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On the order of vanishing of modular $L$-functions at the critical point

par Henryk Iwaniec

1. Introduction

The nonvanishing of $L$-functions at special points is an attractive area of research in contemporary number theory, see [7]-[11]. One example is the Rankin-Selberg zeta-function $L(f \otimes g_j, s)$ associated with a holomorphic cusp form $f$ of weight 2 and Maass cusp forms $g_j$ of eigenvalue $\lambda_j = s_j(1 - s_j)$. In this case the nonvanishing of $L(f \otimes g_j, s)$ at $s = s_j$ plays a rôle in the work of R. Phillips and P. Sarnak [6] on deformations of groups and was proved to be true for infinitely many cusp forms $g_j$ by J.-M. Deshouillers and H. Iwaniec [3]. Another example is the Birch-Swinnerton-Dyer conjecture which asserts that the rank of the group of rational points on an elliptic curve $E$ defined over $\mathbb{Q}$ is equal to the order of vanishing of the associated Hasse-Weil $L$-function $L(s, E)$ at $s = 1$ (the center of the critical strip).

Recently V.A. Kolyvagin [4] has proved that the group of rational points on a modular elliptic curve $E$ is finite if $L(1, E) \neq 0$ and that the $L$-function $L(s, E, \chi_d)$ twisted by a suitable real character $\chi_d$ has simple zero at $s = 1$. The latter condition was subsequently proved to hold true for infinitely many discriminants $d$ by D. Bump, S. Friedberg and J. Hoffstein [2] and independently by K. Murty and R. Murty [5]. In these notes we establish (from scratch) quantitative results on Kolyvagin’s condition.

2 - Statement of results

Let $E$ be a modular elliptic curve defined over $\mathbb{Q}$ and

$$L(s, E) = \sum_{n=1}^{\infty} a_n n^{-s}$$

be the Hasse-Weil $L$-function associated with $E$. Thus

$$f(z) = \sum_{n=1}^{\infty} a_n e(nz)$$

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is a cusp form of weight 2 which is a newform of level \(N\), where \(N\) is the conductor of \(E\). The \(L\)-function is entire and it satisfies the functional equation

\[
\left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s)L(s, E) = w \left(\frac{\sqrt{N}}{2\pi}\right)^{2-s} \Gamma(2-s)L(2-s, E),
\]

where \(w = \pm 1\). We are interested in curves \(E\) for which \(L(1, E) \neq 0\), so the functional equation holds with the sign \(w = 1\). The twisted \(L\)-function

\[
L(s, E, \chi_d) = \sum_{n=1}^{\infty} a_n \chi_d(n)n^{-s},
\]

where \(\chi_d\) is a real primitive character to modulus \(d\) prime to \(N\) is also entire and it satisfies the functional equation

\[
\left(\frac{d\sqrt{N}}{2\pi}\right)^s \Gamma(s)L(s, E, \chi_d) = w_d \left(\frac{d\sqrt{N}}{2\pi}\right)^{2-s} \Gamma(2-s)L(2-s, E, \chi_d)
\]

with the sign \(w_d = w\chi_d(-N)\). In the sequel we let \(d\) range over the set

\[
\mathcal{D} = \{d : 0 < d \equiv -\nu^2 (\text{mod } 4N) \text{ for some } \nu \text{ prime to } 4N\}
\]

and we let \(\chi_d(n) = \left(\frac{-d}{n}\right)\) be the Kronecker symbol. Thus if \(d\) is squarefree \(\chi_d\) is the primitive character to the modulus \(d\) which is associated with the imaginary quadratic field \(\mathbb{Q}(\sqrt{-d})\). Every prime dividing \(N\) splits in \(\mathbb{Q}(\sqrt{-d})\). Moreover we have \(w_d = -1\), so by (1) it follows that

\[
L(1, E, \chi_d) = 0.
\]

