Rached Mneimné
Frédéric Testard

On products of singular elements


<http://www.numdam.org/item?id=JTNB_1991__3_2_337_0>
On products of singular elements

by Rached Mneimné and Frédéric Testard

Some rings, like the ring $M(n, K)$ of square matrices, do not contain irreducible elements: any singular element $x$ can be written as the product $x = yz$ of two singular elements $y$ and $z$. We shall call these rings $S$-rings. Our first purpose in this paper is to exhibit some examples of $S$-rings. For instance, we give a necessary and sufficient condition ensuring that $\mathbb{Z}/n\mathbb{Z}$ is an $S$-ring.

More generally, let us denote by $S_i(R)$ (or just $S_i$ if no confusion is possible) the set of elements of a ring $R$, which can be written as the product of $i$ singular elements; the sequence $(S_i)$ is decreasing (we only consider rings where left invertibility is equivalent to right invertibility) and moreover the ring $R$ is an $S$-ring if and only if $S_1 = S_2$. We denote by $S_\infty$ the intersection of all the $S_i$; when the sequence $(S_i)$ is stationary ($S_i = S_k$ whenever $i \geq k$), we have $S_\infty = S_k$ if $k$ is the first index $i$ such that $S_i = S_{i+1}$. There is a natural operation of the group $GL(R)$ of all invertible elements of the ring $R$ on the set $S_i$ defined by: $(g, x) \mapsto gx$ for $g \in GL(R)$ and $x \in S_i$, where $gx$ is the product in $R$ of the two elements $g$ and $x$. This defines clearly an operation of $GL(R)$ on $S_i$, hence also on $S_i \setminus S_{i+1}$ (elements of $S_i$ which do not belong to $S_{i+1}$). Other natural operations could have been considered: $(g, x) \mapsto xg^{-1}$ or $(g, x) \mapsto gxg^{-1}$ or the following operation of $GL(R) \times GL(R)$ on $S_i$ given by $((g_1, g_2), x) \mapsto g_1 x g_2^{-1}$. When the ring $R$ is commutative, these operations bring nothing new. This is the case of the ring $K[A]$ of polynomial expansions of the matrix $A \in M(n, K)$ for which we dispose of a particularly nice description of the orbits of $GL(A)$ ($= GL(K[A])$)-(Part 3).

In part 2, we study in an elementary way the ring $K[A]$ by giving a necessary and sufficient condition in order that the matrix $A$ could be written as $P(A)Q(A)$, where $P$ and $Q$ are polynomials, with $P(A)$ and $Q(A)$ two singular matrices (i.e. $A \in S_2(K[A])$).

Part 4 is devoted to the solution of the following non trivial problem: given any matrix $A$, what is the maximal number $n(A)$ of singular and
permutable matrices $A_i$ such that $A = A_1 \cdots A_m$? A simple observation allows us to answer the same problem, for $A$ and $A_i$ bistochastic.

1. Examples of $S$-rings

We begin with an easy criterion

**Lemma 1.** Let $E$ and $F$ be two rings and $E \times F$ be their product ring; then $E \times F$ is an $S$-ring if and only if $E$ and $F$ are $S$-rings. In particular, any finite product of fields is an $S$-ring.

*Proof* Consider a singular element $(x, y)$ in $E \times F$. For instance, $x$ is not invertible. We can find $x_1$ and $x_2$ two singular elements in $E$ so that $x = x_1 x_2$; then $(x, y) = (x_1, y)(x_2, 1)$ is the product of two singular elements of $E \times F$. Conversely, suppose that $E \times F$ is an $S$-ring and take $x$, any singular element in $E$. There exist two singular couples $(x_1, y_1)$ and $(x_2, y_2)$ so that $(x, 1) = (x_1, y_1)(x_2, y_2)$. Since $y_1 y_2 = 1$, $x_1$ and $x_2$ are not invertible, and $E$ is an $S$-ring; the same argument works for $F$.

**Lemma 2.** Let $p$ be a prime and $\alpha$ be a positive integer. The ring $R = \mathbb{Z}/p^\alpha \mathbb{Z}$ is an $S$-ring if and only if $\alpha = 1$.

*Proof* If $\alpha = 1$, the ring $R$ is a field and there is no problem; otherwise the class of $p$ cannot be the product of two singular classes since it would imply $p - p^{2k} = cp^\alpha$ where $k$ and $c$ are integers, which is impossible if $\alpha \geq 2$.

**Proposition 1.** Let $R = \mathbb{Z}/n\mathbb{Z}$; the ring $R$ is an $S$-ring if and only if $n = p_1 \cdots p_k$ where the $p_i$ are distinct primes.

*Proof* If $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, the rings $R$ and $\prod_{i=1}^k (\mathbb{Z}/p_i^{\alpha_i} \mathbb{Z})$ are isomorphic. The conclusion follows easily from lemmas 1 and 2.

**Proposition 2.** Let $X$ be a topological space and $R = C(X, \mathbb{R})$ be the ring of all continuous mappings from $X$ to $\mathbb{R}$. Then $R$ is an $S$-ring.

