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Turning points at infinity and stability of detonations

Mark Williams

Abstract

We begin by looking at a few simple examples of turning points in systems of ODEs depending on parameters, and then focus on the difficult case where the turning point occurs at infinity. We explain how turning points at infinity arise in a problem of detonation stability that was studied by J. J. Erpenbeck in the 1960s. In this problem the relevant system of ODEs describes the evolution of high frequency perturbations of a detonation profile, and the parameters on which the system depends are the perturbation frequencies. The resolution of the problem requires an analysis of the turning point at infinity that is uniform with respect to the parameters. This is joint work with Olivier Lafitte and Kevin Zumbrun.

Contents

1. Introduction

When studying the stability of structures arising in fluid mechanics, like shocks, detonations, vortex sheets, or boundary layers for example, one is often led to consider systems of ODEs depending on frequencies as parameters. Typically, the frequencies ζ arise as Fourier transform variables which appear after the original system (say, the Navier-Stokes equations) is linearized about the special structure whose stability is being studied (say, a viscous shock profile $s(x)$ depending on $x \in \mathbb{R}$), and then Fourier-transformed with respect to all variables except one, x, which is now the independent variable in a system of ODEs depending on ζ. Indeed, it is often the case that the hardest part of the stability analysis is to obtain a good understanding of how certain classes of solutions of this system of ODEs (for example, solutions that decay as $x \to +\infty$) vary as the parameters ζ vary. This analysis is greatly complicated by the presence of turning points, which are values of x where eigenvalues cross and the matrix governing the system of ODEs becomes singular. The study of turning points is the subject of a large literature going way back [14].

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Turning points in general vary with the frequency parameters and, in problems on unbounded domains, it is possible for turning points to occur "at infinity" for certain values of the parameters. We begin this talk with some examples of how this can happen and then formulate the main mathematical question addressed here, which in rough form is the following: When a turning point of a system of ODEs occurs at infinity for a particular frequency parameter ζ_0, how can one obtain a uniform understanding of the behavior of solutions for ζ near ζ_0? The kind of "understanding" we seek is, for example, an explicit expansion of the solution in terms of special functions that is uniformly valid in a given parameter domain containing (or near) ζ_0.

Next, we explain how this question arises in detonation theory, in particular, in connection with work on the stability of strong detonation profiles that was begun by Erpenbeck [1, 5, 4] in the 1960s. In [5, 4] Erpenbeck identified two basic classes of detonation profiles, those of increasing (type I) and decreasing type (type D). Our main result, given in Theorem 3.1 and proved in detail in [9], shows that there is a *uniform* high frequency cutoff for stability of type D detonations; in other words, there exists an $M > 0$ such that type D detonations are stable with respect to perturbations of frequency magnitude $\geq M$, independently of the frequency direction. We conclude by discussing some of the ideas used in our uniform analysis of the turning point at infinity that arises in the study of type D detonations.

2. Turning points

For $h > 0$ small, consider the ODE for the unknown $\phi(x)$

$$h\frac{d\phi}{dx} = \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix} \phi \text{ on } \mathbb{R}. \tag{2.1}$$

The matrix has eigenvalues $\pm\sqrt{x}$, which are real for $x > 0$, pure imaginary for $x < 0$. The eigenvalues cross at $x = 0$ and the matrix becomes singular there; we then call $x = 0$ a (finite) *turning point*. This system is equivalent to the scalar second order equation $h^2 w_{xx} = xw$, which is Airy's equation. The function w satisfies $h^2 w_{xx} = xw$ if and only if $\phi := \begin{pmatrix} w \\ hw_x \end{pmatrix}$ satisfies (2.1).[1]

If we introduce a complex parameter α and consider the system

$$h\frac{d\phi}{dx} = \begin{pmatrix} 0 & 1 \\ x-\alpha & 0 \end{pmatrix} \phi, \tag{2.2}$$

as an ODE on the complex plane, the new turning point of course depends on α: $x(\alpha) = \alpha$.