Our aim is to prove that \(L(s, E, \chi_d)\) has a simple zero at \(s = 1\), i.e. \(L'(1, E, \chi_d) \neq 0\) for infinitely many \(d\) in \(\mathcal{D}\). To this end we shall evaluate two sums of type

\[
S_4(Y) = \sum_{d \in \mathcal{D}, d \leq Y} |L'(1, E, \chi_d)|^4
\]

and

\[
S_1(Y) = \sum_{d \in \mathcal{D}} L'(1, E, \chi_d)F(d/Y),
\]

where \(\sum^b\) means that the summation is restricted to squarefree numbers and \(F\) is a smooth function, compactly supported in \(\mathbb{R}^+\) with positive mean value.
Theorem. For any \( c > 0 \) and \( Y \geq 1 \) we have
\[
S_4(Y) \ll Y^2 + \varepsilon
\]
and
\[
S_1(Y) = \alpha Y \log Y + \beta Y + O(Y^{13/14 + \varepsilon})
\]
with some constants \( \alpha \neq 0 \) and \( \beta \) which depend on the curve \( E \) and the test function \( F \).

Corollary. Suppose \( e > 0 \) and \( Y > c(e) \). Then \( L'(1, E, \chi_d) \neq 0 \) for at least \( Y^{2/3 - \varepsilon} \) real primitive characters \( \chi_d \) to modulus \( d \in D, d \leq Y \).

3. Estimates for the coefficients of \( f \)

The Fourier coefficients \( a_n \) of the cusp form \( f \) are multiplicative. More exactly, for \( \text{Re } s > 3/2 \) we have the Euler product
\[
L(s, E) = \prod_p (1 - \alpha_p p^{-s})^{-1} (1 - \beta_p p^{-s})^{-1}
\]
with \( \alpha_p = 0, \pm 1, \beta_p = 0 \) if \( p | N \) and \( |\alpha_p| = |\beta_p| = p^{1/2} \) if \( p \nmid N \). In the latter case the result was proved by M. Eichler and P. Deligne. It yields the following bound for the coefficient \( a_n \) (known as the Ramanujan conjecture)
\[
|a_n| \leq n^{1/2} \tau(n),
\]
where \( \tau(n) \) denotes the divisor function, \( \tau(n) \ll n^{\varepsilon} \). This bound can be slightly improved on average. Indeed, arguing as G. Hardy and E. Hecke with Parseval's formula and using the boundedness of \( yf(z) \) we get
\[
\sum_{m \leq M} |a_m|^2 \ll M^2.
\]
Similarly we get
\[
\sum_{m \leq M} a_m e(\alpha m) \ll M \log M
\]
for any real \( \alpha \) and \( M \geq 2 \), the implied constant depending on \( f \) only. In this section we derive three variations on (10).
LEMMA I. Let $\alpha$ be real and $\psi$ be a periodic function of period $r$. We then have

\begin{equation}
\sum_{m \leq M} a_m \psi(m)e(\alpha m) \ll \Psi M \log M,
\end{equation}

where

\[ \Psi = \frac{1}{r} \sum_{a \pmod{r}} \left| \sum_{b \pmod{r}} \psi(b)e\left(\frac{ab}{r}\right) \right|. \]

Moreover, if $|\psi| \leq 1$ and $s$ is a positive integer then we have

\begin{equation}
\sum_{m \leq M, (m, s) = 1} a_m \psi(m)e(\alpha m) \ll \tau(s)r^{1/2}M \log M
\end{equation}

and

\begin{equation}
\sum_{m \leq M, (m, s) = 1} b^s a_m \psi(m)e(\alpha m) \ll \tau(s)r^{1/2}M (\log M)^7
\end{equation}

PROOF: The sum on the left-hand side of (11) is equal to

\[ \frac{1}{r} \sum_{a \pmod{r}} \left( \sum_{b \pmod{r}} \psi(b)e\left(\frac{ab}{r}\right) \right) \sum_{m \leq M} a_m e((\alpha - \frac{a}{r})m), \]

whence the inequality (11) follows by (10). If $|\psi| \leq 1$ we obtain $\Psi \leq r^{1/2}$ by Cauchy’s inequality. For the proof of (12) we can assume that $(r, s) = 1$ by changing $\psi$ suitably. Then we apply (11) for $\psi \chi_0$ in place of $\psi$, where $\chi_0$ is the principal character to the modulus $s$. We obtain

\[ \Psi = \frac{1}{rs} \sum_{a \pmod{r}} \left| \sum_{b \pmod{r}} \psi(b)e\left(\frac{ab}{r}\right) \right| \]