*Proof* The function $f$ is singular in $R$ if and only if it vanishes at some point of $X$. When it happens, the same is true for the two continuous mappings $f_1 = f^{1/3}$ and $f_2 = f^{2/3}$ and $f = f_1 f_2$.

**Proposition 3.** Let $R$ be the ring of all germs of $C^\infty$ real functions on a neighbourhood of zero. Then $f \in S_i \iff f(0) = f'(0) = \cdots = f^{(i-1)}(0) = 0$. 
Proof Let us recall that a germ is an equivalence class with respect to the relation: $f \mathcal{R} g$ if and only if $f = g$ on a neighbourhood of zero. An element $f$ of $R$ is singular if and only if $f(0) = 0$ and a straightforward application of Leibniz’s derivation rule shows that if $f = f_1 \cdots f_i$ is the product of $i$ singular elements, the function $f$ and its $i - 1$ first derivatives vanish at $0$. Conversely, if this is true, Taylor’s formula gives, for $x$ small enough:

$$f(x) = \frac{x^i}{(i-1)!} \int_0^1 (1-t)^{i-1} f^{(i)}(tx) dt$$

and the conclusion follows.

Remark 1: This result provides an example of a ring where the sequence $S_i$ is not stationary and does not “converge” to 0. Indeed the well known $C^\infty$-function $f(x) = \exp(-1/x^2)$ whenever $x \neq 0$, clearly belongs to all the $S_i$ without being 0. The explanation lies in the fact that the ring $R$ of germs of $C^\infty$ functions which is a local ring ($S_1$ is an ideal, hence the unique maximal ideal) is not noetherian: indeed, in a local noetherian ring, the intersection $\bigcap S_i$ is equal to $\{0\}$ as it results trivially from Krull’s theorem (see e.g. Atiyah-Macdonald: Introduction to Commutative Algebra p.110 - Addison-Wesley 1969).

**Proposition 4.** Let $K$ be a field and $R = M(n, K)$ be the ring of square matrices $n \times n$ with coefficients in $K$. Then $R$ is an $S$-ring.

Proof Let $A \in R$ be a singular matrix and $r < n$ be the rank of $A$. We know that $A$ is equivalent to the matrix $J_r = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ where $I_r$ denotes the identity matrix of order $r$, i.e. there exist two invertible matrices $P$ and $Q$ such that $A = PJ_rQ$. Since $J_r^2 = J_r$, we get $A = XY$ where $X = PJ_r$ and $Y = J_rQ$ are singular matrices.

**Corollary 1.** The ring of bistochastic matrices of order $n$ is an $S$-ring.

Proof Recall that a matrix $M = (a_{i,j})$ is bistochastic if there exists $d$ in $K$ such that $\forall i, \sum_j a_{ij} = d$ and $\forall j, \sum_i a_{ij} = d$. It is easy to prove that $M$ is bistochastic if and only if $M(H) \subseteq H$ and $M(D) \subseteq D$ where $H$ denotes the hyperplane of $K^n$ equipped with its canonical basis $\{e_1, \ldots, e_n\}$, of equation $\sum_i x_i = 0$ and $D$ is the one dimensional subspace generated by $\sum_i e_i$. Hence, there exists an invertible matrix $P$, independent of $M$, satisfying $M = P \begin{bmatrix} A & 0 \\ 0 & \lambda \end{bmatrix} P^{-1}$; where $A$ is an element of $M(n - 1, K)$. This defines an isomorphism between the ring of bistochastic matrices and $M(n - 1, K) \times K$ and the conclusion follows from lemma 1.
2. Singular polynomial decompositions of matrices

From now on, \( A \) will denote a square matrix, \( P \) and \( Q \) will be polynomials.

**Proposition 5.** The singular matrix \( A \) can be written as \( P(A)Q(A) \), where \( P(A) \) and \( Q(A) \) are singular if and only if 0 is a simple root of the minimal polynomial of \( A \).

**Proof** Let us recall that the minimal polynomial of \( A \) is the unitary generator \( \pi \) of the ideal of all polynomials which vanish at \( A \). The roots of \( \pi \) in the field \( K \) are the eigenvalues of \( A \) in \( K \). In particular, 0 is a root of \( \pi \) since \( A \) is singular.

The sufficient condition is easy to prove: one can write, \( 0 = \pi(A) = \lambda A + AQ(A) \) with \( \lambda \neq 0 \), \( Q \) being a polynomial vanishing at 0; so that \( A = (-A/\lambda)Q(A) \) and the conclusion follows, since \( Q(A) \) is singular (\( Q(A) \) admits \( Q(0) = 0 \) as an eigenvalue). Conversely, if \( A = P(A)Q(A) \), the minimal polynomial of \( A \) divides the polynomial \( X - P(X)Q(X) \): it is enough to prove that 0 is a simple root of \( X - P(X)Q(X) \). Let us first remark that the equality \( A = P(A)Q(A) \) remains true for any matrix \( B \) similar to \( A \), so that, considering an upper triangular matrix \( B \) similar to \( A \), (we could need to extend the ground field) we get \( \lambda_i = P(\lambda_i)Q(\lambda_i) \) for any eigenvalue \( \lambda_i \) of \( A \) this implies that if \( \lambda_i \neq 0 \), \( P(\lambda_i) \neq 0 \) and \( Q(\lambda_i) \neq 0 \), so necessarily, since \( P(A) \) and \( Q(A) \) are singular, \( P(0) = Q(0) = 0 \) and the required conclusion follows easily.

