-1} \log^3 y \log^2 Qy)$$

$$= \frac{1}{3} (\log^3 Qy - \log^3 Q) + c \log^2 Q + c \log Q + c + \mathcal{O}(y^{-1} \log^3 y \log^2 Qy) \ .$$

In the same way, we find that

$$\sum_{h \leq y} a_1(h) h^{-1} \log Qh = \frac{1}{2} \beta_1 (\log^2 Qy - \log^2 Q) + c \log Q + c +$$

$$+ \mathcal{O}(y^{-1} \log^3 y \log Qy)$$

and

$$\sum_{h \leq y} a_0(h) h^{-1} = \beta_0 \log y + c + \mathcal{O}(y^{-1} \log^3 y) \ .$$

Collecting all these in (2.18) our lemma follows.

Lastly we evaluate some integrals involving the function $g(\nu)$.

LEMMA 5. *We have*

$$\int_0^\infty g(\nu) d\nu = 0 \ ,$$

$$\int_0^\infty g(\nu) \log \nu \, d\nu = -\sqrt{\pi} 2^{-7/2} \ .$$

Proof. It is known that [13, §13.24]

$$\int_0^\infty J_k(\nu)\nu^{s-k-1}d\nu = \Gamma\left(\frac{s}{2}\right)2^{s-k-1}/\Gamma\left(k-\frac{s}{2}+1\right)$$

for $0 < \operatorname{Re} s < \operatorname{Re} k + 1/2$. Hence

$$\int_0^\infty \left(\nu^{-k}J_k(\nu) - (2k+1)\nu^{-k-1}J_{k+1}(\nu)\right)\nu^s d\nu$$

$$= -s2^{s-k-1}\Gamma\left(\frac{s+1}{2}\right)/\Gamma\left(k-\frac{s}{2}+\frac{3}{2}\right)$$

for $-1 < \operatorname{Re} s < \operatorname{Re} k - 1/2$. Setting $k = 3/2$ and in view of (2.5) we have

$$(2.20) \qquad \int_0^\infty g(\nu)\nu^s d\nu = -s2^{s-5/2}\Gamma\left(\frac{s+1}{2}\right)/\Gamma\left(3-\frac{s}{2}\right)$$

for $-1 < \operatorname{Re} s < 1$. On putting $s = 0$ we get $\int_0^\infty g(\nu)d\nu = 0$. The remaining integral is equal to

$$\frac{d}{ds}\left(\int_0^\infty g(\nu)\nu^s d\nu\right)\Big|_{s=0}$$

which can be evaluated by differentiating the right hand side of (2.20).

3. Proof of Theorem 2

We shall now complete the proof of Theorem 2 by evaluating the double sum

$$(3.1) \qquad\qquad T = \sum_{m<n\leq M} u_{m,n}$$

in Lemma 3, where

$$u_{m,n} = d(m)d(n)(mn)^{-3/4}g\left(4\pi(\sqrt{n}-\sqrt{m})\sqrt{X}\right).$$

In view of Lemma 3, we can allow errors of order up to $X^{-1/2}$ in the course of our analysis.

First of all, we consider those terms $u_{m,n}$ for which $m < n/2$. In this case $\sqrt{n} - \sqrt{m} \asymp \sqrt{n}$ so that, by (2.6)

$$g\big(4\pi(\sqrt{n} - \sqrt{m})\sqrt{X}\big) \ll (nX)^{-1} .$$

The contribution to T from these $u_{m,n}$ is therefore

$$\ll X^{-1} \sum_{m<n\leq M} d(m)d(n)(mn)^{-3/4}n^{-1} \ll X^{-1},$$

which is acceptable.