\[ \sum_{c \pmod{s}} \left| \sum_{d \pmod{s}} \chi_0(d)e\left(\frac{cd}{s}\right) \right| \ll \frac{r^{1/2}}{s} \sum_{c \pmod{s}} \sum_{d \pmod{s}} d = \tau(s)r^{1/2}, \]

which gives (12). Finally we derive (13) from (12). The sum on the left-hand side of (13) is equal to

\[ \sum_{\nu^2m \leq M, (\nu m, s) = 1} \mu(\nu) a_{\nu^2m} \psi(\nu^2m)e(\alpha \nu^2 m) \]

\[ = \sum_{(\nu, s) = 1} \sum_{\lambda | \nu^\infty} \mu(\nu) a_{\nu^2\lambda} \sum_{m \leq M/\nu^2\lambda} \sum_{(m, \nu s) = 1} a_m \psi(\nu^2\lambda m)e(\alpha \nu^2 \lambda m) \]

\[ \ll \tau(s)r^{1/2}M (\log M) \sum_{(\nu, s) = 1} \sum_{\lambda | \nu^\infty} |a_{\nu^2\lambda}| \frac{\tau(\nu)}{\nu^2\lambda}. \]
Hence (13) follows by (8).

4. Approximate formulas for $L'(1, E, \chi_d)$

We shall express $L'(1, E, \chi_d)$ in terms of the rapidly convergent sums

$$A(X, \chi) = \sum_{n=1}^{\infty} a_n \chi(n)n^{-1} V \left( \frac{2\pi n}{X} \right),$$

where $V$ is the incomplete gamma function defined by

$$V(X) = \int_{X}^{\infty} e^{-t}t^{-1}dt = \frac{1}{2\pi i} \int_{(3/4)} \frac{\Gamma(s)}{s} X^{-s}ds.$$

We have

$$A(X, \chi_d) = \frac{1}{2\pi i} \int_{(3/4)} L(1 + s, E, \chi_d) \frac{\Gamma(s)}{s} \left( \frac{2\pi}{X} \right)^s ds.$$

Moving the integration to the line $\text{Re} \ s = -3/4$ we pass a simple pole at $s = 0$ with residuum $L'(1, E, \chi_d)$ by virtue of (2). On the other hand the integral over the line $\text{Res} = -3/4$ is equal to $-A(d^2 NX^{-1}, \chi_d)$ by the functional equation (1). This gives

$$L'(1, E, \chi_d) = A(X, \chi_d) + A(d^2 NX^{-1}, \chi_d)$$

for any $X > 0$ and $d$ in $\mathcal{D}$ which is squarefree. In particular we have

$$L'(1, E, \chi_d) = 2A(d\sqrt{N}, \chi_d).$$

By (9) we infer trivially that $A(X, \chi_d) \ll X^{1/2}$ for any $X > 0$ and inserting this to (14) we obtain

$$L'(1, E, \chi_d) = A(X, \chi_d) + O(dX^{-1/2}).$$

5. Estimation of the fourth moment of $L'(1, E, \chi_d)$

By the large sieve inequality (see [1]) together with (8) we get

$$\sum_{d \leq Y} \sum_{\chi \mod d}^* |A(X, \chi)|^4 \ll (X + Y)^{2+r}.$$
On the other hand by (14) we have for any $d \in \mathcal{D}, d \leq Y$, $d$ squarefree that

$$|L'(1, E, \chi_d)|^4 \ll \int_1^{NY} |A(X, \chi_d)|^4 X^{-1} dX.$$  

Combining both results we infer the upper bound (5) for $S_4(Y)$.