Exact solutions of (2.2) on $\mathbb{C}$ can be written in terms of Airy functions. Airy functions can also be used to describe solutions locally near the turning point $x(\alpha) = \alpha$ of slightly perturbed systems

$$h\frac{d\phi}{dx} = \begin{pmatrix} 0 & 1 \\ (x-\alpha) + hr(x,\alpha,h) & 0 \end{pmatrix} \phi, \tag{2.3}$$

where r is some analytic function of x [14].

2.1. How can a turning point be at infinity?

Consider the *model problem*

$$h\frac{d\phi}{dx} = \begin{pmatrix} 0 & 1 \\ e^{-2x} & 0 \end{pmatrix} \phi, \tag{2.4}$$

where now we regard $x = +\infty$ as a turning point for the same reasons as before.

The model problem with the extra parameter α^2

$$h\frac{d\phi}{dx} = \begin{pmatrix} 0 & 1 \\ e^{-2x} + \alpha^2 & 0 \end{pmatrix} \phi \tag{2.5}$$

exhibits finite turning points $x(\alpha)$ converging to $+\infty$ as $\alpha \to 0$ along the imaginary axis, as well as a turning point at ∞ when $\alpha = 0$.

[1] The system (2.1) exhibits the simplest kind of finite turning point; the analysis becomes more difficult as the power of x in (2.1) is increased [14, 13].

Problem 1. Construct solutions $\phi(x, \alpha, h)$ of this problem on $[0, \infty)$ that decay to zero as $x \to +\infty$. Describe behavior in a way that is *uniform* with respect to small $h > 0$ and small α.

2.2. Connection with Bessel's equation

Consider the scalar equation equivalent to the last system:

$$h^2 \frac{d^2 w}{dx^2} = (e^{-2x} + \alpha^2) w.$$

Under the change of variable $t = e^{-x}$ this becomes

$$h^2 (t^2 w_{tt} + t w_t) = (t^2 + \alpha^2) w. \tag{2.6}$$

Thus, the turning point at $x = +\infty$ for $\alpha = 0$ is mapped to $t = 0$, which is a *regular singular point* for (2.6).

Setting $z = h^{-1} t$ we obtain

$$h^2 [z^2 \frac{d^2 w}{dz^2} + z \frac{dw}{dz}] = (h^2 z^2 + \alpha^2) w.$$

Dividing by h^2 and setting $\beta := \frac{\alpha}{h}$, the equation becomes *Bessel's equation*:

$$z^2 w_{zz} + z w_z - (z^2 + \beta^2) w = 0.$$

It turns out that the resolution of the following variation on Problem 1 is (nearly equivalent to) the main step in the solution of the detonation stability question we discuss in the next section.

Problem 2. Consider the perturbed problem

$$h \frac{d\phi}{dx} = \begin{pmatrix} 0 & 1 \\ (e^{-2x} + \alpha^2) + h r(x, \alpha, h) & 0 \end{pmatrix} \phi. \tag{2.7}$$

Construct solutions $\phi(x, \alpha, h)$ of this problem on $[0, \infty)$ that decay to zero as $x \to +\infty$. Describe behavior in a way that is *uniform* with respect to small $h > 0$ and small α near the imaginary axis. Part of the problem is to decide when $hr(x, \alpha, h)$ can be regarded as a *perturbation*. For example, we'd better have $r(+\infty, \alpha, h) = 0$, at least!

Based on the earlier model problem we expect that Bessel's equation should be important in the analysis of this problem, and that a parameter like $\beta = \alpha / h$ will be a key parameter in the analysis.

