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Résumé

La méthode des vortex est une approche théorique et numérique couramment utilisée afin d'implémenter le mouvement d'un fluide parfait, dans laquelle le tourbillon est approché par une somme de points vortex, de sorte que les équations d'Euler se réécrivent comme un système d'équations différentielles ordinaires. Une telle méthode est rigoureusement justifiée dans le plan complet, grâce aux formules explicites de Biot et Savart. Dans un domaine extérieur, nous remplaçons également le bord imperméable par une collection de points vortex, générant une circulation autour de l'obstacle. La densité de ces points est choisie de sorte que le flot demeure tangent au bord sur certains points intermédiaires aux paires de tourbillons adjacents sur le bord. Dans ce travail, nous proposons une justification rigoureuse de cette méthode dans des domaines extérieurs. L'une des principales difficultés mathématiques étant que le noyau de Biot-Savart définit un opérateur intégral singulier lorsqu'il est restreint à une courbe. Par souci de simplicité et de clarté, nous traitons seulement le cas du disque unité dans le plan, approché par un maillage de points uniformément répartis. La version complète et générale de notre travail est disponible en [1].

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Abstract

The vortex method is a common numerical and theoretical approach used to implement the motion of an ideal flow, in which the vorticity is approximated by a sum of point vortices, so that the Euler equations read as a system of ordinary differential equations. Such a method is well justified in the full plane, thanks to the explicit representation formulas of Biot and Savart. In an exterior domain, we also replace the impermeable boundary by a collection of point vortices generating the circulation around the obstacle. The density of these point vortices is chosen in order that the flow remains tangent at midpoints between adjacent vortices. In this work, we provide a rigorous justification for this method in exterior domains. One of the main mathematical difficulties being that the Biot-Savart kernel defines a singular integral operator when restricted to a curve. For simplicity and clarity, we only treat the case of the unit disk in the plane approximated by a uniformly distributed mesh of point vortices. The complete and general version of our work is available in [1].

1. Introduction

Numerical methods describing the evolution of a flow have many practical interests in engineering and applications. It is therefore important to justify that given methods provide good approximations of analytic solutions. The goal of this proceeding is to validate the vortex method in exterior domains for the two-dimensional Euler equations.

1.1. The Euler equations in exterior domains

The motion of an incompressible ideal fluid filling a domain $\Omega \subset \mathbb{R}^2$ is governed by the Euler equations:

$$\begin{cases} \partial_t u + u\cdot\nabla u + \nabla p = 0 & \text{in} \quad (0,\infty)\times\Omega, \\ \operatorname{div} u = 0 & \text{in} \quad [0,\infty)\times\Omega, \\ u\cdot n = 0 & \text{on} \quad [0,\infty)\times\partial\Omega, \\ u(0,\cdot) = u_0 & \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

where $u = (u_1(t, x_1, x_2), u_2(t, x_1, x_2))$ is the velocity, $p = p(t, x_1, x_2)$ the pressure and n the unit inward normal vector. There is an impressive literature about the study of this system, first for physical motivations and second because it provides elegant mathematical problems at the boundaries of elliptic theory, dynamical systems, convex geometry and partial differential equations. The richness of these equations is due to the role of the vorticity:

$$\omega(t,x) := \operatorname{curl} u(t,x) = \partial_1 u_2 - \partial_2 u_1.$$

Indeed, taking the curl of the momentum equation, we note that this quantity satisfies a transport equation:

$$\partial_t \omega + u\cdot\nabla\omega = 0 \quad \text{in} \quad (0,\infty)\times\Omega. \tag{1.2}$$

From this form, we deduce many conservation properties which allow to establish the wellposedness of the Euler equations in several different settings (standard references can be found in [6, 13]). Therefore, one of the key steps in the analysis of (1.1) consists in reconstructing the velocity u from the vorticity ω by solving the following elliptic problem:

$$\begin{cases} \operatorname{div} u = 0 & \text{in} \quad \Omega, \\ \operatorname{curl} u = \omega & \text{in} \quad \Omega, \\ u \cdot n = 0 & \text{on} \quad \partial\Omega, \\ u \to 0 & \text{as} \quad x \to \infty, \end{cases} \tag{1.3}$$

where $\omega \in C_c^{0,\alpha}(\Omega)$, for some $0 < \alpha \leq 1$.

In the case of the full plane $\Omega = \mathbb{R}^2$, any solution of

$$\operatorname{div} u = 0 \text{ in } \mathbb{R}^2, \quad \operatorname{curl} u = \omega \text{ in } \mathbb{R}^2, \quad u \to 0 \text{ as } x \to \infty, \tag{1.4}$$

satisfies

$$\Delta u = \nabla^\perp \omega \quad \text{in } \mathbb{R}^2,$$

which easily yields

$$u = K_{\mathbb{R}^2}[\omega] = \mathcal{F}^{-1} \frac{i\xi^\perp}{|\xi|^2} \mathcal{F}\omega.$$

Here, the superscript $\perp$ denotes the rotation by $\pi/2$, that is $(x_1, x_2)^\perp = (-x_2, x_1)$. It follows, employing standard results on Fourier multipliers, that $K_{\mathbb{R}^2}$ has bounded extensions from L^p to $\dot{W}^{1,p}$, for any $1 < p < \infty$. Furthermore, writing $\Phi(x) = -\frac{1}{2\pi} \log|x|$ the fundamental solution of the Laplacian in $\mathbb{R}^2$, it holds that (see e.g. [7])

$$u = K_{\mathbb{R}^2}[\omega] = -\Phi * \left(\nabla^\perp \omega\right) = -\nabla^\perp \left(\Phi * \omega\right) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2} \omega(y) dy \in C^1\left(\mathbb{R}^2\right). \tag{1.5}$$

We refer to [5, p. 249] for a justification of the C^1-regularity of $K_{\mathbb{R}^2}[\omega]$.

When $\Omega = \{x \in \mathbb{R}^2,\ |x| > 1\}$ is the exterior of the unit disk, there are an infinite number of solution of (1.3), because of the harmonic vector field:

$$H(x) = \frac{1}{2\pi} \frac{x^\perp}{|x|^2},$$

which verifies

$$\operatorname{div} H = 0 \text{ in } \Omega, \quad \operatorname{curl} H = 0 \text{ in } \Omega, \quad H \cdot n = 0 \text{ on } \partial\Omega, \quad H \to 0 \text{ as } x \to \infty.$$

Thus, in order to reconstruct uniquely the velocity in terms of the vorticity, the standard idea consists in prescribing the circulation:

$$\oint_{\partial\Omega} u \cdot \tau \, ds = \gamma,$$

where $\gamma \in \mathbb{R}$ and $\tau := n^\perp$ is the tangent vector to $\partial\Omega$. This constraint is natural because Kelvin's theorem implies then that the circulation of u around an obstacle is a conserved quantity for the Euler equations. With this additional condition, it holds now true that there exists a unique solution u of

$$\begin{cases} \operatorname{div} u = 0 & \text{in} \quad \Omega, \\ \operatorname{curl} u = \omega & \text{in} \quad \Omega, \\ u \cdot n = 0 & \text{on} \quad \partial\Omega, \\ u \to 0 & \text{as} \quad x \to \infty, \\ \oint_{\partial\Omega} u \cdot \tau \, \mathrm{d}s = \gamma, \end{cases} \tag{1.6}$$

where $\omega \in C_c^{0,\alpha}(\Omega)$, for some $0 < \alpha \leq 1$, and $\gamma \in \mathbb{R}$ (see e.g. [12, Prop. 2.1]).

To solve this elliptic problem, we introduce the Green function with Dirichlet boundary condition $G_\Omega : \ \Omega \times \Omega \to \mathbb{R}$ as a function verifying:

$$\begin{aligned} G_\Omega(x, y) &= G_\Omega(y, x) \quad && \forall (x, y) \in \Omega^2, \\ \Delta_x G_\Omega(x, y) &= \delta(x - y) \quad && \forall (x, y) \in \Omega^2, \\ G_\Omega(x, y) &= 0 && \forall (x, y) \in \partial\Omega \times \Omega, \end{aligned}$$

where δ denotes the Dirac function centered at the origin. In the case of the exterior of the unit disk $D := \overline{B(0, 1)}$, we have an explicit formula:

$$G_\Omega(x, y) = \frac{1}{2\pi} \ln \frac{|x - y|}{|x - y^*||y|},$$

with the notation $y^* = \frac{y}{|y|^2}$, for any $y \in \mathbb{R}^2 \setminus \{0\}$. This expression allows us to write explicitly the solution of (1.6) (for all details, we refer e.g. to [12]):

$$\begin{aligned} u(x) = K_\Omega[\omega](x) + \alpha H(x) &:= \int_\Omega \nabla_x^\perp G_\Omega(x, y) \omega(y) \, dy + \alpha H(x) \\ &= \frac{1}{2\pi} \int_\Omega \left(\frac{x - y}{|x - y|^2} - \frac{x - y^*}{|x - y^*|^2} \right)^\perp \omega(y) \, dy + \frac{\alpha}{2\pi} \frac{x^\perp}{|x|^2} \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x - y)^\perp}{|x - y|^2} \left(\omega(y) - \frac{1}{|y|^4} \omega(y^*) \right) dy + \frac{\alpha}{2\pi} \frac{x^\perp}{|x|^2} \in C^1\left(\overline{\Omega}\right), \end{aligned} \tag{1.7}$$

where we have set

$$\alpha = \gamma + \int_\Omega \omega(y) \, dy.$$

Note that the total mass of the vorticity is also a conserved quantity of incompressible ideal two-dimensional flows.