**Remark 2:** An equivalent way to characterize such matrices is the following: 0 is a simple root of the minimal polynomial if and only if \( \ker(A) = \ker(A^2) \).

**Remark 3:** Let \( R \) be the ring \( K[A] \); it results from the proof of proposition 5 that if \( A \in S_2 \), then \( A \in S_i \), \( \forall i \) (once we have written \( A = (-A/\lambda)Q(A) \), we obtain \( A = (A/\lambda^2)Q(A)Q(A) \), and so on). We will understand the situation much better in the following section (see Remark 8).

**Corollary 2.** For \( A = B^k \), there exist polynomials \( P \) and \( Q \) so that \( A = P(A)Q(A) \) with \( P(A) \) and \( Q(A) \) singular matrices if and only if 0 is a root of the minimal polynomial of \( B \) of order \( \leq k \).

**Proof** This is an easy consequence of the fact already noticed in remark 2, that the order of 0 in the minimal polynomial of a matrix \( M \) is the first step where the increasing sequence \( \ker(M^i) \) becomes stationnary: we have \( \ker(B^k) = \ker(A) \subset \ker(B^{k+1}) \subset \cdots \subset \ker(B^{2k}) = \ker(A^2) \).
3. The ring $K[A]$ for itself

In this section it will be assumed that the field $K$ is algebraically closed, although most results can be stated in a more general context; let us recall that the ring $R = K[A] = \{ P(A), P \in K[X] \}$ is isomorphic to the quotient ring $K[X]/(\pi)$, where $(\pi)$ denotes the principal ideal generated by the minimal polynomial of $A$. Writing $\pi$ in the form $\pi(X) = \prod_i (X - \lambda_i)^{\alpha_i} (\lambda_i \in K, \alpha_i \in \mathbb{N}^*)$ it follows from the Chinese remainder theorem (or from an adequate computation of the dimension of the underlying vector spaces) that $K[A]$ is isomorphic to the product ring $\prod_i K[X]/(X - \lambda_i)^{\alpha_i}$, so that we obtain, as for the ring $\mathbb{Z}/n\mathbb{Z}$, a first result:

**Proposition 6.** The ring $K[A]$ is an $S$-ring if and only if $A$ is diagonalizable.

**Proof** This is again a straightforward consequence of lemma-1, once we know that a matrix $A$ can be reduced to the diagonal form if and only if the minimal polynomial of $A$ has simple roots.

**Remark 4:** If $K$ is no more algebraically closed, we can replace the statement of proposition 6 by the more general one: the ring $K[A]$ is an $S$-ring if and only if $A$ is semisimple (i.e. diagonalizable over an extension $K'$ of $K$).

**Remark 5:** It is not worthless to note that an element $M = P(A)$ of the ring $R = K[A]$ is invertible if and only if $\det(M) \neq 0$ or still, if and only if $P(X)$ and $\pi(X)$ are coprime: the first criterion results for instance, from a direct application of Cayley-Hamilton theorem; as for the second it is, in view of the isomorphism $K[A] \cong K[X]/(\pi)$, a consequence of Bezout theorem.

Before we start the study of the sets $S_i$ for the ring $K[A]$, together with their $GL(A)$-action, we give a general lemma which can be more easily stated if the underlying set of the group $GL(R)$ of a ring $R$ is denoted by $S_0(R)$:

**Lemma 3.** Let $E$ and $F$ be two rings and $E \times F$ be their product ring. Then, for $n \geq 1$

$$S_n(E \times F) = \bigcup S_i(E) \times S_j(F) \text{ the union being taken over } i + j \geq n.$$ 

**Proof** Let $x = x_1 \cdots x_i$ be an element of $S_i(E)$ and $y = y_1 \cdots y_j$ an element of $S_j(F)$ where all the $(x_k, y_k)$ are singular unless $i = 0$ or $j = 0$. We write $(x, y) = (x_1, 1) \cdots (x_i, 1)(1, y_1) \cdots (1, y_j)$; the element $(x, y)$
belongs to $S_{i+j}(E \times F) \subset S_n$, since $i + j \geq n \geq 1$. Conversely, let $(x, y) = (x_1, y_1) \cdots (x_n, y_n)$ be an element of $S_n(E \times F)$ where all the couples $(x_i, y_i)$ are singular. We write $(x, y) = (x_1 \cdots x_n, y_1 \cdots y_n)$ and we denote by $i$ the number (possibly equal to 0) of $x_k$ which are singular in $E$, so there are $(n-i)$ elements $x_k$ which are invertible; the corresponding $y_k$ are necessarily singular, so that at least $j \geq n - i$ elements among the $y_k$ are singular and $y \in S_j(F)$; the result then follows from the hypothesis $x \in S_i(E)$.