6. An approximate formula for the first moment of $L'(1, E, \chi_d)$

By (15) we obtain

$$S_1(Y) = 2 \sum_{d \in \mathcal{D}} b A(d\sqrt{N}, \chi_d) F\left(\frac{d}{Y}\right).$$

Now we relax the condition that $d$ is squarefree by introducing the factor $\sum_{a \mid d} \mu(a)$, then we split the sum according to whether $a \leq A$ or $a > A$ and in the latter case we return to squarefree numbers by extracting square divisors of $a^{-2}d$. We obtain $S_1(Y) = S + R$, say, where

$$S = 2 \sum_{a \leq A, \ (a, 4N) = 1} \mu(a) \sum_{d \in \mathcal{D}} A(a^2d\sqrt{N}, \chi_{a^2d}) F\left(\frac{a^2d}{Y}\right)$$

and

$$R = 2 \sum_{(b, 4N) = 1} \left( \sum_{a \mid b, \ a > A} \mu(a) \right) \sum_{d \in \mathcal{D}} b A(b^2d\sqrt{N}, \chi_{b^2d}) F\left(\frac{b^2d}{Y}\right).$$

Here $A$ is a large number to be chosen later. In the term $A(X, \chi_{b^2d})$ with $X = b^2d\sqrt{N}$ we return to $L'(1, E, \chi_d)$ by reversing the arguments as follows

$$A(X, \chi_{b^2d}) = \sum_{(n, b) = 1} a_n \chi_d(n) n^{-1} V\left(\frac{2\pi n}{X}\right)$$

$$= \sum_{k \mid b} \sum_{\ell \mid b} \alpha_k \beta_{\ell} \chi_d(k\ell) \frac{\mu(k) \mu(\ell)}{k\ell} A\left(\frac{X}{k\ell}, \chi_d\right)$$

$$= L'(1, E, \chi_d) \prod_{p \mid b} \left(1 - \chi_d(p) \frac{\alpha_p}{p}\right) \left(1 - \chi_d(p) \frac{\beta_p}{p}\right) + O(\tau(b)dX^{-\frac{1}{2}})$$
the second line being obtained by (7) and the third line by (16). Finally
applying (5) and the Hölder inequality we conclude that

\[(17) \quad R \ll \sum_{b} \left( \sum_{a \mid b, a > A} 1 \right) \left( b^{-\frac{3}{4}} Y^{\frac{1}{2}} + b^{-4} Y^{\frac{3}{2}} \right) Y^\epsilon \ll \left( A^{-\frac{3}{4}} Y^{\frac{1}{2}} + A^{-3} Y^{\frac{3}{2}} \right) Y^\epsilon. \]

7. A transformation of \( S \)

It remains to evaluate \( S \). For \((a, 4N) = 1\) and \( d \in D \) we have

\[ A(a^2 d \sqrt{N}, \chi_{a^2 d}) = \sum_{(n,a)=1} a_n n^{-1} \chi_d(n) V(2\pi n/a^2 d \sqrt{N}). \]

Every \( n \) can be written uniquely as the product \( n = k\ell^2 m \), where \( k \) has prime factors in \( 4N \), \( \ell m \) is prime to \( 4N \) and \( m \) is squarefree. For \( n \) written this way and \( d \) in \( D \) we have \( \chi_d(n) = \chi_d(m) \) subject to \((d, \ell) = 1\). The last condition is detected by the familiar formula of Möbius giving

\[ S = 2 \sum_{n=1}^{A} \mu(a) \sum_{n=1}^{A} a_n n^{-1} \chi_{d}(m) F \left( \frac{a^2 dq}{Y} \right) V \left( \frac{2\pi n}{a^2 dq \sqrt{N}} \right). \]

Next, by means of Gauss sums we write

\[ \chi_{d}(m) = \bar{\epsilon}_m m^{-\frac{1}{2}} \sum_{1 \leq |r| < m} \chi_{N r}(m) e \left( \frac{4N r d}{m} \right), \]

where \( \epsilon_m = 1 \) if \( m \equiv 1(\text{mod } 4) \), \( \epsilon_m = i \) if \( m \equiv -1(\text{mod } 4) \) and \( 4N4N \equiv 1(\text{ mod } m) \). This gives

\[ S = 2 \sum_{a \leq A} \sum_{n=1}^{A} \mu(a) n^{-1} \bar{\epsilon}_m m^{-\frac{1}{2}} \sum_{2 \leq |r| < m} \chi_{N r}(m) \sum_{d \in D} F \left( \frac{a^2 dq}{Y} \right) V \left( \frac{2\pi n}{a^2 dq \sqrt{N}} \right) e \left( \frac{4N r d}{m} \right). \]

where

\[ \sum_{d \in D} = \sum_{d \in D} F \left( \frac{a^2 dq}{Y} \right) V \left( \frac{2\pi n}{a^2 dq \sqrt{N}} \right) e \left( \frac{4N r d}{m} \right). \]