3. Turning points at infinity in detonation theory

In the 1960s J.J. Erpenbeck studied [1, 5, 4, 3] the stability of certain solutions of the *Zeldovitch-von Neumann-Döring* (ZND) system of combustion equations (reactive compressible Euler coupled to a reaction rate equation):

$$\begin{aligned} &\partial_t v + \mathbf{u} \cdot \nabla v - v \nabla \cdot \mathbf{u} = 0 \\ &\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + v \nabla p = 0 \\ &\partial_t S + \mathbf{u} \cdot \nabla S = -r \Delta F / T \\ &\partial_t \lambda + \mathbf{u} \cdot \nabla \lambda = r. \end{aligned} \tag{3.1}$$

The unknowns in this 6×6 system are $(v, \mathbf{u}, S, \lambda)$ (specific volume, particle velocity $\mathbf{u} = (u^x, u^y, u^z)$, entropy, fraction of reactant) and the thermodynamic functions are the pressure $p = p(v, S, \lambda)$, temperature T, free energy ΔF, and reaction rate r. The ZND system is probably the most widely studied model of combustion in the detonation theory literature [6, 12].

Erpenbeck was interested in the stability of special solutions of this system given by steady, planar, *strong detonation profiles* $P(x) = (v, u, 0, 0, S, \lambda)$. Here the spatial variables are (x, y, z) and $P(x)$ is a weak solution of the ZND system depending only on $x \in \mathbb{R}$ with a jump, called the *von Neumann shock*, at $x = 0$.

The profile $P(x)$ varies with x and is subsonic, $0 < u(x) < c(x)$, in the *reaction zone* $[0, \infty)$; in the *quiescent zone* $(-\infty, 0]$, the profile $P(x)$ is constant and supersonic. The two constant states $P(\pm 0)$ defining the von Neumann shock actually give a stationary shock solution of the Euler equations without the reaction equation. In studying the stability of $P(x)$, Erpenbeck assumed that *the von Neumann shock is uniformly stable as an Euler shock*, so that he could focus on effects due to the reaction.

In $x \leq 0$, the quiescent zone, we have $\lambda = 1$. Moreover, the profile $P(x)$ has a well-defined endstate $P(+\infty)$ for which $\lambda(+\infty) = 0$ and

$$|P(x) - P(+\infty)| \leq Ce^{-\mu x} \text{ for some } \mu > 0. \tag{3.2}$$

Erpenbeck linearized the nonlinear system about this profile. He was perhaps the first person to realize, both here and in the study of Euler shocks [2], that the nonlinear system must be linearized not just with respect to the state variables, but also with respect to the front.

Taking the Laplace transform of the linearized system with respect to time, and the Fourier transform with respect to the transverse spatial variables (y, z), he obtained a 6×6 ODE in x that we write as

$$\begin{aligned} &(a)\ \frac{d\phi}{dx} = \mathcal{G}(x, \tau, \eta)\phi + F_1(x, \tau, \eta) \text{ for } x \in [0, \infty), \\ &(b)\ \mathcal{B}(\tau, \eta)\phi|_{x=0} = F_2(\tau, \eta). \end{aligned} \tag{3.3}$$

Here $\tau \in \mathbb{C}$ with $\Re\tau \geq 0$ is dual to time, η is dual to (y, z), and the x dependence enters only through $P(x)$. The boundary conditions are the linearized, transformed Rankine-Hugoniot conditions.

Using a rotational symmetry one can reduce the above system to a 5×5 system on $[0, \infty)$:

$$\begin{aligned} &(a)\ \frac{d\phi}{dx} = G(x, \tau, |\eta|)\phi + f_1(x, \tau, |\eta|) \text{ for } x \in [0, \infty) \\ &(b)\ B(\tau, |\eta|)\phi|_{x=0} = f_2(\tau, |\eta|). \end{aligned} \tag{3.4}$$

To resolve the stability question one must understand how solutions of this problem depend on the frequency parameters $(\tau, |\eta|)$. For example, (violent) instability is indicated by the existence of solutions to (3.4) that decay to zero as $x \to +\infty$ for frequencies $(\tau, |\eta|)$ with $\Re\tau > 0$. Such frequencies correspond to perturbations of $P(x)$ with transverse frequency magnitude $|\eta|$ that grow exponentially in time.