In conclusion, the Euler equations around the obstacle D can be seen as the transport of the vorticity (1.2) by the velocity field u defined by (1.7). This property conveniently allows the use of various mathematical theories. It is therefore crucial to develop efficient and robust methods to rebuild the velocity field u from the vorticity ω or an approximation of it. In particular, for the sake of applications, we are now going to focus on the theoretical and numerical approximation of (1.7).

1.2. The vortex method

In the full plane $\mathbb{R}^2$, when the initial vorticity is close to be concentrated at N given points $\{x_i^0\}_{i=1}^N \subset \mathbb{R}^2$, i.e. $\omega(t = 0) \sim \sum_{i=1}^N \gamma_i \delta_{x_i^0}$ in some suitable sense, Marchioro and Pulvirenti [14] have shown that the corresponding solution of the Euler equations in the full plane has a vorticity which remains close to a combination of Dirac masses

$\omega(t) \sim \sum_{i=1}^N \gamma_i \delta_{x_i(t)}$ (in some suitable sense) where the centers $\{x_i\}_{i=1}^N$ verify a system of ODE's, called the point vortex system:

$$\begin{cases} \dot{x}_i(t) = \dfrac{1}{2\pi} \displaystyle\sum_{j\neq i} \gamma_j \dfrac{(x_i(t) - x_j(t))^\perp}{|x_i(t) - x_j(t)|^2}, \\ x_i(0) = x_i^0. \end{cases} \tag{1.8}$$

Here, the point vortex $\gamma_i \delta_{x_i(t)}$ moves under the velocity field produced by the other point vortices.

It turns out that this Lagrangian formulation is much easier to handle numerically than the Eulerian formulation (1.2). Indeed, standard numerical methods on (1.2) generate an "inherent numerical viscosity" and some quantities which should be conserved instead decrease (see e.g. [11, 17]). Actually, smoothing the Biot-Savart kernel by mollifying $\frac{x^\perp}{|x|^2}$ in (1.8) gives a more stable system, called the vortex-blob method (i.e. approximation of the vorticity by Dirac masses and regularization of the kernel). The stability and the convergence as $N \to \infty$ of the vortex-blob and point vortex methods have been extensively studied: in [3] for the vortex-blob method when the initial vorticity is bounded, in [9] for the point vortex method for smooth initial data and in [16] for both methods and for weak solutions as e.g. a vortex sheet (see also the textbook [4]).

However, all these works use the explicit formula of the Biot-Savart law in the full plane (1.5) where the flow $\frac{(x-x_i)^\perp}{2\pi|x-x_i|^2}$ is identified with $K_{\mathbb{R}^2}[\delta_{x_i}]$. In an exterior domain, the Biot-Savart law is much more complicated. A possible approach could be to use the explicit formula (1.7) in order to adapt the previous vortex methods. But such an approach would only be useful in the exterior of the disk. Indeed, if we consider that Ω is the exterior of a compact, simply connected subset of $\mathbb{R}^2$, we can implicitly adapt formula (1.7) thanks to conformal mappings, which has some theoretical interest, but this approach yields serious practical difficulties, for there are very few explicit Riemann mappings available.

Our alternative strategy consists in approximating the impermeable boundary of the exterior domain by a collection of point vortices $\sum_{i=1}^N \frac{\gamma_i^N(t)}{N} \delta_{x_i}$, where the vortex positions $\{x_i\}_{i=1}^N$ are fixed but the density of points $\{\gamma_i^N\}_{i=1}^N$ now evolves with time and is chosen in order that the resulting velocity field remains tangent at midpoints on the boundary between the x_i's. Note that this approach appears sometimes in physics and engineering books (see e.g. [2, 8]).

To this end, we introduce u_P the solution of (1.4) in the full plane, which is explicitly given by (1.5):

$$u_P := K_{\mathbb{R}^2}[\omega] \in C^1\left(\mathbb{R}^2\right) \subset C^1\left(\overline{\Omega}\right), \tag{1.9}$$

and the remainder velocity field u_R defined by:

$$u_R := u - u_P \in C^0\left(\overline{\Omega}\right) \cap C^1\left(\Omega\right). \tag{1.10}$$

As ω is compactly supported in Ω we get by the Stokes formula that $\oint_{\partial\Omega} u_P \cdot \tau \,\mathrm{d}s = \int_{B(0,1)} \operatorname{curl} u_P = \int_{B(0,1)} \omega = 0$. Hence, it is readily seen that u_R solves

$$\begin{cases} \operatorname{div} u_R = 0 & \text{in} \quad \Omega, \\ \operatorname{curl} u_R = 0 & \text{in} \quad \Omega, \\ u_R \cdot n = -u_P \cdot n & \text{on} \quad \partial\Omega, \\ u_R \to 0 & \text{as} \quad x \to \infty, \\ \oint_{\partial\Omega} u_R \cdot \tau \,\mathrm{d}s = \gamma. \end{cases} \tag{1.11}$$

In particular, u_R is harmonic in Ω and therefore it is smooth in Ω, i.e. $u_R \in C^\infty(\Omega)$ (see [7, Corollary 8.11] or [10]).

The vortex method for the exterior domain Ω is essentially an approximation procedure of u_R by point vortices on $\partial\Omega$.

Thus, let now $(x_1^N, x_2^N, \ldots, x_N^N)$ be the positions of N distinct point vortices on the boundary $\partial\Omega$. In the case of the disk, $\partial\Omega = \partial B(0,1) = \{(\cos\theta, \sin\theta) \in \mathbb{R}^2 \;:\; \theta \in [0, 2\pi)\}$, we consider

$$0 = \theta_1^N < \theta_2^N < \cdots < \theta_N^N < 2\pi \text{ such that } x_i^N = (\cos\theta_i^N, \sin\theta_i^N). \tag{1.12}$$

We further introduce some intermediate points on the boundary, for each $i = 1, \ldots, N-1$:

$$\tilde{\theta}_i^N \in (\theta_i^N, \theta_{i+1}^N), \quad \tilde{x}_i^N := (\cos\tilde{\theta}_i^N, \sin\tilde{\theta}_i^N). \tag{1.13}$$

The method consists in approximating the solution u_R to (1.11) by a suitable flow

$$u_{\text{app}}^N(x) := \frac{1}{2\pi} \sum_{j=1}^N \frac{\gamma_j^N}{N} \frac{(x - x_j^N)^\perp}{|x - x_j^N|^2} = K_{\mathbb{R}^2}\Big[\sum_{j=1}^N \frac{\gamma_j^N}{N} \delta_{x_j^N}\Big], \tag{1.14}$$

whose vorticity is precisely made of N point vortices with density $\left\{\frac{\gamma_i^N}{N}\right\}_{i=1}^N$ on the boundary $\partial\Omega$.

It is to be emphasized that this approximation is consistent with and motivated by the physical idea that the circulation around the obstacle (here, the unit disk $B(0,1)$) is created by a collection of vortices on the boundary of the obstacle, i.e. a vortex sheet on the boundary.

However, it is a priori not obvious that such a flow u_{app}^N can be made a good approximation of u_R. Nevertheless, note that u_{app}^N already naturally satisfies

$$\begin{cases} \operatorname{div} u_{\text{app}}^N = 0 & \text{in} \quad \Omega, \\ \operatorname{curl} u_{\text{app}}^N = 0 & \text{in} \quad \Omega, \\ u_{\text{app}}^N \to 0 & \text{as} \quad x \to \infty. \end{cases}$$

Therefore, the key idea lies in enforcing that the boundary and circulation conditions be satisfied as $N \to \infty$ by setting $\gamma^N = (\gamma_1^N, \ldots, \gamma_N^N) \in \mathbb{R}^N$ to be the solution of the following system of N linear equations:

$$\begin{aligned} &\frac{1}{2\pi} \sum_{j=1}^N \frac{\gamma_j^N}{N} \frac{(\tilde{x}_i^N - x_j^N)^\perp}{|\tilde{x}_i^N - x_j^N|^2} \cdot n(\tilde{x}_i^N) = -[u_P \cdot n](\tilde{x}_i^N), \quad \text{for all } i = 1, \ldots, N-1, \\ &\sum_{i=1}^N \frac{\gamma_i^N}{N} = \gamma. \end{aligned} \tag{1.15}$$

It will be shown later on, under suitable hypotheses on the placement of point vortices, that the above system always has a solution γ^N. The fact that u_{app}^N is a good approximation of u_R is precisely the content of our main theorem below (see Theorem 1.1). Clearly, it will then follow that u is well approximated by $u^N_{\rm app} + K_{\mathbb{R}^2}[\omega]$, which will conclude the rigorous justification of the vortex method for the boundary of the exterior of a disk. Other more complicated non-smooth exterior domains are investigated in the full version of our work [1].