Remark 6: The lemma can be easily extended by induction to the case of a finite product of rings $E_1, \ldots, E_l$.

Remark 7: For $n \geq 2$, the indexation in lemma 3 could be replaced by $i + j = n$. (For $n = 1$, this is no more true because the factor $S_1 \times S_1$ cannot be taken into account). In the case of $k$ rings, we get the same for $n \geq k$.

**Proposition 7.** Let $R = K[A]$ and $\pi(X) = \prod_i (X - \lambda_i)^{\alpha_i}$, $i = 1, \ldots, r$ the minimal polynomial of $A$, then $S_\infty = S_\rho$ where $\rho = \sum_i (\alpha_i - 1) + 1$.

**Proof** Since the sets $S_i$ behave well under ring isomorphisms, we look at the problem in the ring $R = \prod_i R_i$, where $R_i$ denotes the quotient ring $K[X]/(X - \lambda_i)^{\alpha_i}$. Let $x = (x_1, \ldots, x_r)$ belong to $S_\rho(R)$; we shall prove that one of the components of $x$ is zero, this will imply clearly that $x \in S_\infty$. From lemma 3, we have $x_j \in S_{\beta_j}(R_j)$, where $\sum_j \beta_j \geq \rho$, so that one of the $\beta_i$, say $\beta_k$ is $\geq \alpha_k$ (otherwise, we would have $\sum_j \beta_j \leq \sum_j (\alpha_j - 1) < \rho$) which ensures $x_k \in S_{\alpha_k}(R_k) = \{0\}$. To end the proof, we notice that the element $x = ((X - \lambda_1)^{\alpha_1-1}, \ldots, (X - \lambda_r)^{\alpha_r-1})$ is in $S_{\rho-1}$ but not in $S_\rho$ (no component of $x$ is equal to zero !)

Again Lemma 3 will be of use to establish the following criterion:

**Proposition 8.** An element $P(A)$ in the ring $R = K[A]$ belongs to $S_2$ if and only if $P$ vanishes at at least two eigenvalues not necessary distinct of $A$ or at an eigenvalue of order one in the minimal polynomial of $A$.

**Proof** We keep the notation introduced in the precedent proof; the isomorphism between the ring $R = K[A]$ and the ring $\prod_i R_i$ is given by $P(A) \mapsto P_i$ where $P_i$ denotes the class of the polynomial $P(X)$ in the quotient $R_i$. Hence, the element $P(A)$ belongs to $S_2$ if and only if one among the $P_i$ belongs to $S_2(R_i)$ or at least two among the $P_i$, say $P_t$ and $P_s$, belong to $S_1(R_t)$ and $S_1(R_s)$ respectively, the second alternative implies clearly that the polynomial $P$ is divisible by $(X - \lambda_t)$ and by $(X - \lambda_s)$, the first alternative means that $P$ is divisible by $(X - \lambda_i)^2$ if $\alpha_i \geq 2$ or $P_i = 0$ if $\alpha_i = 1$. 


Remark 8: We understand now better the proposition 5 and the remark 3: to say that $A$ belongs to $S_2$ means that the polynomial $X$ (which cannot vanish at two eigenvalues of $A$!) vanishes at an eigenvalue of order 1 in $\pi_A$; since 0 is its only root, this means that 0 is a simple root of $\pi_A$. The image in the product $\prod R_i$ has one of its components 0 so, belongs to $S_\infty$.

Remark 9: A necessary and sufficient condition in order that an element $P(A)$ belongs to $S_3$ could be stated: the polynomial must vanish at at least three roots, or must be divisible by $(X - \lambda)^2$ where $\lambda$ is a root of order 2 of $\pi_A$, or vanish at a simple root of $\pi_A$. The proof is left to the reader.

Our purpose until the end of this section will be the study of the orbits of $GL(A)$ on the $S_i$. We begin with the case $\pi(X) = (X - \lambda)^\alpha$ (i.e. $A = \lambda I + N$, $N$ nilpotent). In this case $S_\alpha = \{0\} \subset S_{\alpha - 1} \subset \cdots \subset S_1$ (strict inclusions). For $i = 1, \ldots , \alpha - 1$, an element of $R = K[X]/(X - \lambda)^\alpha$ belongs to $S_i \setminus S_{i+1}$ if and only if it can be written as $(X - \lambda)^i Q(X)$, $Q$ and $\pi$ being mutually prime, which means in view of remark 5, that it belongs to the orbit of $(X - \lambda)^i$. This proves that the $S_i \setminus S_{i+1}$ along with $S_\alpha$ are the orbits of $GL(R)$ acting on $S_1$; in particular, there are $\alpha$ orbits.

The following lemma will permit us to compute the number of orbits in the general case:

**Lemma 4.** Let $G_i$ denotes the group of invertible elements of the ring $E_i$, $i = 1, \ldots , k$ and let $E$ be the product ring. Then $GL(E)$ is isomorphic to the product $\prod GL(E_i)$. Moreover, if $\alpha_i$ is the number of orbits of $G_i$ acting on $S_1(E_i)$, then the number of orbits of $GL(E)$ on $S_1(E)$ is given by $(\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_r + 1) - 1$.