3.1. Erpenbeck's stability function

Erpenbeck defined a *stability function* (what we now call an *Evans function*) $V(\tau, |\eta|)$ such that $V(\tau, |\eta|) = 0$ implies the existence of an exponentially growing solution at the frequency $(\tau, |\eta|)$. To determine whether or not $V(\tau, \eta)$ vanishes for a given choice of (τ, η), one *could* attempt to construct decaying solutions to the interior equation (3.3)(a) on $[0, \infty)$, and then try to "see" if there are any that satisfy the boundary conditions at $x = 0$.

Instead, Erpenbeck showed that one could detect zeros of $V(\tau, |\eta|)$ more simply by studying decaying solutions of the homogenous, *transposed* interior equation:

$$\frac{d\theta}{dx} = -G^t(x, \tau, |\eta|)\theta = (|\eta|\Phi_0(x, \zeta) + \Phi_1(x))\theta. \tag{3.5}$$

Here ζ with $\Re\zeta \geq 0$ is defined by $\tau = |\eta|\zeta$.

Erpenbeck knew (or expected) that $V(\tau, |\eta|)$ would vanish for many frequencies when $|\eta|$ was of moderate size, so he focused on stability for large $|\eta|$. For such η the eigenvalues and eigenvectors of $\Phi_0(x, \zeta)$ are clearly decisive. Call these $\mu_j(x, \zeta)$ and $T_j(x, \zeta)$, $j = 1, \ldots, 5$.

For $\Re\zeta > 0$ the eigenvalues of $\Phi_0(x, \zeta)$ satisfy

$$\Re\mu_1(x, \zeta) < 0, \quad \Re\mu_j(x, \zeta) > 0 \text{ for } j = 2, \ldots, 5. \tag{3.6}$$

In view of (3.6) it is not surprising that

$$\frac{d\theta}{dx} = (|\eta|\Phi_0(x, \zeta) + \Phi_1(x))\theta \tag{3.7}$$

has a *one-dimensional* space of solutions decaying to zero as $x \to \infty$. Let us choose a nontrivial solution in this subspace $\theta(x, \zeta, |\eta|)$ and refer to it as *the* decaying solution.

Erpenbeck showed that for any given $|\eta|$ and $\tau = \zeta|\eta|$, *the stability function* $V(\tau, |\eta|)$ *is not* 0 *provided the decaying solution* $\theta(x, \zeta, |\eta|)$*, evaluated at* $x = 0$*, is parallel (or nearly parallel) to* $T_1(0, \zeta)$, the eigenvector of $\Phi_0(0, \zeta)$ associated to $\mu_1(0, \zeta)$.

Erpenbeck distinguished two basic types of detonations:

$$\begin{aligned} &- \text{ \textit{type D}, for which } \frac{d}{dx}(c^2-u^2)<0 \text{ on } [0,\infty) \\ &- \text{ \textit{type I}, for which } \frac{d}{dx}(c^2-u^2)>0 \text{ on } [0,\infty). \end{aligned} \tag{3.8}$$

These detonations are "basic" in the sense that some stability questions for more general detonations can be analyzed by decomposition of $P(x)$ into concatenations of these two types.

3.2. Erpenbeck's conclusions

In 1965 Erpenbeck provided convincing evidence, by a combination of rigorous and non-rigorous arguments, that:

(a) for some choices of ζ, detonations of type I are *violently unstable* for transverse frequencies of arbitrarily large magnitude $|\eta|$.

(b) for type D detonations there is a $\zeta-$*dependent* cutoff magnitude $M(\zeta)$ such that $V(\zeta|\eta|,|\eta|)\neq 0$ for $|\eta|\geq M(\zeta)$. This applied to all ζ with $\Re\zeta\geq 0$, *except for two special frequencies* ζ_0 and ζ_∞ on the imaginary axis, which his methods could not treat.

In 2012 [7] we gave rigorous proofs, with some simplifications and extensions, of Erpenbeck's 1965 conclusions. For type D detonations, the 2012 paper still provided *only* a ζ-dependent high frequency cutoff for stability, and still *failed* to treat the two exceptional frequencies.