Notice that it is now also possible to combine the vortex method for the boundary of an exterior domain with the aforementioned vortex method in the whole plane in order to obtain a full and dynamic vortex method for an exterior domain. To this end, we consider an approximation of the initial vorticity ω_0 by a combination of point vortices $\sum_{k=1}^M \alpha_k \delta_{y_k(0)}$. Then, the position $y_k(t)$ of each point vortex is let evolve under the influence of the vector field created by the remaining vortices $\sum_{p\neq k} \alpha_p \delta_{y_p(t)}$ (with possible regularization of the kernel) and the fixed vortices on the boundary $\sum_{i=1}^N \frac{\gamma_i^N(t)}{N}\delta_{x_i^N}$, where the variable vortex density $\gamma_i^N(t)$ is computed through (1.15) where u_P is replaced by $= K_{\mathbb{R}^2}[\sum_{k=1}^M \alpha_k \delta_{y_k(t)}]$.

Finally, it is to be emphasized that the main novelty of this method, when compared to the standard point vortex and vortex-blob methods, is the computation of γ^N though (1.15) allowing the construction of an approximate flow $K_{\mathbb{R}^2}[\omega] + u^N_{\rm app}$ which only requires the use of the Biot-Savart kernel in the whole plane and does not resort to (1.7).

1.3. Main result

For simplicity, we only consider in this work the stationary case where the points $\{x_i^N\}_{i=1}^N$ and $\{\tilde{x}_i^N\}_{i=1}^N$ are uniformly distributed on the unit circle:

$$\theta_i^N = \frac{(i-1)2\pi}{N} \quad \text{and} \quad \tilde{\theta}_i^N = \frac{(i-\frac{1}{2})2\pi}{N} \quad \forall i = 1,\dots,N. \tag{1.16}$$

Our main result states that the approximate flow $u^N_{\rm app}$, constructed through the procedure (1.15), is a good approximation of u_R:

Theorem 1.1. *Let $\omega \in C_c^{0,\alpha}(\Omega)$ (with $0 < \alpha \leq 1$) and $\gamma \in \mathbb{R}$ be given. For any $N \geq 2$, we consider the uniformly distributed mesh* (1.16) *and u_P defined in* (1.9).

Then, the system (1.15) *admits a unique solution $\gamma^N \in \mathbb{R}^N$. Moreover, for any closed set $K \subset \Omega$ there exists a constant $C = C(K)$ independent of N such that*

$$\|u_R - u^N_{\rm app}\|_{L^\infty(K)} \leq \frac{C}{N^2},$$

where $u^N_{\rm app}$ is given by (1.14) *in terms of γ^N and u_R is the continuous flow* (1.10).

Notice that in this particular case of the unit disk, we also have an explicit formula for u_R thanks to (1.7):

$$\begin{aligned} u_R(x) &= -\frac{1}{2\pi}\int_\Omega \frac{(x-y^*)^\perp}{|x-y^*|^2}\omega(y)\,dy + \frac{\alpha}{2\pi}\frac{x^\perp}{|x|^2} \\ &= -\frac{1}{2\pi}\int_{B(0,1)} \frac{(x-y)^\perp}{|x-y|^2}\frac{1}{|y|^4}\omega(y^*)\,dy + \frac{\alpha}{2\pi}\frac{x^\perp}{|x|^2}. \end{aligned} \tag{1.17}$$

Numerically, we indeed verify that the system (1.15) is always invertible, and that the L^∞-norm, on any compact set K, of the difference of $u^N_{\rm app}$ (given in (1.14)) with u_R (given in (1.17)) decreases as $1/N^2$, which is exactly the rate obtained in

Theorem 1.1. This rate is therefore optimal, at least from the numerical viewpoint. It would be interesting to obtain a rigorous proof of optimality.

The remainder of this article is composed of four parts. In the following section, we establish important representation formulas for the solution of (1.11), which will be used in the proof of our main theorem, and we show the link between our problem and the circular Hilbert transform. Then, in Section 3, we prove that the linear system (1.15) is invertible. In Section 4, we establish that $(u_R - u^N_{\text{app}}) \cdot n|_{\partial\Omega}$ converges to zero in a weak sense. Finally, in the last section, we deduce that such a weak convergence implies the conclusion of Theorem 1.1.

Thus, the goal of this article is to give a first and simpler poof of validity of the vortex method when restricted to the particular case of the disk where the points $\{x_i^N, \tilde{x}_i^N\}_{i=1}^N$ are uniformly distributed. The full general case of an arbitrary exterior domain and more generally distributed meshes is treated in [1]. Therein, we also consider the time dependence of the flow and non-zero velocities at infinity.

Remark. Removing the harmonic part $x^\perp/|x|^2$ and the circulation condition in (1.6) and (1.11), the main result can be readily adapted to describe an ideal fluid inside the unit disk (see [1] for more details).

2. Boundary vortex sheets and the circular Hilbert transform

We present now two distinct representation formulas – other than (1.17) – for the solution u_R of (1.11), which will be crucial for the justification of Theorem 1.1 and whose understanding will shed light on the approximation of u_R by point vortices on the boundary $\partial\Omega$.

Recall that we are considering some given vorticity $\omega \in C_c^{0,\alpha}(\Omega)$, with $0 < \alpha \leq 1$, and $\gamma \in \mathbb{R}$, and wish to construct a velocity field $u_R \in C^0\left(\overline{\Omega}\right) \cap C^1(\Omega)$ solving (1.11).

We show now that it is possible to express the solution to (1.11) as a vortex sheet on the boundary $\partial\Omega$, which, again, is consistent with the physical idea that the flow around an obstacle is produced by a boundary layer of vortices.

More precisely, we claim that u_R can be expressed as a boundary vortex sheet:

$$\begin{aligned} v(x) =& K_{\mathbb{R}^2}\left[g\delta_{\partial\Omega}\right] = \frac{1}{2\pi}\int_{\partial\Omega} \frac{(x-y)^\perp}{|x-y|^2} g(y)dy \\ =& -\frac{1}{2\pi}\int_{\partial\Omega} \frac{(x-y)\cdot\tau(y)}{|x-y|^2} n(y)g(y)dy \\ &+ \frac{1}{2\pi}\int_{\partial\Omega} \frac{(x-y)\cdot n(y)}{|x-y|^2}\tau(y)g(y)dy \in C^\infty\left(\mathbb{R}^2\setminus\partial\Omega\right), \end{aligned} \tag{2.1}$$

for some suitable $g \in C^{0,\alpha}(\partial\Omega)$, with $0 < \alpha \leq 1$. Notice that (1.14) is essentially a discretization of (2.1).

Indeed, the theory of single and double layer potentials (or of Cauchy integrals, see [15]) instructs us that, for a smooth boundary $\partial\Omega$ and for any $g \in C^{0,\alpha}(\partial\Omega)$, the flow defined by (2.1) is continuous up to the boundary $\partial\Omega$ (see [15, Chap. 2, § 16]), that is $v \in C\left(\overline{\Omega}\right) \cup C\left(\Omega^c\right)$, and that the limiting values of v on $\partial\Omega$ are given by (see [15, Chap. 2, § 17])

$$\lim_{\substack{x\to x_0\in\partial\Omega\\ x\in\Omega\cup\overline{\Omega}^c}}\frac{1}{2\pi}\int_{\partial\Omega}\frac{(x-y)\cdot\tau(y)}{|x-y|^2}n(y)g(y)dy=\frac{1}{2\pi}\int_{\partial\Omega}\frac{(x_0-y)\cdot\tau(y)}{|x_0-y|^2}n(y)g(y)dy,$$

$$\lim_{\substack{x\to x_0\in\partial\Omega\\ x\in\Omega}}\frac{1}{2\pi}\int_{\partial\Omega}\frac{(x-y)\cdot n(y)}{|x-y|^2}\tau(y)g(y)dy=\frac{1}{2\pi}\int_{\partial\Omega}\frac{(x_0-y)\cdot n(y)}{|x_0-y|^2}\tau(y)g(y)dy+\frac{1}{2}\tau(x_0)g(x_0),$$

$$\lim_{\substack{x\to x_0\in\partial\Omega\\ x\in\overline{\Omega}^c}}\frac{1}{2\pi}\int_{\partial\Omega}\frac{(x-y)\cdot n(y)}{|x-y|^2}\tau(y)g(y)dy=\frac{1}{2\pi}\int_{\partial\Omega}\frac{(x_0-y)\cdot n(y)}{|x_0-y|^2}\tau(y)g(y)dy-\frac{1}{2}\tau(x_0)g(x_0),$$

where the integral in the right-hand side of the first equation above is defined in the sense of Cauchy's principal value (note that, in the remaining equations, all integrals are defined in the usual sense).