We put \( \Delta = \min(1/2, a^2 q Y^\epsilon - 1) \) and split \( S = S_0 + S_1 + S_2 \), where \( S_0, S_1, S_2 \) denote the partial sums restricted by the conditions \( r = 0, 0 < |r| < \Delta m, \Delta m \leq |r| < m/2 \) respectively.
8. Estimates for $S_2$ and $S_1$

**Lemma 2.** Suppose $g(x)$ is a smooth and integrable function on $\mathbb{R}$ with derivatives $g^{(j)}(x) \ll (|x| + X)^{-j}$ for all $j \geq 1$ the implied constant depending on $j$ only. Suppose $\alpha$ is real and $q$ is a positive integer such that $\alpha q$ is not an integer. We then have

$$
\sum_{n \equiv v \pmod{q}} g(n)e(\alpha n) \ll \frac{X}{q} \left( \frac{q}{X ||\alpha q||} \right)^j
$$

for any $j \geq 2$, the implied constant depending on $j$ only.

**Proof:** By Poisson’s formula the sum is equal to

$$
\frac{1}{q} \sum_{n=-\infty}^{\infty} e \left( \frac{uv}{q} \right) \hat{g} \left( \alpha - \frac{u}{q} \right),
$$

where $\hat{g}(y)$ denotes the Fourier transform of $g(x)$. We have

$\hat{g}(y) \ll X(Xy)^{-j}$ by the partial integration $j$ times, whence (18) follows by trivial summation over $u$.

To estimate $S_2$ we sum over $d$ first by an appeal to (18). For any $j \geq 2$ we get $\sum_{d} \ll (n + Y)^{-j}$, whence $S_2 \ll 1$.

To estimate $S_1$ we sum over $m$ first using (13) and partial summation together with the relation

$$
e \left( \frac{4Nrd}{m} \right) = e \left( \frac{rd}{4Nm} - \frac{mrd}{4N} \right)
$$

and then we sum over $r$ trivially getting

$$
\sum_{0<|r|<\Delta m} a_n n^{-1} \varepsilon_m m^{-\frac{1}{2}} \chi_{N\epsilon q}(m)V \left( \frac{2\pi n}{a^2dq\sqrt{N}} \right) e \left( \frac{4Nrd}{m} \right) 
\ll k^{-\frac{1}{3}} \ell^{-3} a^3q^2Y^{\epsilon-\frac{1}{2}}.
$$

Hence we conclude that

$$
S_1 \ll \sum_{a \leq A} \sum_{kr^2} \sum_{q \mid r} \sum_{d} F \left( \frac{a^2dq}{Y} \right) k^{-\frac{1}{3}} \ell^{-3} a^3q^2Y^{\epsilon-\frac{1}{2}} \ll A^2Y^{\epsilon+\frac{1}{2}}.
$$
9. Evaluation of $S_0$

Since $r = 0$ we have $\chi_{N\ell^q}(m) = 0$ for all $m > 1$ and the terms with $m = 1$ yield

$$S_0 = 2 \sum_{a \leq A} \mu(a) \sum_{n = k\ell^2} a_n n^{-1} \sum_{q | \ell} \mu(q) \sum_{d \in D} F \left( \frac{a^2 dq}{Y^2} \right) V \left( \frac{2\pi n}{a^2 dq \sqrt{N}} \right).$$

We split the summation over $d$ into residue classes modulo $4N$. Each class contributes

$$\frac{Y}{4Na^2q} \int F(t)V \left( \frac{2\pi n}{t\sqrt{NY}} \right) dt + O \left( \left( 1 + \frac{n}{Y} \right)^{-j} \right)$$

for any $j \geq 2$, and the number of relevant classes is

$$\gamma(4N) = \# \{ d(\text{mod} \ 4N) : d \equiv -\nu^2(\text{mod} \ 4N), (\nu, 4N) = 1 \}.$$ 