Different methods were needed to obtain a *uniform* high frequency cutoff for stability of type D detonations and to treat the exceptional frequencies. As we'll see, the exceptional frequency ζ_0 is associated with a turning point at $x=0$, and the frequency ζ_∞ is associated with a turning point at $x=+\infty$.

3.3. Our main result: a *uniform* high frequency cutoff for stability of type D detonations

In the following theorem the assumptions are the same as those made by Erpenbeck in 1965.

Theorem 3.1 ([9]). *Assume:*

(H1) *All the thermodynamic functions appearing in the ZND equations (pressure, temperature, reaction rate, etc.) are real-analytic functions of their arguments* (v,S,λ)*, and the detonation profile* $P(x)$ *is of type D and real-analytic on* $[0,\infty)$*.*

(H2) *In the reaction equation,* $\partial_t\lambda+u\cdot\operatorname{grad}\lambda=r$*, the rate function* $r=r(v,S,\lambda)$ *satisfies* $r|_{\lambda=0}=0$*,* $r_\lambda<0$*.*

(H3) *The von Neumann shock is uniformly stable.*

Then there is a uniform *transverse frequency magnitude* M *such that if* $|\eta|\geq M$*, we have* $V(\zeta|\eta|,|\eta|)\neq 0$ *for* all ζ *with* $\Re\zeta\geq 0$.

Recall that to show $V(\zeta|\eta|,|\eta|)\neq 0$ for a particular $(\zeta,|\eta|)$, it is enough to show that the decaying solution of

$$\frac{d\theta}{dx}=(|\eta|\Phi_0(x,\zeta)+\Phi_1(x))\theta \tag{3.9}$$

on $[0,\infty)$ is nearly parallel to $T_1(0,\zeta)$ at $x=0$, i.e., that $\theta(0,\zeta,|\eta|)$ is nearly parallel to $T_1(0,\zeta)$.

Henceforth, let us set $h=\frac{1}{|\eta|}$, so $\tau=\zeta|\eta|=\zeta/h$, and write (3.9) as

$$h\frac{d\theta}{dx}=(\Phi_0(x,\zeta)+h\Phi_1(x))\theta. \tag{3.10}$$

With slight abuse we will now write $V(\zeta,h)$ instead of $V(\tau,|\eta|)$ and $\theta(x,\zeta,h)$ instead of $\theta(x,\zeta,|\eta|)$.

Thus, finding a uniform cutoff for stability amounts to finding $h_0>0$ such that for all ζ with $\Re\zeta\geq 0$ and all $0<h\leq h_0$ we have $V(\zeta,h)\neq 0$.

The eigenvalues $\{\mu_1,\mu_2\}$ of $\Phi_0(x,\zeta)$ satisfy:

$$\mu_2(x,\zeta)-\mu_1(x,\zeta)=k(x)s(x,\zeta), \tag{3.11}$$

where $s(x,\zeta) = \sqrt{\zeta^2 + (c^2(x) - u^2(x))}$ and $k(x) > C > 0$ for all x. Note: the eigenvalues μ_1 and μ_2 *cross* when $s = 0$. This can only happen when ζ lies in the *interval of turning point frequencies* given by:

$$\pm[\zeta_\infty, \zeta_0] := \pm\left[i\sqrt{(c^2 - u^2)(+\infty)}, i\sqrt{(c^2 - u^2)(0)}\right]. \tag{3.12}$$

Observe that when $\zeta = \zeta_\infty$, $s(x, \zeta_\infty)$ vanishes at $x = +\infty$, while for $\zeta = \zeta_0$, $s(x, \zeta_0)$ vanishes at $x = 0$. As ζ moves from ζ_∞ to ζ_0, the turning point moves from $+\infty$ to 0. The main difficulty in the proof is to obtain representations of $\theta(0, \zeta, h)$ that are *uniformly valid* for small h and for ζ near a turning point frequency. We focus now on the most interesting (and hardest) case of all, namely when ζ lies in a small neighborhood $\omega \ni \zeta_\infty$.