Hence, we deduce that

$$\lim_{\substack{x\to x_0\in\partial\Omega\\ x\in\Omega}}v(x)=\frac{1}{2\pi}\int_{\partial\Omega}\frac{(x_0-y)^\perp}{|x_0-y|^2}g(y)dy+\frac{1}{2}\tau(x_0)g(x_0),$$

and

$$\lim_{\substack{x\to x_0\in\partial\Omega\\ x\in\overline{\Omega}^c}}v(x)=\frac{1}{2\pi}\int_{\partial\Omega}\frac{(x_0-y)^\perp}{|x_0-y|^2}g(y)dy-\frac{1}{2}\tau(x_0)g(x_0),$$

where, again, the integrals in the right-hand sides above are defined in the sense of Cauchy's principal value.

Therefore, we conclude that the flow $v(x)$ given by (2.1) defines the unique solution $u_R(x)\in C^0\left(\overline{\Omega}\right)\cap C^1(\Omega)$ of (1.11) if and only if $g\in C^{0,\alpha}(\partial\Omega)$ satisfies

$$\frac{1}{2\pi}\int_{\partial\Omega}\frac{(x-y)^\perp}{|x-y|^2}\cdot n(x)g(y)dy=u_R\cdot n(x)=-u_P\cdot n(x),\quad\text{for every } x\in\partial\Omega,\quad(2.2)$$

and

$$\begin{aligned}\int_{\partial\Omega}g(x)dx&=\int_{\partial\Omega}\left(\frac{1}{2\pi}\int_{\partial\Omega}\frac{(x-y)^\perp}{|x-y|^2}g(y)dy+\frac{1}{2}\tau(x)g(x)\right)\cdot\tau(x)dx\\&\quad-\int_{\partial\Omega}\left(\frac{1}{2\pi}\int_{\partial\Omega}\frac{(x-y)^\perp}{|x-y|^2}g(y)dy-\frac{1}{2}\tau(x)g(x)\right)\cdot\tau(x)dx\\&=\int_{\partial\Omega}u_R\cdot\tau(x)dx-\int_{\overline{\Omega}^c}\operatorname{curl}v(x)dx=\gamma.\end{aligned}\quad(2.3)$$

Again, we insist on the fact that the representation formula (2.1) for the solution of system (1.11) only involves the usual Biot-Savart kernel in the whole plane.

The existence of such a density $g\in C^{0,\alpha}(\partial\Omega)$ satisfying conditions (2.2) and (2.3) for any suitable given data is nontrivial, which we address now in the case of the unit disk only.

To this end, note that the singularity of the Biot-Savart kernel satisfies, for all $x, y \in \partial B(0,1)$, that

$$\frac{(x-y)^{\perp}}{|x-y|^2} \cdot n(x) = \frac{-y^{\perp} \cdot x}{|x-y|^2} = \frac{-\cos\left(\frac{\pi}{2} + \phi - \theta\right)}{4\sin^2\left(\frac{\phi-\theta}{2}\right)} = \frac{\sin(\phi-\theta)}{4\sin^2\left(\frac{\phi-\theta}{2}\right)} = -\frac{1}{2}\cot\left(\frac{\theta-\phi}{2}\right), \tag{2.4}$$

where $x = (\cos\theta, \sin\theta)$ and $y = (\cos\phi, \sin\phi)$, which is nothing but the kernel of the circular Hilbert transform.

Therefore, system (2.2)-(2.3) can be recast as

$$\begin{aligned} &\int_0^{2\pi} \cot\left(\frac{\theta-\phi}{2}\right) g(\phi) d\phi = f(\theta), \quad \text{for every } \theta \in [0, 2\pi], \\ &\int_0^{2\pi} g(\phi) d\phi = \gamma, \end{aligned} \tag{2.5}$$

where the 2π-periodic function $g \in C^{0,\alpha}([0,2\pi])$, for some $0 < \alpha \leq 1$, is the unknown and the 2π-periodic function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(\theta) = 4\pi[u_P \cdot n](\cos\theta, \sin\theta) \in C^{\infty}([0,2\pi]). \tag{2.6}$$

As u_P is smooth and divergence free, we note by the Stokes formula that

$$\int_0^{2\pi} f = 0. \tag{2.7}$$

Clearly, solving system (2.5) amounts to inverting the circular Hilbert transform

$$Hg(\theta) = \int_0^{2\pi} \cot\left(\frac{\theta-\phi}{2}\right) g(\phi) d\phi = -i \sum_{k \in \mathbb{Z}} \operatorname{sign}(k) \hat{g}(k) e^{ik\theta},$$

which is a well-known involution on the space of zero-mean periodic functions in $L^2([0,2\pi])$, that is to say

$$H^2 g(\theta) = -4\pi^2 \left(g(\theta) - \frac{1}{2\pi}\int_0^{2\pi} g(\phi) d\phi\right), \quad \text{for all } g \in L^2([0,2\pi]). \tag{2.8}$$

It therefore follows that the solution to (2.5) (or, equivalently, to (2.2)-(2.3)) is given by

$$g(\theta) = \frac{-1}{4\pi^2} Hf(\theta) + \frac{\gamma}{2\pi} = \frac{-1}{\pi} H[u_P \cdot n](\theta) + \frac{\gamma}{2\pi} \in C^{\infty}([0,2\pi]),$$

and is smooth, for f is smooth, whereby, in view of (2.1), we obtain the following representation formula on the exterior of a disk:

$$u_R(x) = -\frac{1}{2\pi^2} \int_{\partial B(0,1)} \frac{(x-y)^{\perp}}{|x-y|^2} H[u_P \cdot n](y) dy + \frac{\gamma}{4\pi^2} \int_{\partial B(0,1)} \frac{(x-y)^{\perp}}{|x-y|^2} dy.$$

In particular, we deduce, by comparing the above identity with (1.17) and by uniqueness of solutions to system (1.11), that it holds

$$\frac{1}{2\pi} \int_{\partial B(0,1)} \frac{(x-y)^{\perp}}{|x-y|^2} dy = \frac{x^{\perp}}{|x|^2}, \quad \text{for every } x \in \Omega. \tag{2.9}$$

Thus, we finally conclude that

$$u_R(x) = -\frac{1}{2\pi^2} \int_{\partial B(0,1)} \frac{(x-y)^{\perp}}{|x-y|^2} H[u_P \cdot n](y) dy + \frac{\gamma}{2\pi} \frac{x^{\perp}}{|x|^2}. \tag{2.10}$$

It turns out that there is yet another convenient representation formula for the flow u_R, which is a variant of the boundary vortex sheet (2.1).

More precisely, we claim now that in the exterior of a disk, u_R can also be expressed as:

$$\begin{aligned} w(x) =& \frac{1}{2\pi}\int_{\partial\Omega} \frac{x-y}{|x-y|^2} h(y)dy + \frac{\gamma}{2\pi}\frac{x^\perp}{|x|^2} \\ =& \frac{1}{2\pi}\int_{\partial\Omega} \frac{(x-y)\cdot n(y)}{|x-y|^2} n(y)h(y)\,dy + \frac{1}{2\pi}\int_{\partial\Omega} \frac{(x-y)\cdot \tau(y)}{|x-y|^2} \tau(y)h(y)\,dy \\ &+ \frac{\gamma}{2\pi}\frac{x^\perp}{|x|^2} \in C^\infty\left(\mathbb{R}^2 \setminus \partial\Omega\right), \end{aligned} \tag{2.11}$$

for some suitable $h \in C^{0,\alpha}(\partial\Omega)$, with $0 < \alpha \leq 1$.

As before, the theory of single and double layer potentials instructs us that, for a smooth boundary $\partial\Omega$ and for any $h \in C^{0,\alpha}(\partial\Omega)$, the flow w is continuous up to the boundary $\partial\Omega$, that is $w \in C\left(\overline{\Omega}\right) \cup C(\Omega^c)$, and that the limiting values of w on $\partial\Omega$ are given by

$$\begin{aligned} \lim_{\substack{x\to x_0\in\partial\Omega \\ x\in\Omega\cup\overline{\Omega}^c}} \frac{1}{2\pi}\int_{\partial\Omega} \frac{(x-y)\cdot\tau(y)}{|x-y|^2}\tau(y)h(y)\,dy &= \frac{1}{2\pi}\int_{\partial\Omega} \frac{(x_0-y)\cdot\tau(y)}{|x_0-y|^2}\tau(y)h(y)\,dy, \\ \lim_{\substack{x\to x_0\in\partial\Omega \\ x\in\Omega}} \frac{1}{2\pi}\int_{\partial\Omega} \frac{(x-y)\cdot n(y)}{|x-y|^2}n(y)h(y)\,dy &= \frac{1}{2\pi}\int_{\partial\Omega} \frac{(x_0-y)\cdot n(y)}{|x_0-y|^2}n(y)h(y)\,dy \\ &\quad + \frac{1}{2}n(x_0)h(x_0), \\ \lim_{\substack{x\to x_0\in\partial\Omega \\ x\in\overline{\Omega}^c}} \frac{1}{2\pi}\int_{\partial\Omega} \frac{(x-y)\cdot n(y)}{|x-y|^2}n(y)h(y)\,dy &= \frac{1}{2\pi}\int_{\partial\Omega} \frac{(x_0-y)\cdot n(y)}{|x_0-y|^2}n(y)h(y)\,dy \\ &\quad - \frac{1}{2}n(x_0)h(x_0), \end{aligned}$$

where the integral in the right-hand side of the first equation above is defined in the sense of Cauchy's principal value (note that, in the remaining equations, all integrals are defined in the usual sense).