For the remaining terms $u_{m,n}$ in T, we write $n = m+h$ with $1 \leq h \leq m$. Then

$$T = \sum_{h\leq M/2} \sum_{h\leq m\leq M-h} u_{m,m+h} + \mathcal{O}(X^{-1}) .$$

For $h \leq m$, we have $4\pi\big(\sqrt{m+h} - \sqrt{m}\big)\sqrt{X} \asymp 2\pi h\sqrt{X/m}$ so that, by (2.6) again

$$(3.2) \qquad g\big(4\pi(\sqrt{m+h} - \sqrt{m})\sqrt{X}\big) \ll mh^{-2}X^{-1}$$

and each term $u_{m,m+h}$ satisfies

$$u_{m,m+h} \ll M^{\varepsilon}m^{-3/2}mh^{-2}X^{-1} .$$

Thus, the contribution to T from those $u_{m,m+h}$ with $h > \sqrt{M}$ is $\ll X^{-1}M^{\varepsilon}$ and the error caused by extending the upper limit for the summation on m to M is $\mathcal{O}(X^{-1}M^{-1/2+\varepsilon})$. Hence we have

$$T = \sum_{h\leq \sqrt{M}} \sum_{h\leq m\leq M} u_{m,m+h} + \mathcal{O}(X^{-1}M^{\varepsilon}) .$$

For simplicity let

$$D_h = h^2 XL^{-8} .$$

Then we can further write

$$(3.3)$$
$$T = \sum_{h\leq \sqrt{M}} \sum_{h\leq m\leq \min(D_h,M)} + \sum_{h\leq X^3L^4} \sum_{D_h<m\leq M} + \mathcal{O}(X^{-1}M^{\varepsilon})$$
$$= \sum\nolimits_1 + \sum\nolimits_2 + \mathcal{O}(X^{-1}M^{\varepsilon}) ,$$

say. Using the same bound (3.2), each term $u_{m,m+h}$ in $\sum_1$ is

$$\ll \left(d(m)^2 + d(m+h)^2\right)\left(m(m+h)\right)^{-3/4} mh^{-2}X^{-1}$$
$$\ll d(m)^2 m^{-1/2} h^{-2} X^{-1} + d(m+h)^2(m+h)^{-1/2} h^{-2} X^{-1} ,$$

since $m+h \asymp m$ for $1 \le h \le m$. An application of the well-known estimate

$$\sum_{m \le y} d(m)^2 m^{-1/2} \ll \sqrt{y} \log^3 y \qquad \text{for} \quad y > 1 ,$$

then yields

$$\sum_1 \ll X^{-1} \sum_{h \le \sqrt{M}} h^{-2} \sqrt{\min(D_h, M)} \log^3 M \ll X^{-1/2} .$$

Putting this into (3.3), we have

$$(3.4) \qquad \begin{aligned} T = &\sum_{h \le X^3 L^4} \sum_{D_h < m \le M} d(m)d(m+h)\left(m(m+h)\right)^{-3/4} g(\theta_{m,m+h}) \\ &+ \mathcal{O}(X^{-1/2}) \end{aligned}$$

with $\theta_{m,m+h} = 4\pi\left(\sqrt{m+h} - \sqrt{m}\right)\sqrt{X}$.

Next, we transform the above inner sum over m into an integral. By (2.12), (2.13) and Riemann Stieltjes integration we have

$$(3.5) \qquad \begin{aligned} \sum_{D_h < m \le M} &= \int_{D_h}^{M} \left(y(y+h)\right)^{-3/4} g(\theta_{y,y+h}) d\psi_h(y) \\ &= \int_{D_h}^{M} \left(y(y+h)\right)^{-3/4} g(\theta_{y,y+h}) I_h'(y) dy \\ &\quad + \left[\left(y(y+h)\right)^{-3/4} g(\theta_{y,y+h}) E_h(y)\right]_{D_h}^{M} \\ &\quad - \int_{D_h}^{M} E_h(y) \frac{d}{dy}\left\{\left(y(y+h)\right)^{-3/4} g(\theta_{y,y+h})\right\} dy \\ &= W_1(h) + W_2(h) + W_3(h) , \end{aligned}$$

say. We bound $W_2(h)$ by using (2.15) and the trivial estimate $g(\nu) \ll 1$. Whence

$$W_2(h) \ll M^{-3/2} M^{5/6+\varepsilon} + D_h^{-3/2} D_h^{5/6+\varepsilon} \ll D_h^{-2/3+\varepsilon} \ll h^{-4/3} X^{-2/3+\varepsilon} .$$