Hence

$$S_0 = \gamma(4N)Y \sum_{n = k\ell^2} \frac{a_n \varphi(\ell)}{2Nn\ell} \left( \sum_{a \leq A,(a,4N\ell)=1} \mu(a)a^{-2} \right) \int F(t)V \left( \frac{2\pi n}{t\sqrt{NY}} \right) dt + O \left( AY^r + \frac{1}{2} \right)$$

$$= c_N Y \int F(t)B(t\sqrt{NY}) dt + O((AY^\frac{1}{2} + A^{-1}Y)Y^r),$$

where

$$c_N = \frac{3\gamma(4N)}{\pi^2N} \prod_{p|4N} \left( 1 - \frac{1}{p^2} \right)$$

and

$$B(X) = \sum_{n = k\ell^2} \frac{b_n}{n} V \left( \frac{2\pi n}{X} \right)$$

with

$$b_n = a_n \prod_{p|n, \ p \neq 4N} \left( 1 + \frac{1}{p} \right).$$

To evaluate the series $B(X)$ we appeal to analytic properties of the zeta-function

$$L(s) = \sum_{n = k\ell^2} b_n n^{-s}.$$
The required properties are inherited from the properties of the Rankin-Selberg zeta-function

\[ H(s) = \sum_{n=1}^{\infty} a_n^2 n^{-s}. \]

The Rankin-Selberg zeta-function is meromorphic on \( \mathbb{C} \), holomorphic on \( \text{Re } s > 1 \) except for a simple pole at \( s = 2 \) with residuum

\[ H = \lim_{s \to 2} H(s) > 0, \]

and it satisfies a functional equation which connects \( H(s) \) with \( H(2 - s) \). Moreover, as shown by G. Shimura [12] the function

\[ L(s, \text{sym}^2) = \frac{\zeta(2s)}{\zeta(s)} H(s + 1) \]

is entire. By the Phragmén-Lindelöf principle, using the functional equation, it follows that

\[ L(s, \text{sym}^2) \ll |s| \text{ if } \text{Re } s > 1/2. \]

Since \( L(s) \) agrees with \( L(2s - 1, \text{sym}^2)/\zeta(4s - 2) = H(2s)/\zeta(2s - 1) \) up to an Euler product \( P(s) \), say, which converges absolutely in \( \text{Re } s > 3/4 \) we conclude that \( L(s) \) is holomorphic in \( \text{Re } s > 3/4 \), it satisfies

\[ L(s) \ll |s|^2 \text{ if } \text{Re } s > 3/4 \]

and that

\[ L(1) = HP(1) \neq 0. \]

Now by the contour integration we get

\[ \mathcal{B}(X) = \frac{1}{2\pi i} \int_{(3/4)} L(s + 1) \frac{\Gamma(s)}{s} \left( \frac{X}{2\pi} \right)^s ds \]

\[ = \lim_{s \to 0} L(s + 1) \frac{\Gamma(s)}{s} \left( \frac{X}{2\pi} \right)^s + \frac{1}{2\pi i} \int_{(-1/4)} \]

\[ = L(1) \left( \log \frac{X}{2\pi} - \gamma \right) + L'(1) + O(X^{-1/4}) \]

by the expansion \( \Gamma(s) = s^{-1} - \gamma + \ldots \), where \( \gamma \) is the Euler constant. Integrating against \( F(t) \) we conclude that

\[ S_\alpha = \alpha Y \log Y + \beta Y + O((AY^{-1/2} + A^{-1}Y)Y') \]
with

$$\alpha = c_N L(1) \int F(t) dt \neq 0$$

and

$$\beta = c_N \int F(t) \left[ L(1) \left( \log \frac{t \sqrt{N}}{2 \pi} - \gamma \right) + L'(1) \right] dt.$$  

10. Evaluation of the first moment of $L'(1, E, \chi_d)$. Conclusion

Collecting the established evaluations we infer that

$$S_1(Y) = S_0 + S_1 + S_2 + R = \alpha Y \log Y + \beta Y + O((AY^{3/2} + A^{-1}Y + A^2Y^{1/2} + A^{-2}Y^{3/4} + A^{-3}Y^{3/2})Y^e)$$

which gives (6) on taking $A = Y^{3/14}$.

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