Hence, we deduce that

$$\lim_{\substack{x\to x_0\in\partial\Omega \\ x\in\Omega}} w(x) = \frac{1}{2\pi}\int_{\partial\Omega} \frac{x_0-y}{|x_0-y|^2}h(y)dy + \frac{1}{2}n(x_0)h(x_0) + \frac{\gamma}{2\pi}\frac{x_0^\perp}{|x_0|^2},$$

and

$$\lim_{\substack{x\to x_0\in\partial\Omega \\ x\in\overline{\Omega}^c}} w(x) = \frac{1}{2\pi}\int_{\partial\Omega} \frac{x_0-y}{|x_0-y|^2}h(y)dy - \frac{1}{2}n(x_0)h(x_0) + \frac{\gamma}{2\pi}\frac{x_0^\perp}{|x_0|^2},$$

where, again, the integrals in the right-hand sides above are defined in the sense of Cauchy's principal value.

Therefore, we conclude that the flow $w(x)$ given by (2.11) defines the unique solution $u_R(x) \in C^0\left(\overline{\Omega}\right) \cap C^1(\Omega)$ of (1.11) if and only if $h \in C^{0,\alpha}(\partial\Omega)$ satisfies

$$\frac{1}{2\pi}\int_{\partial\Omega} \frac{x-y}{|x-y|^2}\cdot n(x)h(y)dy + \frac{1}{2}h(x) = u_R\cdot n(x) = -u_P\cdot n(x), \quad \text{for every } x\in\partial\Omega, \tag{2.12}$$

and

$$\begin{aligned}\int_{\partial\Omega} h(x)dx &= \int_{\partial\Omega}\left(\frac{1}{2\pi}\int_{\partial\Omega}\frac{x-y}{|x-y|^2}h(y)dy+\frac{1}{2}n(x)h(x)+\frac{\gamma}{2\pi}\frac{x^\perp}{|x|^2}\right)\cdot n(x)dx\\ &\quad-\int_{\partial\Omega}\left(\frac{1}{2\pi}\int_{\partial\Omega}\frac{x-y}{|x-y|^2}h(y)dy-\frac{1}{2}n(x)h(x)+\frac{\gamma}{2\pi}\frac{x^\perp}{|x|^2}\right)\cdot n(x)dx\\ &=\int_{\partial\Omega}u_R\cdot n(x)dx-\int_{\overline{\Omega}^c}\operatorname{div} w(x)dx=-\int_{\partial\Omega}u_P\cdot n(x)dx=0.\end{aligned}\tag{2.13}$$

Note that the circulation condition

$$\begin{aligned}\int_{\partial\Omega}u_R\cdot\tau(x)dx &= \int_{\partial\Omega}\left(\frac{1}{2\pi}\int_{\partial\Omega}\frac{x-y}{|x-y|^2}h(y)dy+\frac{1}{2}n(x)h(x)+\frac{\gamma}{2\pi}\frac{x^\perp}{|x|^2}\right)\cdot\tau(x)dx\\ &=\int_{\partial\Omega}\left(\frac{1}{2\pi}\int_{\partial\Omega}\frac{x-y}{|x-y|^2}h(y)dy-\frac{1}{2}n(x)h(x)\right)\cdot\tau(x)dx+\gamma\\ &=\int_{\overline{\Omega}^c}\operatorname{curl}\left(\frac{1}{2\pi}\int_{\partial\Omega}\frac{x-y}{|x-y|^2}h(y)dy\right)dx+\gamma=\gamma,\end{aligned}$$

is automatically satisfied.

The existence of such a density $h\in C^{0,\alpha}(\partial\Omega)$ satisfying conditions (2.12) and (2.13) for any suitable given data is nontrivial, which we address now in the case of the unit disk only.

To this end, note that the singularity of the Biot-Savart kernel satisfies, for all $x,y\in\partial B(0,1)$, that

$$\frac{x-y}{|x-y|^2}\cdot n(x)=\frac{1-y\cdot x}{|x-y|^2}=\frac{1-\cos(\phi-\theta)}{4\sin^2\left(\frac{\phi-\theta}{2}\right)}=\frac{1}{2},$$

where $x=(\cos\theta,\sin\theta)$ and $y=(\cos\phi,\sin\phi)$.

Therefore, it is readily seen that system (2.12)-(2.13) is uniquely solved by

$$h(\theta)=-2u_P\cdot n(\theta)\in C^\infty([0,2\pi]),$$

whereby, in view of (2.11), we obtain the following representation formula on the exterior of a disk:

$$u_R(x)=-\frac{1}{\pi}\int_{\partial B(0,1)}\frac{x-y}{|x-y|^2}(u_P\cdot n)(y)dy+\frac{\gamma}{2\pi}\frac{x^\perp}{|x|^2}.\tag{2.14}$$

It then follows, by comparing (2.14) with (2.10) and by uniqueness of solutions to system (1.11), that

$$\int_{\partial B(0,1)}\frac{x-y}{|x-y|^2}(u_P\cdot n)(y)dy=\frac{1}{2\pi}\int_{\partial B(0,1)}\frac{(x-y)^\perp}{|x-y|^2}H[u_P\cdot n](y)dy,\quad\text{for every } x\in\Omega,$$

whence we infer that, replacing $u_P\cdot n$ by $H\varphi$ in view of the arbitrariness of zero-mean boundary data in (1.11) and using the inversion of the Hilbert transform (2.8),

$$\begin{aligned}\int_{\partial B(0,1)}\frac{x-y}{|x-y|^2}H\varphi(y)dy &= \frac{1}{2\pi}\int_{\partial B(0,1)}\frac{(x-y)^\perp}{|x-y|^2}H^2\varphi(y)dy\\ &=-2\pi\int_{\partial B(0,1)}\frac{(x-y)^\perp}{|x-y|^2}\left(\varphi(y)-\frac{1}{2\pi}\int_{\partial B(0,1)}\varphi(z)dz\right)dy,\\ &\qquad\text{for every } x\in\Omega.\end{aligned}$$

Hence, for $y = (\cos\phi, \sin\phi) \in \partial B(0,1)$ and $x \in \Omega$, we have by (2.9):

$$\begin{aligned}\int_0^{2\pi}\int_0^{2\pi} \frac{x-y}{|x-y|^2}\cot\left(\frac{\phi-\theta}{2}\right)\varphi(\theta)d\theta d\phi &= -2\pi\int_0^{2\pi}\frac{(x-y)^\perp}{|x-y|^2}\varphi(\phi)d\phi \\ &\quad + 2\pi\frac{x^\perp}{|x|^2}\int_{\partial B(0,1)}\varphi(z)dz, \\ \int_0^{2\pi}\int_0^{2\pi} \frac{x-z}{|x-z|^2}\cot\left(\frac{\theta-\phi}{2}\right)\varphi(\phi)d\phi d\theta &= -2\pi\int_0^{2\pi}\frac{(x-y)^\perp}{|x-y|^2}\varphi(\phi)d\phi \\ &\quad + 2\pi\frac{x^\perp}{|x|^2}\int_0^{2\pi}\varphi(\phi)d\phi,\end{aligned}$$

where $z = (\cos\theta, \sin\theta)$. Finally, by the arbitrariness of φ, we conclude

$$\frac{1}{2\pi}\int_{\partial B(0,1)}\frac{x-z}{|x-z|^2}\cot\left(\frac{\phi-\theta}{2}\right)dz = \frac{(x-y)^\perp}{|x-y|^2} - \frac{x^\perp}{|x|^2}, \tag{2.15}$$

for every $x \in \Omega$ and $y \in \partial B(0,1)$, which will be useful later on. Once again, we insist on the fact that the above integral is defined is the sense of Cauchy's principal value.

Remark. As explained in the introduction, our goal is to justify that u^N_{app} (1.14) is a good discretization of the formulation (2.10). In fact, it would be easier, at least numerically, to discretize (2.14) which would give us a direct approximation of u_R without inverting large matrices (related to the computation of the inverse Hilbert transform H^{-1}; see Section 3).

For more general geometries of Ω, we prove in [1] that there also exist densities g and h satisfying the above conditions (2.2), (2.3), (2.12) and (2.13). However, (2.14) does not hold anymore and so, the processes to get g or h involve similar difficulties.