**Proof** The assertion concerning $GL(E)$ is trivial. As for the second, we begin with the case $n = 2$. Considering the action of $G_1 \times G_2$ on the set of singular elements of $R_1 \times R_2$, we can divide the orbits in three kinds: orbits of elements $(x, y)$ where $x$ and $y$ are singular, orbits of elements $(x, y)$ where $x$ is singular and $y$ is invertible and finally, orbits of those elements $(x, y)$ where $x$ is invertible and $y$ singular. There are clearly $\alpha_1 \alpha_2$ orbits of the first kind, $\alpha_1$ of the second type and $\alpha_2$ of the third, which gives $\alpha_1 \alpha_2 + \alpha_1 + \alpha_2 = (\alpha_1 + 1)(\alpha_2 + 1) - 1$, first and last. An induction argument will do with the general case.

**Proposition 9.** The action of $GL(A)$ on the set of singular elements of $K[A]$ determines $\prod_i (\alpha_i + 1) - 1$ orbits, $i = 1, \ldots , r$ if the minimal polynomial is given by $\prod_i (X - \lambda_i)^{\alpha_i}$.

**Proof** it is an immediate consequence of lemma 4 and the discussion.
COROLLARY 3. The non empty sets $S_i \setminus S_{i+1}$, along with $S_{\infty}$ are exactly the orbits of $GL(A)$ acting on $K[A]$ if and only if the matrix $A$ can be written $A = \lambda I + N$, where $\lambda \in K$ and $N$ nilpotent.

Proof We have already established the sufficient condition. Conversely, our hypothesis implies that, in view of proposition 7 and 8,

$$\sum_i (\alpha_i - 1) + 1 = \prod_i (\alpha_i + 1) - 1$$

which is possible only if $r = 1$, that is $A = \lambda I + N$.

COROLLARY 4. Let $A$ have $r$ distinct eigenvalues, then $A$ is diagonalisable if and only if the number of orbits on the set of singular elements is $2^r - 1$.

Proof This is clear since the condition is equivalent to $\alpha_i = 1, \forall i$.

PROPOSITION 10. Let $S_{\infty} = S_{\rho}$ in the ring $R = K[A]$ and suppose that $K[A]$ is not an $S$-ring (i.e. $\rho \geq 2$), then the number of orbits of $GL(A)$ acting on the non-empty set $S_1 \setminus S_2$ is exactly the number of multiple roots of the minimal polynomial $\pi_A$. Moreover, the non-empty set $S_{\rho-1} \setminus S_{\rho}$ is exactly an orbit in the singular set.

Proof We keep use of the isomorphism $R \cong \prod_i R_i$ with its $GL(A) \cong \prod_i GL(R_i)$ action; an element $(x_1, \ldots, x_r)$ belongs to $S_1 \setminus S_2$ if and only if all the $x_i$ but one, say $x_k$ are invertible and $x_k$ belongs to $S_1(R_k) \setminus S_2(R_k)$; this set is hence non empty and a $GL(R_k)$-orbit. We get so a correspondence between the orbit of the element $(x_1, \ldots, x_r)$ and the necessary multiple eigenvalue $\alpha$. As for the second assertion, we first make use of lemma 3: the element $(x_1, \ldots, x_r)$ belongs to $S_{\rho-1} \setminus S_{\rho}$ if $x_i \in S_{\beta_i}(R_{\beta_i})$ and $\sum_i \beta_i \geq \rho - 1$ and no $x_i$ is zero (cf. proof of proposition 7), that is $\beta_i \leq \alpha_i - 1$; since $\rho - 1 = \sum_i (\alpha_i - 1)$, we get $\beta_i = \alpha_i - 1$, for every $i$. But each $S_{\alpha_i-1}(R_i) \setminus S_{\alpha_i}(R_i)$ is an orbit (even if $\alpha_i = 1$; see our convention of notation preceding lemma 3), the conclusion follows.

Remark 10: More generally, it is not difficult to establish that there is a one-to-one correspondence between the orbits in $S_k \setminus S_{k-1}$ and the $r$-uples $(a_1, \ldots, a_r)$ for which $a_1 + \cdots + a_r = k$ and $0 \leq a_i \leq \alpha_i - 1$ for every $i$. This gives for example in the case of a matrix $A$ with minimal polynomial $\pi_A(X) = X^3(X+1)^4(X-1)^3$ (here $\rho = 2 + 3 + 2 + 1 = 8$ and the number of orbits is 79) exactly 3 orbits in $S_1 \setminus S_2$, 6 orbits in $S_2 \setminus S_3$, 8 orbits in
\[ S_3 \setminus S_4, \text{8 orbits in } S_4 \setminus S_5, \text{6 orbits in } S_5 \setminus S_6, \text{3 orbits in } S_6 \setminus S_7, \text{one orbit in } S_7 \setminus S_8 \text{ and } 44 \text{ orbits in } S_8. \]