For $W_3(h)$, by [13, §3.2] we have

$$g'(\nu) = -\nu^{-3/2} J_{5/2}(\nu) + 4\nu^{-5/2} J_{7/2}(\nu) \ll \nu \quad \text{for} \quad \nu \geq 0 \ ,$$

since $J_k(\nu) \ll \nu^k$. Hence, by $g(\nu) \ll 1$ and (2.15) we have

$$W_3(h) \ll \int_{D_h}^{M} y^{5/6+\varepsilon} \Big\{ y^{-5/2} + y^{-3/2} \big|\theta_{y,y+h}\big| \big|\frac{d}{dy}\theta_{y,y+h}\big| \Big\} dy$$

$$\ll \int_{D_h}^{M} \{ y^{-5/3+\varepsilon} + y^{-2/3+\varepsilon} hy^{-1/2} X^{1/2} hy^{-3/2} X^{1/2} \} dy$$

$$\ll h^{-4/3} X^{-2/3+\varepsilon} \ .$$

In view of (3.4) and (3.5), the contribution to T from $W_2(h)$ and $W_3(h)$ is therefore

$$\ll \sum_{h \leq X^3 L^4} h^{-4/3} X^{-2/3+\varepsilon} \ll X^{-2/3+\varepsilon} \ ,$$

which is again acceptable. Thus,

$$(3.6) \quad T = \sum_{h \leq X^3 L^4} \int_{D_h}^{M} \big(y(y+h)\big)^{-3/4} g(\theta_{y,y+h}) I'_h(\dot{y}) dy + \mathcal{O}(X^{-1/2}) \ .$$

To evaluate the inner integral, we begin by making the change of variable

$$\omega = \theta_{y,y+h} = 4\pi \big(\sqrt{y+h} - \sqrt{y}\big)\sqrt{X} \ .$$

Then

$$y = 4\pi^2 X \omega^{-2} h^2 - \frac{1}{2}h + (64\pi^2 X)^{-1}\omega^2 = 4\pi^2 X \omega^{-2} h^2 \big(1 + \mathcal{O}(\omega^2 X^{-1} h^{-1})\big) \ ,$$

so that

$$\big(y(y+h)\big)^{-3/4} = \big(4\pi^2 X \omega^{-2} h^2 - (64\pi^2 X)^{-1}\omega^2\big)^{-3/2}$$

$$= (2\pi h)^{-3} X^{-3/2} \omega^3 \big(1 + \mathcal{O}(\omega^4 X^{-2} h^{-2})\big)$$

and

$$\frac{dy}{d\omega} = -8\pi^2 X \omega^{-3} h^2 \big(1 + \mathcal{O}(\omega^4 X^{-2} h^{-2})\big) \ .$$

Moreover, by (2.16)

$$I'_h(y) = 4a_2(h) \log^2\left(2\pi\sqrt{X}\omega^{-1}h\right) + 2a_1(h)\log(2\pi\sqrt{X}\omega^{-1}h) + a_0(h)$$
$$+ \mathcal{O}\big(\omega^2 X^{-1}h^{-1}(|a_2(h)|L + |a_1(h)|)\big).$$

Set

$$(3.7)\qquad u_1 = 4\pi\big(\sqrt{M+h} - \sqrt{M}\big)\sqrt{X} = 2\pi h X^{-3} + \mathcal{O}(h^2 X^{-10})$$

and

$$(3.8)\qquad u_2 = 4\pi\big(\sqrt{D_h+h} - \sqrt{D_h}\big)\sqrt{X} = 2\pi L^4 + \mathcal{O}(h^{-1}X^{-1}L^{12}).$$

Then with the help of all these estimates we find that

$$\int_{D_h}^{M} \big(y(y+h)\big)^{-3/4} g(\theta_{y,y+h}) I'_h(y)\,dy$$
$$= \frac{1}{\pi\sqrt{X}} \int_{u_1}^{u_2} g(\omega)h^{-1}\big\{4a_2(h)\log^2\left(2\pi\sqrt{X}\omega^{-1}h\right)$$
$$+ 2a_1(h)\log\left(2\pi\sqrt{X}\omega^{-1}h\right) + a_0(h)\big\}d\omega + \mathcal{O}\big(h^{-2}X^{-3/2+\varepsilon}\big).$$