3. Solving system (1.15) and the discrete circular Hilbert transform

Using (2.4) and considering the angles $\{\theta_i^N\}$ and $\{\tilde\theta_i^N\}$ associated to $\{x_i^N\}$ and $\{\tilde x_i^N\}$ (see (1.12)-(1.13)), the system (1.15) of N equations can be recast as

$$\begin{aligned}&\frac{1}{N}\sum_{j=1}^N \gamma_j^N \cot\left(\frac{\tilde\theta_i^N - \theta_j^N}{2}\right) = f(\tilde\theta_i^N), \quad \text{for all } i = 1,\dots,N-1, \\ &\frac{1}{N}\sum_{i=1}^N \gamma_i^N = \gamma,\end{aligned} \tag{3.1}$$

where $\gamma^N = (\gamma_1^N,\dots,\gamma_N^N) \in \mathbb{R}^N$ is the unknown and f is defined in (2.6). Loosely speaking, solving system (3.1) amounts to inverting a discrete Hilbert transform on the circle. Indeed, (3.1) clearly is a discretization of (2.5).

From now on, we will also conveniently denote the matrices:

$$A_{N-1,N} := \left(\cot\left(\frac{\tilde\theta_i^N - \theta_j^N}{2}\right)\right)_{1\le i\le N-1, 1\le j\le N} \quad \text{and } A_N := \left(\cot\left(\frac{\tilde\theta_i^N - \theta_j^N}{2}\right)\right)_{1\le i,j\le N},$$

and we will make use of the following notations for $z \in \mathbb{R}^N$:

$$\begin{aligned}
\|z\|_{\ell^p} &:= \left(\frac{1}{N}\sum_{i=1}^N |z_i|^p\right)^{1/p}, \quad \text{for any } p \in [1,\infty),\\
\|z\|_{\ell^\infty} &:= \max_{i=1,\ldots,N} |z_i|,\\
\langle z\rangle &:= \frac{1}{N}\sum_{i=1}^N z_i.
\end{aligned}$$

Note that, with this normalization of the norms, we have:

$$\|z\|_{\ell^p} \le \|z\|_{\ell^q}, \text{ for any } 1 \le p \le q \le \infty.$$

Finally, for the uniformly distributed mesh (1.16), notice that, by odd symmetry of the cotangent function,

$$\sum_{1\le j\le N} \cot\left(\frac{\tilde\theta_i^N - \theta_j^N}{2}\right) = 0, \qquad \text{and} \qquad \sum_{1\le j\le N} \cot\left(\frac{\tilde\theta_j^N - \theta_i^N}{2}\right) = 0, \tag{3.2}$$

for each $i = 1,\ldots,N$. In fact, it can be shown (see [1]) that the only possible mesh satisfying (3.2) and $\theta_1^N = 0$ is necessarily given by (1.16).

Remark. These cancellations will be used several times in the following proofs and are related with the continuous version $\int_0^{2\pi} \cot\left(\frac{\phi-\theta}{2}\right)d\theta = 0$. As the oddness of the cotangent function plays a crucial role to define the Cauchy's principale value, the symmetry of the points $(\theta_i^N, \tilde\theta_i^N)$ is important to get a suitable discretization of the Hilbert transform.

The first result in this section is a precise ℓ^2-estimate on A_N for the uniformly distributed mesh (1.16).

Proposition 3.1. *Consider the uniformly distributed mesh* $(\theta_1^N,\ldots,\theta_N^N), (\tilde\theta_1^N,\ldots,\tilde\theta_N^N) \in [0,2\pi)^N$ *defined by* (1.16).

Then, for any $z \in \mathbb{R}^N$, *we have that*

$$\|z - \langle z\rangle \mathbf{1}\|_{\ell^2} = \frac{1}{N}\|A_N z\|_{\ell^2},$$

where $\mathbf{1} = (1,\ldots,1) \in \mathbb{R}^N$.

Proof. First, we compute

$$\begin{aligned}
N\|A_N z\|_{\ell^2}^2 &= \sum_{1\le k\le N}\left|\sum_{1\le j\le N}\cot\left(\frac{\tilde\theta_k^N-\theta_j^N}{2}\right)z_j\right|^2\\
&= \sum_{1\le k\le N}\sum_{1\le i,j\le N}\cot\left(\frac{\tilde\theta_k^N-\theta_i^N}{2}\right)\cot\left(\frac{\tilde\theta_k^N-\theta_j^N}{2}\right)z_i z_j\\
&= -\frac{1}{2}\sum_{1\le i,j\le N}(z_i-z_j)^2\sum_{1\le k\le N}\cot\left(\frac{\tilde\theta_k^N-\theta_i^N}{2}\right)\cot\left(\frac{\tilde\theta_k^N-\theta_j^N}{2}\right)\\
&\quad + \sum_{1\le i,k\le N}|z_i|^2\cot\left(\frac{\tilde\theta_k^N-\theta_i^N}{2}\right)\sum_{1\le j\le N}\cot\left(\frac{\tilde\theta_k^N-\theta_j^N}{2}\right).
\end{aligned}$$

Note that the last sum in the right-hand side is equal to zero by (3.2).

As for the remaining term above, we use the following elementary relation, valid for any a, b such that $a, b, a - b \notin \pi\mathbb{Z}$:

$$\cot a \cot b = \cot(b - a)[\cot a - \cot b] - 1,$$

to write

$$\begin{aligned} N\|A_N z\|_{\ell^2}^2 = &- \frac{1}{2} \sum_{1 \le i \neq j \le N} (z_i - z_j)^2 \cot\left(\frac{\theta_i^N - \theta_j^N}{2}\right) \sum_{1 \le k \le N} \left[\cot\left(\frac{\tilde{\theta}_k^N - \theta_i^N}{2}\right) - \cot\left(\frac{\tilde{\theta}_k^N - \theta_j^N}{2}\right)\right] \\ &+ \frac{N}{2} \sum_{1 \le i \neq j \le N} (z_i - z_j)^2 \\ =& \frac{N}{2} \sum_{1 \le i, j \le N} (z_i - z_j)^2, \end{aligned}$$

where we have also used (3.2).

Finally, the last sum is easily recast as

$$\begin{aligned} \frac{N}{2} \sum_{1 \le i, j \le N} (z_i - z_j)^2 &= N \sum_{1 \le i, j \le N} (z_i - \langle z \rangle)^2 - N \sum_{1 \le i, j \le N} (z_i - \langle z \rangle)(z_j - \langle z \rangle) \\ &= N^2 \sum_{1 \le i \le N} (z_i - \langle z \rangle)^2 = N^3 \|z - \langle z \rangle \mathbf{1}\|_{\ell^2}^2. \end{aligned}$$

We have therefore obtained that

$$N\|A_N z\|_{\ell^2}^2 = N^3 \|z - \langle z \rangle \mathbf{1}\|_{\ell^2}^2,$$

which ends the proof of the proposition. □

The preceding proposition allows us to get the existence and the uniqueness of the solution to (3.1).

Corollary 3.2. *Consider the uniformly distributed mesh* $(\theta_1^N, \ldots, \theta_N^N), (\tilde{\theta}_1^N, \ldots, \tilde{\theta}_N^N) \in [0, 2\pi)^N$ *defined by* (1.16).

Then, for any $v \in \mathbb{R}^{N-1}$ *and* $\gamma \in \mathbb{R}$*, the following problem:*

$$z \in \mathbb{R}^N, \quad \frac{1}{N} A_{N-1,N} z = v, \quad \langle z \rangle = \gamma, \tag{3.3}$$

has a unique solution. Moreover, this solution satisfies:

$$\|z\|_{\ell^1} \le \|z\|_{\ell^2} \le \|v\|_{\ell^2} + |\gamma| + \sqrt{N}\,|\langle v \rangle| \le \|v\|_{\ell^\infty} + |\gamma| + \sqrt{N}\,|\langle v \rangle|\,. \tag{3.4}$$

Proof. Let us define

$$\begin{aligned} \Phi \ : \quad & \mathbb{R}^N \to \mathbb{R}^{N+1} \\ & z \mapsto \begin{pmatrix} \frac{1}{N} A_N z \\ \langle z \rangle \end{pmatrix}, \end{aligned}$$

which is an injective linear mapping (see Proposition 3.1). Therefore, Φ is bijective from $\mathbb{R}^N$ onto Im Φ.