Computing all the orbits in \( S_1 \setminus S_\rho \), we need to know all the \((a_1, \ldots, a_r)\) such that \( \forall i \geq 0: a_i \leq \alpha_i - 1 \) and \( 1 \leq a_1 + \cdots + a_r \leq \rho - 1 \). This last inequality is a consequence of the first \( r \) inequalities, so there are \((\alpha_1 \cdots \alpha_r - 1)\) orbits in \( S_1 \setminus S_{\infty} \) and by substraction \( \prod (\alpha_i + 1) - (\alpha_1 \cdots \alpha_r) \) orbits in \( S_{\infty} \) (result which is valid even if \( \rho = 1 \)). It is now easy to solve the following:

Exercise 1: Prove that if \( A \) has exactly \( k \) distinct roots with \( k \geq 2 \), then \( A \) is diagonalisable if and only if there are \( 2^k - 1 \) orbits of \( GL(A) \) on \( S_\infty \). (Compare with corollary 4).

4. Permutable decompositions of singular matrices

If \( A \) is a singular matrix, we define \( n(A) \) as the upper bound of the numbers \( m \) of singular permutative matrices \( A_i \) such that \( A = A_1 \cdots A_m \). In order to compute the number \( n(A) \) for a given matrix \( A \), we need to introduce a special class of operators characterized by the following:

**Proposition 11.** For a given matrix acting on the finite dimensional vector space \( E = K^n \), it is equivalent to say:

a) \( \dim \ker(A^2) = 2 \dim \ker(A) \)

b) the Jordan cells of \( A \) associated with the eigenvalue 0 are of order \( \geq 2 \)

c) \( \ker(A) \subset \im(A) \)

d) the matrix \( A \) is similar to a matrix \( \begin{bmatrix} 0 & X \\ 0 & Y \end{bmatrix} \) written with respect to a direct decomposition of \( E = \ker(A) \oplus G \) where the linear operators

\[
X : G \xrightarrow{A} E \xrightarrow{pr_1} \ker(A) \quad Y : G \xrightarrow{A} E \xrightarrow{pr_2} G
\]

satisfy \((\alpha)\) \( \ker(X) \oplus \ker(Y) = G \) and \((\beta)\) \( X \) is onto.

**Proof** The equivalence between a) and b) results from the classical Jordan decomposition; the one between a) and c) is a direct consequence of the Frobenius injection \( \varphi : \ker(A^2)/\ker(A) \rightarrow \ker(A) \) given by \( \overline{x} \mapsto A(x) \); thus a) is equivalent to say that \( \varphi \) is surjective, which is exactly c). We prove now a) \( \Rightarrow \) d): let \( C_1 \) be a complementary subspace of \( \ker(A) \) in \( \ker(A^2) \) and \( C_2 \) be a complementary subspace of \( \ker(A^2) \) in \( E \) and write \( G = C_1 \oplus C_2 \) -we have already noticed that the restriction of \( A \) to \( C_1 \) is an isomorphism between \( C_1 \) and \( \ker(A) \); the same is true for the restriction of \( X \) to \( C_1 \),
since these restrictions are equal. It follows that $X$ is onto and that $C_1$ and $\ker(X)$ are complementary in $G$. We need only to prove that $C_1 = \ker(Y)$; it is clear that $C_1 \subset \ker(Y)$, moreover, if $A^+$ denotes the restriction of $A$ to $G$, $A^+$ is one-to-one so $\dim(C_1) + \dim(C_2) = \rk(A^+) = \rk \begin{bmatrix} X \\ Y \end{bmatrix} = \rk([X \\ Y]) = \rk(X) + \rk(Y) = \dim(C_1) + \rk(Y)$ and we are done.

Finally let us prove $d) \Rightarrow a)$: the matrix $A^2$ is similar to $\begin{bmatrix} 0 & XY \\ 0 & Y^2 \end{bmatrix}$ and with respect to the direct decomposition $E = \ker(A) \oplus G$, to say that the vector column $\begin{bmatrix} u \\ v \end{bmatrix}$ is in $\ker(A^2)$ means that $v \in \ker(Y) \cap \ker(XY)$ and $u$ is arbitrary in $\ker(A)$; but $\ker(Y) = \ker(Y)^2 \cap \ker(XY)$ if $\ker(X) \cap \ker(Y) = \{0\}$ (easy) so that $v \in \ker(Y)$. We end the proof by noting that since $X$ is onto and $G = \ker(X) \oplus \ker(Y)$, we have in fact $\dim \ker(Y) = \dim \ker(A)$.

We are able to state the main result of this section:

**Proposition 12.** The number $n(A)$ is finite if and only if $A$ satisfies the equivalent properties given in proposition 11. In which case $n(A) = \dim \ker(A)$.