3.4. Some ideas of the proof

First one shows, using the analyticity assumption (and the *scalar* reaction equation), that the profile $P(x)$ actually has a convergent expansion

$$P(x) = P_0 + P_1 e^{-\mu x} + P_2 e^{-2\mu x} + \dots \text{ for some } \mu > 0, \tag{3.13}$$

and thus that $P(x)$ *extends analytically to the half-plane* $H(M_0) = \{x \in \mathbb{C} : \Re x > M_0\}$ for some large enough $M_0 > 0$.

Now, the eigenvalues of $\Phi_0(x, \zeta)$ have the property that for ζ near ζ_∞ and $x \in H(M_0)$, $\{\mu_1, \mu_2\}$ *are well separated from* $\{\mu_3, \mu_4, \mu_5\}$ for all (x, ζ).

This allows us to construct a 5×5 conjugator $Y(x, \zeta, h)$ such that the map $\theta = Y(x, \zeta, h)\phi$ *exactly* transforms the system (3.9) to block diagonal form on the wedge $W(M_0, \frac{\pi}{4}) = \{x \in \mathbb{C} : |\arg(x - M_0)| < \frac{\pi}{4}\}$:

$$h\frac{d}{dx}\phi = \begin{pmatrix} A_{11}(x,\zeta,h) & 0 \\ 0 & A_{22}(x,\zeta,h) \end{pmatrix}\phi, \tag{3.14}$$

where the blocks A_{11} and A_{22} are 2×2 and 3×3, with eigenvalues near $\{\mu_1, \mu_2\}$ and $\{\mu_3, \mu_4, \mu_5\}$, respectively.

The entries of the conjugator Y satisfy differential equations on the infinite wedge that are solved by contraction arguments. The analytic extension of $P(x)$ to $H(M_0)$ gives essential flexibility in choosing suitable paths of integration. [2]

Write $\phi = (\phi_1, \phi_2)$. The block A_{11} has eigenvalues close to the crossing eigenvalues $\mu_1(x, \zeta)$, $\mu_2(x, \zeta)$. Thus, for ζ near ζ_∞ we have *reduced* the problem of constructing the decaying solution $\theta(x, \zeta, h)$ on $[M_0, +\infty)$ *to constructing the decaying solution of the* 2×2 *system* $\frac{d}{dx}\phi_1 = A_{11}(x, \zeta, h)\phi_1$.

This 2×2 system can be written as an equivalent scalar, second-order equation

$$\begin{aligned} &h^2 w_{xx} = (C(x,\zeta) + hr(x,\zeta,h))w, \quad \text{where} \\ &C(x,\zeta) = (\mu_2 - \mu_1)^2(x,\zeta) = [\zeta^2 + (c^2(x) - u^2(x))]k^2(x), \end{aligned} \tag{3.15}$$

and $0 < C_1 \leq |k(x)| \leq C_2$. We focus on solving this equation on an infinite strip of the form $T_{M,R} := \{x \in \mathbb{C} : \Re x \geq M, |\Im x| \leq R\}$. Note that for M large enough, $T_{M,R} \subset W(M_0, \frac{\pi}{4})$.

The perturbation $r(x, \zeta, h)$, which can be expressed in terms of components of $P(x)$ and the conjugator $Y(x, \zeta, h)$, satisfies $r(+\infty, \zeta, h) = 0$; so there is hope the "perturbation" is not too big.

Moreover, the coefficient of w in $h^2 w_{xx} = (C(x,\zeta) + hr(x,\zeta,h))w$ *vanishes* at $(x, \zeta) = (+\infty, \zeta_\infty)$; there is a *turning point at* $x = +\infty$.