Moreover, noting that, for any $z \in \mathbb{R}^N$, we have

$$\langle A_N z \rangle = \frac{1}{N} \sum_{j=1}^N z_j \sum_{i=1}^N \cot\left(\frac{\tilde{\theta}_i^N - \theta_j^N}{2}\right) = 0,$$

by (3.2), and that $\dim(\mathrm{Im}\ \Phi) = N$, we conclude

$$\mathrm{Im}\ \Phi = \left\{ u \in \mathbb{R}^{N+1}\ :\ \sum_{i=1}^{N} u_i = 0 \right\}.$$

Now, let $v \in \mathbb{R}^{N-1}$ and $\gamma \in \mathbb{R}$ be fixed. There exists a unique v_N such that $\begin{pmatrix} v \\ v_N \\ \gamma \end{pmatrix} \in \mathrm{Im}\ \Phi$, namely $v_N = -\sum_{i=1}^{N-1} v_i$. With this v_N, we then deduce the existence of $z \in \mathbb{R}^N$ such that $\Phi(z) = \begin{pmatrix} v \\ v_N \\ \gamma \end{pmatrix}$. In particular z is a solution to (3.3) and, by Proposition 3.1, it holds that

$$\begin{aligned} \|z - \gamma \mathbf{1}\|_{\ell^2} = \|z - \langle z\rangle \mathbf{1}\|_{\ell^2} = \frac{1}{N}\|A_N z\|_{\ell^2} &= \left(\frac{1}{N}\sum_{i=1}^{N}|v_i|^2\right)^{1/2} \\ &= \left(\frac{1}{N}\sum_{i=1}^{N-1}|v_i|^2 + \frac{1}{N}\left|\sum_{i=1}^{N-1} v_i\right|^2\right)^{1/2}. \end{aligned}$$

As $\|z\|_{\ell^2} - |\gamma| \leq \|z - \gamma\mathbf{1}\|_{\ell^2}$ and

$$\begin{aligned} \left(\frac{1}{N}\sum_{i=1}^{N-1}|v_i|^2 + \frac{1}{N}\left|\sum_{i=1}^{N-1} v_i\right|^2\right)^{1/2} &\leq \left(\frac{1}{N}\sum_{i=1}^{N-1}|v_i|^2\right)^{1/2} + \frac{1}{\sqrt{N}}\left|\sum_{i=1}^{N-1} v_i\right| \\ &= \sqrt{\frac{N-1}{N}}\|v\|_{\ell^2} + \frac{N-1}{\sqrt{N}}\left|\langle v\rangle\right|, \end{aligned}$$

we conclude that

$$\|z\|_{\ell^2} \leq \|v\|_{\ell^2} + |\gamma| + \sqrt{N}\left|\langle v\rangle\right|.$$

Finally, concerning the uniqueness of a solution to (3.3), let us consider z and $\tilde{z}$ two solutions of (3.3). Then, $\Phi(z - \tilde{z}) = \begin{pmatrix} 0_{\mathbb{R}^{N-1}} \\ x \\ 0 \end{pmatrix}$ (for some $x \in \mathbb{R}$) belongs to Im Φ if only if $x = 0$. By injectivity of Φ, we conclude that necessarily $z = \tilde{z}$, thereby completing the proof of the corollary. □

4. Weak convergence of the discrete circular Hilbert transform

The results in this section will serve to show that $(u_R - u^N_{\mathrm{app}}) \cdot n|_{\partial\Omega}$ vanishes in a weak sense.

The following elementary lemma is a reminder about standard estimates on the rate of convergence of Riemann sums.

Lemma 4.1. *Consider the uniformly distributed mesh* $(\theta_1^N, \dots, \theta_N^N), (\tilde{\theta}_1^N, \dots, \tilde{\theta}_N^N) \in [0, 2\pi)^N$ *defined by* (1.16) *and let* g *be a smooth periodic function.*

Then, for any $0 < \alpha \leq 1$ *and* $k = 0, 1$,

$$\left| \int_0^{2\pi} g(\theta)\mathrm{d}\theta - \frac{2\pi}{N}\sum_{i=1}^{N} g(\tilde{\theta}_i^N) \right| \leq \frac{C}{N^{k+\alpha}}\|g\|_{C^{k,\alpha}},$$

for some independent constant $C > 0$.

Proof. First, a standard estimate yields, setting $\theta_{N+1}^N = 2\pi$,

$$\begin{aligned}\left|\int_0^{2\pi} g(\theta)\mathrm{d}\theta - \frac{2\pi}{N}\sum_{i=1}^N g(\tilde\theta_i^N)\right| &= \left|\sum_{i=1}^N \left(\int_{\theta_i^N}^{\theta_{i+1}^N} g(\theta)\mathrm{d}\theta - \frac{2\pi}{N} g(\tilde\theta_i^N)\right)\right| \\ &\leq \sum_{i=1}^N \left|\int_{\theta_i^N}^{\theta_{i+1}^N} \left(g(\theta) - g(\tilde\theta_i^N)\right)\mathrm{d}\theta\right| \\ &\leq \frac{(2\pi)^{1+\alpha}}{N^\alpha} \sup_{x,y\in[0,2\pi]} \frac{|g(x)-g(y)|}{|x-y|^\alpha} \leq \frac{(2\pi)^{1+\alpha}}{N^\alpha} \|g\|_{C^{0,\alpha}},\end{aligned}$$

which establishes the lemma when $k=0$.

For the case $k=1$, recalling $\tilde\theta_i^N = \frac{\theta_i^N + \theta_{i+1}^N}{2}$, one finds that

$$\begin{aligned}\left|\int_0^{2\pi} g(\theta)\mathrm{d}\theta - \frac{2\pi}{N}\sum_{i=1}^N g(\tilde\theta_i^N)\right| &\leq \sum_{i=1}^N \left|\int_{\theta_i^N}^{\theta_{i+1}^N} \left(g(\theta) - g(\tilde\theta_i^N)\right)\mathrm{d}\theta\right| \\ &= \frac{\pi}{N}\sum_{i=1}^N \left|\int_0^1 \left(g\left(\tilde\theta_i^N + \frac{\pi}{N}t\right) + g\left(\tilde\theta_i^N - \frac{\pi}{N}t\right) - 2g(\tilde\theta_i^N)\right)\mathrm{d}t\right| \\ &= \frac{\pi^2}{N^2}\sum_{i=1}^N \left|\int_0^1\int_0^1 t\left(g'\left(\tilde\theta_i^N + \frac{\pi}{N}st\right) - g'\left(\tilde\theta_i^N - \frac{\pi}{N}st\right)\right)\mathrm{d}t\mathrm{d}s\right| \\ &\leq \frac{\pi^{2+\alpha}}{N^{1+\alpha}} \sup_{x,y\in[0,2\pi]} \frac{|g'(x)-g'(y)|}{|x-y|^\alpha} \leq \frac{\pi^{2+\alpha}}{N^{1+\alpha}} \|g\|_{C^{1,\alpha}},\end{aligned}$$

which concludes the proof of the lemma. □

Proposition 4.2. *Consider the uniformly distributed mesh* $(\theta_1^N, \dots, \theta_N^N), (\tilde\theta_1^N, \dots, \tilde\theta_N^N) \in [0,2\pi)^N$ *defined by* (1.16) *and, according to Corollary 3.2, consider the solution* $\gamma^N = (\gamma_1^N, \dots, \gamma_N^N) \in \mathbb{R}^N$ *to the system* (3.1) *for some periodic function* $f \in C^{k,\alpha}([0,2\pi])$*, where* $k = 0, 1$*,* $0 < \alpha \leq 1$ *and* $k + \alpha \geq \frac{1}{2}$*, with zero mean value* (2.7) *and some* $\gamma \in \mathbb{R}$*. We define the approximation*

$$f_{\text{app}}^N(\theta) := \frac{1}{N}\sum_{j=1}^N \gamma_j^N \cot\left(\frac{\theta - \theta_j^N}{2}\right). \tag{4.1}$$

Then, for any periodic test function $\varphi \in C^{k+1,\alpha}([0,2\pi])$*,*

$$\left|\int_0^{2\pi} (f_{\text{app}}^N - f)\varphi\right| \leq \frac{C}{N^{k+\alpha}} \left(\|f\|_{C^{k,\alpha}} + |\gamma|\right) \|\varphi\|_{C^{k+1,\alpha}},$$

where the singular integrals are defined in the sense of Cauchy's principal value.

Proof. Let $\varphi \in C^\infty([0,2\pi])$ be a periodic test function. Then, we decompose

$$\begin{aligned}\int_0^{2\pi} (f_{\text{app}}^N - f)\varphi &= \left(\int_0^{2\pi} f_{\text{app}}^N\varphi - \frac{2\pi}{N}\sum_{i=1}^N f_{\text{app}}^N(\tilde\theta_i^N)\varphi(\tilde\theta_i^N)\right)\\ &\quad - \left(\int_0^{2\pi} f\varphi - \frac{2\pi}{N}\sum_{i=1}^N f(\tilde\theta_i^N)\varphi(\tilde\theta_i^N)\right)\\ &\quad + \frac{2\pi}{N}\sum_{i=1}^{N-1}\left(f_{\text{app}}^N(\tilde\theta_i^N) - f(\tilde\theta_i^N)\right)\varphi(\tilde\theta_i^N)\\ &\quad + \frac{2\pi}{N}\left(f_{\text{app}}^N(\tilde\theta_N^N) - f(\tilde\theta_N^N)\right)\varphi(\tilde\theta_N^N)\\ &=: D_1 + D_2 + D_3 + D_4.\end{aligned}$$

It is readily seen that D_3 is null, for $f_{\text{app}}^N(\tilde\theta_i^N) = f(\tilde\theta_i^N)$, for all $i = 1,\dots,N-1$, by construction (see (3.1)).