**Proof** The matrix $A$ is similar to a matrix $B$ of the form:

$$B = \begin{bmatrix} B_0 & B_1 & 0 \\ B_1 & B_2 & \ddots \\ 0 & \ddots & B_k \end{bmatrix},$$

the matrix $B_0$ being invertible and each of the matrices $B_i$ being a Jordan cell associated to the eigenvalue 0 (obviously, $k = \dim \ker(A)$ and moreover $B_0$ is absent if $A$ is nilpotent). If one of the $B_i$ is of order 0, the matrix $A$ is similar to $B = \begin{bmatrix} B' & 0 \\ 0 & 0 \end{bmatrix}$ and $B = B_1 \times B_2 \times \cdots \times B_p$, with $B_1 = B$, $B_2 = \cdots = B_p = \begin{bmatrix} I_{n-1} & 0 \\ 0 & 0 \end{bmatrix}$ (with evident notation), all these matrices are singular and permutative, and we can choose $p$ as large as we want: $n(A) = \infty$. When $\dim \ker(A^2) = 2 \dim \ker(A)$, we have $B = B'_1 \times \cdots \times B'_k$ where $B'_1 = \begin{bmatrix} B_0 & 0 \\ B_1 & I_d \\ \vdots & \ddots \\ 0 & \cdots & I_d \end{bmatrix}$ (the blocks $B_0$ and $B_1$ kept unchanged and the others replaced by $I_d$) and
for $i = 2, \ldots, k$, $B'_i = \begin{bmatrix} I_d & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & I_d \end{bmatrix}$ (we replace all the blocks $B_j$ by $I_d$, except $B_i$ which remains unchanged); again these matrices are singular and permutative so $n(A) > \dim \ker(A)$.

We proceed to prove the opposite inequality (in due course we shall need two lemmas). Suppose that $M = X$ given by proposition 11 can be written as a product $N_1 \cdots N_{k+1}$, where the $N_i$ are permutative matrices; we shall show that one of the $N_i$ must be invertible.

Let us write $N_i = \begin{bmatrix} S_i & D_i \\ R_i & C_i \end{bmatrix}$ according to the decomposition of $M$. The first remark is $R_i = 0$. Indeed, since $N_i$ and $M$ commute, $N_i(\ker(M)) \subseteq \ker(M)$, that is $R_i = 0$. It follows that the $S_i$ are permutative and that $S_1 \times S_2 \cdots \times S_{k+1} = 0$.

**Lemma 5.** Let $S_1, \ldots, S_{k+1}$ be permutative matrices of order $k$ satisfying $S_1 \times S_2 \times \cdots \times S_{k+1} = 0$, then after reindexation $S_1 \times S_2 \times \cdots \times S_k = 0$.

**Proof** By induction. The result is trivial for $k = 1$; if $S_{k+1}$ is invertible, the conclusion is clear since we may multiply on the right by its inverse. We may then suppose that the dimension $d$ of the image subspace $\text{im}(S_{k+1})$ is strictly smaller than $n$. If $S'_i$, $i = 1, \ldots, n$, denotes the restriction (everything commute with $S_{k+1}$) of $S_i$ to the subspace $\text{im}(S_{k+1})$, we have $S'_1 \times S'_2 \times \cdots \times S'_k = 0$. This last expression can be thought (by grouping if necessary some operators together) as the null product of $d + 1$ commuting operators in a $d$-dimensional space. By induction hypothesis, we get (after possible reindexation, and reinserting of some possible operators) $S'_1 \times S'_2 \times \cdots \times S'_{k-1} = 0$, and conclude that at the level of the hole space $S_1 \times S_2 \times \cdots \times S_{k-1} \times S_{k+1} = 0$.

Accordingly, we may suppose that $S_1 \times \cdots \times S_k = 0$ and that, denoting the product $N = N_1 \cdots N_k$ by $\begin{bmatrix} H \\ U \end{bmatrix}$ and $N_{k+1}$ by $\begin{bmatrix} R & S \\ 0 & T \end{bmatrix}$, we have

$X = HT = RH + SU \quad (i)$

$Y = UT = TU \quad (ii)$, since $M = NN_{k+1} = N_{k+1}N$.

The last step of the proof will consist of proving that $R$ and $T$ are invertible.
(i) and (ii) imply that \( \ker(T) \subseteq \ker(X) \) and \( \ker(T) \subseteq \ker(Y) \) so that \( \ker(T) = \{0\} \) : \( T \) is invertible. Now since \( T \) is invertible, again (ii) shows that \( \ker(U) = \ker(Y) \) and (i) shows that \( \text{rk}(H) = \text{rk}(X) \).

Keeping the notations of proposition 11, we assert that \( G = \ker(X) \oplus \ker(U) \) and \( G = \ker(H) \oplus \ker(U) \); the first equality is now clear, the second will be established if \( \ker(H) \cap \ker(U) = \{0\} \), but this is easy since \( \ker(H) \cap \ker(U) \subseteq \ker(U) = \ker(Y) \) and by (i) \( \ker(H) \cap \ker(U) \subseteq \ker(X) \). We get now the invertibility of \( R \) from the following lemma:

**Lemma 6.** Consider the diagram:

\[
\begin{array}{ccc}
G & \xrightarrow{u} & G \\
\downarrow{h} & & \downarrow{x} \\
F & \xrightarrow{r} & F \\
\end{array}
\]

and suppose that \( x = r \circ h + s \circ u \) together with \( \ker(h) \) and \( \ker(x) \) in direct summand with \( \ker(u) \) in \( G \), then \( r \) induces an isomorphism between the images of \( h \) and \( x \).