A transformation of the form $t = t(x, \zeta) = f(\zeta)e^{-\mu x/2}$ for some $f(\zeta)$, transforms (3.15) into an equation that is a perturbation of Bessel's equation:

$$\begin{aligned} &h^2(t^2 W_{tt} + tW_t) = (t^2 + \alpha^2)W + \\ &[\alpha^2 t^2 b_1(t,\zeta) + t^4 b_2(t,\zeta) + ht^2 b_3(t,\zeta,h)]W \text{ on } \mathcal{W}, \end{aligned} \tag{3.16}$$

where $\mathcal{W}$, the image of the strip $T_{M,R}$ under the map $t = t(x, \zeta)$, is a bounded wedge in $\{\Re t \geq 0\}$ with vertex at $t = t(+\infty, \zeta) = 0$.

[2]Examples given in [8] show that, in general, analyticity is necessary for the existence of such conjugators.

Bessel's equation is a very singular equation, with a regular singular point at 0, an irregular singular point at ∞, and turning points for certain choices of (α, h). It is a delicate matter to understand what is a tolerable perturbation of such a singular object. The behavior of solutions depends on both the *phase* of α and on the *relative magnitude* of α and h.

The parameter α in (3.16) satisfies $\alpha^2 \approx i(\zeta - \zeta_\infty)$. Setting $\beta = \alpha/h$, we distinguish *3 parameter regimes*:

$$\begin{aligned} I &: |\beta| \geq K, \ \ \arg(\zeta - \zeta_\infty) \in \left[-\frac{\pi}{2}, \frac{\pi}{2} - \delta\right] \\ II &: |\beta| \geq K, \ \ \arg(\zeta - \zeta_\infty) \in \left[\frac{\pi}{2} - \delta, \frac{\pi}{2}\right] \\ III &: |\beta| \leq K. \end{aligned} \tag{3.17}$$

The perturbed Bessel problem (3.16) can be analyzed in Regimes I, II, III by using suitable transformations of dependent and independent variables to reduce (3.16) to the *normal form*

$$W_{\xi\xi} = (u^2\xi^m + \psi(\xi))W, \tag{3.18}$$

where $m = 0$, 1, or -1, respectively, u is a large complex parameter, and ψ depends on the perturbation in (3.16).

In Regime I the correct choice of large parameter is $u = \beta = \alpha/h$ and a basis of solutions of (3.16) can be written in terms of exponentials , $e^{\pm u\xi}$. In Regime II, the large parameter is $u = -i\beta$ and a basis of solutions of (3.16) can be written in terms of Airy functions, $Ai_k(u^{2/3}\xi)$, $k = 1, 2$. In Regime III the large parameter is $u = 1/h$ and solutions are expressed in terms of the modified Bessel functions $I_{\frac{\alpha}{h}}(2u\xi^{1/2})$, $K_{\frac{\alpha}{h}}(2u\xi^{1/2})$.[3]

The validity of the transformations to normal form depends on precise estimates of how the functions $b_1(t, \zeta)$, $b_2(t, \zeta)$, $b_3(t, \zeta, h)$ and their $t-$derivatives grow on the wedge $\mathcal{W}$ and on the dilated wedge $\mathcal{W}/h$.

In each case the many transformations can be unravelled to identify the form of the decaying solution $\theta(x, \zeta, h)$ on the $x-$strip

$$T_{M,R} := \{x \in \mathbb{C} : \Re x \geq M, |\Im x| \leq R\}.$$

This analysis provides *explicit* expressions for $\theta(M, \zeta, h)$, one expression for each of the three regimes, that are uniformly valid for ζ near ζ_∞ and $h \in [0, h_0)$ Since *there are no turning points to the left of* $x = M$, it is then relatively easy to examine θ on $[0, M]$ and check that $\theta(0, \zeta, h)$ is nearly parallel to $T_1(0, \zeta)$. Thus, the stability function $V(\zeta, h)$ is nonvanishing for ζ near ζ_∞ and $h \in [0, h_0)$.

To complete the proof of the theorem, we give in [9] a uniform analysis near each of the *finite* turning point frequencies, a uniform analysis on any bounded set in the ζ half-plane that excludes a neighborhood of the turning point frequencies, and a uniform analysis on $\{|\zeta| \geq K\}$ for K large enough.

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