Next, note that D_2 is the error of approximation of the integral $\int_0^{2\pi} f\varphi$ by its Riemann sum. Therefore, a direct application of Lemma 4.1 yields

$$|D_2| \le \frac{C}{N^{k+\alpha}}\|f\varphi\|_{C^{k,\alpha}} \le \frac{C}{N^{k+\alpha}}\|f\|_{C^{k,\alpha}}\|\varphi\|_{C^{k,\alpha}}. \tag{4.2}$$

As for the term D_1, it is first rewritten, exploiting the symmetry of the cotangent function (see (3.2)), as

$$\begin{aligned}D_1 &= \int_0^{2\pi} f_{\text{app}}^N\varphi - \frac{2\pi}{N}\sum_{i=1}^N f_{\text{app}}^N(\tilde\theta_i^N)\varphi(\tilde\theta_i^N)\\ &= \frac{1}{N}\sum_{j=1}^N\gamma_j^N\int_0^{2\pi}\cot\left(\frac{\theta-\theta_j^N}{2}\right)\varphi(\theta)\,\mathrm{d}\theta - \frac{2\pi}{N^2}\sum_{i,j=1}^N\gamma_j^N\cot\left(\frac{\tilde\theta_i^N-\theta_j^N}{2}\right)\varphi(\tilde\theta_i^N)\\ &= \frac{1}{N}\sum_{j=1}^N\gamma_j^N\int_0^{2\pi}\cot\left(\frac{\theta-\theta_j^N}{2}\right)\left(\varphi(\theta)-\varphi(\theta_j^N)\right)\,\mathrm{d}\theta\\ &\quad - \frac{2\pi}{N^2}\sum_{i,j=1}^N\gamma_j^N\cot\left(\frac{\tilde\theta_i^N-\theta_j^N}{2}\right)\left(\varphi(\tilde\theta_i^N)-\varphi(\theta_j^N)\right)\\ &= \int_0^{2\pi}F(\theta)\mathrm{d}\theta - \frac{2\pi}{N}\sum_{i=1}^N F(\tilde\theta_i^N),\end{aligned}$$

where

$$F(\theta) = \frac{1}{N}\sum_{j=1}^N\gamma_j^N\cot\left(\frac{\theta-\theta_j^N}{2}\right)\left(\varphi(\theta)-\varphi(\theta_j^N)\right).$$

Note that the integrand $\theta\mapsto\cot\left(\frac{\theta-\theta_j^N}{2}\right)(\varphi(\theta)-\varphi(\theta_j^N))$ above is now regular, thus assuring that the Riemann sums converge. It therefore follows from Lemma 4.1 that

$$\begin{aligned}|D_1| &\le \frac{C}{N^{k+\alpha}}\|F\|_{C^{k,\alpha}} \le \frac{C}{N^{k+\alpha}}\left\|\gamma^N\right\|_{\ell^1}\|x\cot x\|_{C^{k,\alpha}([0,\frac{\pi}{2}])}\|\varphi'\|_{C^{k,\alpha}}\\ &\le \frac{C}{N^{k+\alpha}}\left\|\gamma^N\right\|_{\ell^1}\|\varphi\|_{C^{k+1,\alpha}}.\end{aligned}$$

Then, further utilizing estimate (3.4), Lemma 4.1, that $k+\alpha \geq \frac{1}{2}$ and the fact that f has zero mean value (2.7), we infer

$$\begin{aligned}
|D_1| &\leq \frac{C}{N^{k+\alpha}}\left(\|f\|_{L^\infty}+|\gamma|+\sqrt{N}\left|\frac{1}{N-1}\sum_{i=1}^{N-1} f(\tilde{\theta}_i^N)\right|\right)\|\varphi\|_{C^{k+1,\alpha}} \\
&\leq \frac{C}{N^{k+\alpha}}\left(\|f\|_{L^\infty}+|\gamma|+\sqrt{N}\left|\int_0^{2\pi} f(\theta)\mathrm{d}\theta-\frac{2\pi}{N}\sum_{i=1}^{N} f(\tilde{\theta}_i^N)\right|\right)\|\varphi\|_{C^{k+1,\alpha}} \\
&\leq \frac{C}{N^{k+\alpha}}\left(\|f\|_{C^{k,\alpha}}+|\gamma|\right)\|\varphi\|_{C^{k+1,\alpha}}.
\end{aligned} \tag{4.3}$$

Finally, regarding D_4, recalling that, by (3.1) and (3.2),

$$\sum_{i=1}^{N-1} f(\tilde{\theta}_i^N)=\frac{1}{N}\sum_{j=1}^{N}\gamma_j^N\sum_{i=1}^{N-1}\cot\left(\frac{\tilde{\theta}_i^N-\theta_j^N}{2}\right)=-\frac{1}{N}\sum_{j=1}^{N}\gamma_j^N\cot\left(\frac{\tilde{\theta}_N^N-\theta_j^N}{2}\right)=-f_{\mathrm{app}}^N(\tilde{\theta}_N^N),$$

we find

$$\begin{aligned}
D_4 &= \frac{2\pi}{N}\left(f_{\mathrm{app}}^N(\tilde{\theta}_N^N)-f(\tilde{\theta}_N^N)\right)\varphi(\tilde{\theta}_N^N)=-\frac{2\pi}{N}\sum_{i=1}^{N} f(\tilde{\theta}_i^N)\varphi(\tilde{\theta}_N^N) \\
&= \left(\int_0^{2\pi} f(\theta)\mathrm{d}\theta-\frac{2\pi}{N}\sum_{i=1}^{N} f(\tilde{\theta}_i^N)\right)\varphi(\tilde{\theta}_N^N).
\end{aligned}$$

Hence, utilizing Lemma (4.1) again,

$$|D_4| \leq \frac{C}{N^{k+\alpha}}\|f\|_{C^{k,\alpha}}\|\varphi\|_{L^\infty}. \tag{4.4}$$

On the whole, since $D_3=0$, combining (4.2), (4.3) and (4.4), we deduce that

$$\left|\int_0^{2\pi}(f_{\mathrm{app}}^N-f)\varphi\right| \leq \frac{C}{N^{k+\alpha}}\left(\|f\|_{C^{k,\alpha}}+|\gamma|\right)\|\varphi\|_{C^{k+1,\alpha}},$$

which concludes the proof of the proposition. □

5. Proof of Theorem 1.1

We proceed now to the demonstration of our main result – Theorem 1.1 – on the approximation of the boundary of an exterior domain by point vortices.

First, for given $\omega \in C_c^{0,\alpha}$ and $\gamma \in \mathbb{R}$, recall that the full plane flow $u_P \in C^1\left(\overline{\Omega}\right)$ is obtained from (1.9) and that the 2π-periodic function $f \in C^\infty([0,2\pi])$, which has zero mean value (2.7), is defined by (2.6). Therefore, with this given f, according to Corollary 3.2, we find a unique solution $\gamma^N \in \mathbb{R}^N$ of (3.1).

Next, the approximate flow u_{app}^N is introduced by (1.14), which verifies by (2.4):

$$u_{\mathrm{app}}^N(x)\cdot n(x)=-\frac{1}{4\pi}f_{\mathrm{app}}^N(\theta),$$

where $x = (\cos\theta, \sin\theta) \in \partial B(0,1)$ and f_{app}^N is defined by (4.1). Utilizing identity (2.15) to rewrite the discrete Biot-Savart kernel of u_{app}^N, we find that

$$\begin{aligned} u_{\text{app}}^N(x) &= \frac{1}{2\pi}\sum_{j=1}^N \frac{\gamma_j^N}{N}\left(\frac{1}{2\pi}\int_{\partial B(0,1)} \frac{x-z}{|x-z|^2}\cot\left(\frac{\tilde\theta_j^N - \theta}{2}\right)dz + \frac{x^\perp}{|x|^2}\right) \\ &= \frac{-1}{4\pi^2}\int_{\partial B(0,1)} \frac{x-z}{|x-z|^2} f_{\text{app}}^N(z)dz + \frac{\gamma}{2\pi}\frac{x^\perp}{|x|^2}, \quad \text{on } \Omega. \end{aligned}$$

Furthermore, recall that, according to (2.14), the remainder flow u_R can be expressed as

$$u_R(x) = -\frac{1}{4\pi^2}\int_{\partial B(0,1)} \frac{x-y}{|x-y|^2} f(y)dy + \frac{\gamma}{2\pi}\frac{x^\perp}{|x|^2}, \quad \text{on } \Omega,$$

whereby

$$\left(u_R - u_{\text{app}}^N\right)(x) = \frac{1}{4\pi^2}\int_{\partial B(0,1)} \frac{x-y}{|x-y|^2}\left(f_{\text{app}}^N - f\right)(y)dy, \quad \text{on } \Omega.$$

Therefore, in view of Proposition 4.2, we deduce that, for any fixed $x \in \Omega$,

$$\left|\left(u_R - u_{\text{app}}^N\right)(x)\right| \le \frac{C}{N^2}\left\|\frac{x-y}{|x-y|^2}\right\|_{C_y^3} \le \frac{C}{N^2}\sup_{y\in\partial B(0,1)}\left(\frac{1}{|x-y|} + \frac{1}{|x-y|^4}\right),$$

where the constant $C > 0$ only depends on ω and γ. It follows that, for any closed set $K \subset \Omega$,

$$\|u_R - u_{\text{app}}^N\|_{L^\infty(K)} \le \frac{C}{N^2},$$

which concludes the proof of the theorem. □

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