In obtaining the above $\mathcal{O}$-term, we have used $g(\omega) \ll 1$, (3.8) and the observation that $a_j(h) \ll \log^3 h \ll L^3$. The integration limits u_1 and u_2 can be replaced by $2\pi h X^{-3}$ and $2\pi L^4$ respectively, since the error thus caused is

$$\ll X^{-1/2}h^{-1}L^5(h^2X^{-10} + h^{-1}X^{-1}L^{12}) \ll hX^{-21/2}L^5 + h^{-2}X^{-3/2}L^{17}$$
$$\ll h^{-2}X^{-3/2+\varepsilon},$$

by (3.7) and (3.8). Collecting these into (3.6), we get

$$T = \frac{1}{\pi\sqrt{X}} \sum_{h \le X^3 L^4} \int_{2\pi h X^{-3}}^{2\pi L^4} g(\omega)h^{-1}\big\{4a_2(h)\log^2\left(2\pi\sqrt{X}\omega^{-1}h\right)$$
$$+ 2a_1(h)\log\left(2\pi\sqrt{X}\omega^{-1}h\right) + a_0(h)\big\}d\omega + \mathcal{O}(X^{-1/2}).$$

Next we interchange the summation and integration. In view of (2.18) we have

$$(3.9)\quad T = \frac{1}{\pi\sqrt{X}} \int_{2\pi X^{-3}}^{2\pi L^4} g(\omega)\xi\big((2\pi)^{-1}\omega X^3, 2\pi\sqrt{X}\omega^{-1}\big)d\omega + \mathcal{O}(X^{-1/2}).$$

By Lemma 4, and after some simplifications, we have

$$\xi\big((2\pi)^{-1}\omega X^3, 2\pi\sqrt{X}\omega^{-1}\big) = \log\omega \log^2 X + (c_8\log\omega + c_9\log^2\omega)\log X +$$
$$+ c_{10}\log\omega + c_{11}\log^2\omega + c_{12}\log^3\omega + \Phi(X) + \mathcal{O}(\omega^{-1}X^{-3}L^5) \,,$$

where $\Phi(X) = c_{13}\log^3 X + c_{14}\log^2 X + c_{15}\log X + c_{16}$ and $c_8, c_9, \ldots, c_{16}$ are certain constants. Finally inserting this into (3.9) we get

$$(3.10) \qquad T = \frac{1}{\pi\sqrt{X}} \int_{2\pi X^{-3}}^{2\pi L^4} g(\omega)\big\{ \log\omega \log^2 X + (c_8\log\omega + c_9\log^2\omega)\log X$$
$$+ c_{10}\log\omega + c_{11}\log^2\omega + c_{12}\log^3\omega + \Phi(X)\big\}d\omega + \mathcal{O}(X^{-1/2}) \,.$$

It remains to evaluate the integrals

$$K_j = \int_{2\pi X^{-3}}^{2\pi L^4} g(\omega)\log^j\omega d\omega$$

for $j = 0, 1, 2, 3$. Writing

$$K_j = \int_0^\infty g(\omega)\log^j\omega d\omega - \int_0^{2\pi X^{-3}} g(\omega)\log^j\omega d\omega - \int_{2\pi L^4}^\infty g(\omega)\log^j\omega d\omega \,,$$

we see, by (2.6), that the last two integrals are bounded by $X^{-3}L^j$ and L^{-4+j} respectively. Hence, by Lemma 5 we have

$$K_0 \ll L^{-4}, \ K_1 = -\sqrt{\pi}2^{-7/2} + \mathcal{O}(L^{-3})$$

and by (2.6),

$$K_2, K_3 = \text{ constant } + \mathcal{O}(L^{-1}) \,.$$

When these are inserted into (3.10) we obtain

$$T = -2^{-3}(2\pi X)^{-1/2}\log^2 X + c_{17}X^{-1/2}\log X + \mathcal{O}(X^{-1/2}) \,,$$

and Theorem 2 now follows from (3.1) and Lemma 3.

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