*Proof* This is immediate as soon as we consider the restrictions to \( \ker(u) \) of the mappings given on \( G \).

**Corollary 5.** If \( n(A^k) \) is finite then \( n(A^k) = k \cdot n(A) \).

*Proof* Write \( \{0\} \subseteq \ker(A) \subseteq \ker(A^2) \subseteq \cdots \subseteq \ker(A^k) \subseteq \ker(A^{k+1}) \subseteq \cdots \subseteq \ker(A^{2k}) \). Since \( \dim \ker(A^{2k}) = 2 \dim \ker(A^k) \), the Frobenius inequalities:

\[
\dim \ker(A^{k+1}) - \dim \ker(A^k) \leq \dim \ker(A^k) - \dim \ker(A^{k-1})
\]

are in fact equalities so \( \dim \ker(A^k) = k \cdot \dim \ker(A) \).

Remark 11: The preceding corollary shows in particular that if \( n(A) \) is odd, the matrix \( A \) has no square root.

**Proposition 13.** Suppose \( n(A) < \infty \), and let \( A = X_1 \cdots X_m \) a permutative singular maximal decomposition of \( A \) (\( m = n(A) \)), then \( \forall i, \ n(X_i) < \infty \) and is \( = 1 \).

*Proof* We have \( \ker(X_i) \subseteq \ker(A) \subseteq \text{im}(A) \subseteq \text{im}(X_i) \), since the \( X_i \) commute. So \( n(X_i) \) is finite. We proceed, for proving \( n(X_i) = 1 \), by induction on \( m = \dim \ker(A) \); the case \( m = 1 \) is trivial. Write \( A = X_1 \cdot B \) where \( B = X_2 \cdots X_m \); as for \( X_i \), we prove that \( n(B) \) is finite, but \( B \) is
already written as $m-1$ permutative singular matrices, hence $n(B) \geq m-1$.
Remember now that $\ker(B) \subseteq \ker(A)$ so either $\dim \ker(B) = m - 1$ or $m$; we prove that it is not $m$: otherwise, the inclusion $\im(A) \subseteq \im(B)$ would in fact be an equality. Write now: $\im(B) = \im(A) = X_1(\im(B))$. This means that $X_1$ leaves $\im(B)$ invariant, and its restriction to $\im(B)$ is surjective, and hence $\ker(X_1) \cap \im(B) = \{0\}$. But $\ker(X_1) \subseteq \ker(A) \subseteq \im(A) = \im(B)$, so $X_1$ is bijective which is false. We have in fact $\dim \ker(B) = m - 1$, and $n(X_j) = 1 \ \forall j \geq 2$ by induction hypothesis. Since we could have chosen $B = X_1 \cdots X_{m-1}$, the fact $n(X_i) = 1$ is clear.

The next result is a simple application of proposition 12 to permutative decomposition of singular bistochastic matrices: if $A$ is such a matrix we define $n_+(A)$ as the upper bound of the number $m$ of singular permutative bistochastic matrices $A_i$ such that $A = A_1 \cdots A_m$.

**Proposition 14.** For a bistochastic matrix, $n_+(A) = n(A)$.

*Proof* We make again use of the isomorphism between the ring of bistochastic matrices and the product ring $M_{n-1}(K) \times K$, and may suppose $A = \begin{bmatrix} A_1 & 0 \\ 0 & \lambda \end{bmatrix}$ (see the proof of corollary 1); if $\lambda = 0$, $n_+(A) = n(A) = \infty$; and if $n(A) < \infty$ the scalar $\lambda$ is different from 0 (proposition 11 b)) and $n(A) = n(A_1)$ the conclusion follows easily.

We look in this final paragraph to the upper bound $m(A)$ of numbers $k$ such that $A = A_1 \cdots A_k$ where the $A_i$ are singular and quasi-commutative (i.e. $A_i A_j - A_j A_i$ is nilpotent).

**Proposition 15.** $m(A) = \infty, \forall A$.

*Proof* The problem behaves well under base change, and a simple argument similar to the one given at the beginning of the proof of proposition 12, shows that we only need to consider the case when $A$ is a Jordan cell $J_n$ associated to the zero eigenvalue. But if $B = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ & & & 0 \end{bmatrix}$, we have for every $m$, $B^m J_n = BJ_n = J_n$; we get the result by noting that two triangular matrices are quasi-commutative.

Exercises: 2 - Given an arbitrary matrix $A$, prove that there exists an invertible matrix $P$, such that $n(PA) < \infty$.

3 - Prove that if $n(A \otimes B) < \infty$, where $A \otimes B$ is the tensor
product of $A$ and $B$, then either $A$ or $B$ is invertible.

4 - Prove that if $p \geq 2$, then $n(\Lambda^p A) = \infty$. (We have denoted by $\Lambda^p A$ the $p^{th}$ exterior power of $A$).

5 - Prove that the ring of upper triangular matrices is an $S$-ring. Use this fact to give another proof of proposition 15.

References


Université de Paris VII
Département de Mathématiques
Tour 45.55, 5e étage
2, place Jussieu
75251 PARIS Cedex 05

and

Université de NICE
Parc Valrose
06034 NICE Cedex.