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Angel Castro Diego Córdoba Francisco Gancedo

Abstract

We consider the dynamics of an interface given by two incompressible fluids with different characteristics evolving by Darcy's law. This scenario is known as the Muskat problem, being in 2D mathematically analogous to the two-phase Hele-Shaw cell. The purpose of this paper is to outline recent results on local existence, weak solutions, maximum principles and global existence.

1. Introduction

This paper is concerned with the evolution of fluids in porous media which is an important topic in fluid mechanics encountered in engineering, physics and mathematics. This phenomena has been described using the experimental Darcy's law that, in two dimensions, is given by the following momentum equation:

$$\frac{\mu}{\kappa}u = -\nabla p - (0, \mathrm{g}\rho).$$

Here $(x,t) \in \mathbb{R}^2 \times \mathbb{R}^+$, $u = (u_1(x,t), u_2(x,t))$ is the incompressible velocity (i.e. $\nabla\cdot u = 0$), $p = p(x,t)$ is the pressure, μ is the dynamic viscosity, κ is the permeability of the isotropic medium, $\rho = \rho(x,t)$ is the liquid density, and g is the acceleration due to gravity.

The Muskat problem [17] models the evolution of an interface between two fluids with different viscosities and densities in porous media by means of Darcy's law. More precisely, the interface separates the domain Ω^1 and Ω^2 defined by

$$(\mu,\rho)(x_1,x_2,t) = \begin{cases} (\mu^1,\rho^1), & x \in \Omega^1(t) \\ (\mu^2,\rho^2), & x \in \Omega^2(t) = \mathbb{R}^2 - \Omega^1(t), \end{cases}$$

and $\mu^1, \mu^2, \rho^1, \rho^2$ ($\mu^1 \neq \mu^2$ and $\rho^1 \neq \rho^2$) are constants.

This problem has been considered extensively without surface tension, in which case the pressures of the fluids are equal on the interface. Saffman and Taylor [21] made the observation that the one phase version (one of the fluids has zero viscosity) was also known as the Hele-Shaw cell equation, which, in turn, is the zero-specific heat case of the classical one-phase Stefan problem (see [8] and reference there in).

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By adding surface tension to the difference of the pressures at the interface the system regularizes and instabilities do not appear [14]. In [15] the author study Muskat problem with different densities and viscosities in a periodic setting with a top and a bottom. Local well-posedness is shown by reducing the problem to an abstract evolution equation. In addition they obtain exponential stability of some flat equilibrium. In [16] a similar study is performed for the case of three interfaces (fluid-fluid-air).

Let the free boundary be parameterized by

$$\partial\Omega^j(t) = \{z(\gamma,t) = (z_1(\gamma,t), z_2(\gamma,t)) : \gamma \in \mathbb{R}\}$$

where

$$(z_1(\gamma+2k\pi,t), z_2(\gamma+2k\pi,t)) = (z_1(\gamma,t)+2k\pi, z_2(\gamma,t)),$$

with periodicity in the horizontal space variable or an open contour vanishing at infinity

$$\lim_{\gamma\to\infty}(z(\gamma,t)-(\gamma,0)) = 0$$

with the initial data $z(\gamma,0) = z_0(\gamma)$. From Darcy's law, we find that the vorticity is concentrated on the free boundary $z(\gamma,t)$, and is given by a Dirac distribution as follows:

$$w(x,t) = \varpi(\gamma,t)\delta(x-z(\gamma,t)),$$

with $\varpi(\gamma,t)$ representing the vorticity strength i.e. ω is a measure defined by

$$<\omega,\eta>= \int \varpi(\gamma,t)\eta(z(\gamma,t))d\gamma,$$

with $\eta(x)$ a test function.

Then $z(\gamma,t)$ evolves with an incompressible velocity field coming from the Biot-Savart law:

$$u(x,t) = \nabla^{\perp}\Delta^{-1}\omega(x,t).$$

It can be explicitly computed on the contour $z(\gamma,t)$ and is given by the Birkhoff-Rott integral of the amplitude ϖ along the interface curve:

$$BR(z,\varpi)(\gamma,t) = \frac{1}{2\pi}P.V.\int \frac{(z(\gamma,t)-z(\eta,t))^{\perp}}{|z(\gamma,t)-z(\eta,t)|^2}\varpi(\eta,t)d\eta.$$

We close the system with the following formula:

$$\varpi(\gamma,t) = (I+A_\mu T)^{-1}\Big(-2\mathrm{g}\kappa\frac{\rho^2-\rho^1}{\mu^2+\mu^1}\partial_\gamma z_2\Big)(\gamma,t),$$

where

$$T(\varpi) = 2BR(z,\varpi)\cdot\partial_\gamma z, \quad A_\mu = \frac{\mu^2-\mu^1}{\mu^2+\mu^1}.$$

Baker, Meiron and Orszag [2] shown that the adjoint operator T^*, acting on ϖ, is described in terms of the Cauchy integral of ϖ along the curve $z(\gamma,t)$, and whose real eigenvalues have absolute values strictly less than one. This yields that the operator $(I+A_\mu T)$ is invertible so that the system gives an appropriate contour dynamics problem:

$$z_t(\gamma,t) = BR(z,\varpi)(\gamma,t) + c(\gamma,t)\partial_\gamma z(\gamma,t). \tag{1.1}$$

where the term c is in the tangential direction without modifying the geometric evolution of the curve.

The first important question to be asked is whether local-existence is guaranteed. However such a result turns out to be false for general initial data. Rayleigh [20] and Saffman-Taylor [21] gave a condition that must be satisfied in order to have a solution locally in time, namely that the normal component of the pressure gradient jump at the interface has to have a distinguished sign. This is known as the Rayleigh-Taylor condition:

$$\sigma(\gamma,t) = -(\nabla p^2(z(\gamma,t),t) - \nabla p^1(z(\gamma,t),t))\cdot\partial_\gamma^\perp z(\gamma,t).$$

Applying Darcy's law and approaching the boundary, we obtain

$$\sigma(\gamma,t) = \frac{\mu^2-\mu^1}{\kappa}BR(z,\varpi)(\gamma,t)\cdot\partial_\gamma^\perp z(\gamma,t) + \mathrm{g}(\rho^2-\rho^1)\partial_\gamma z_1(\gamma,t).$$

It is easy to check that

$$\frac{\mu^2-\mu^1}{\kappa}BR(z,\varpi)(\gamma,t)\cdot\partial_\gamma^\perp z(\gamma,t) = \frac{1}{4\pi}\partial_\gamma\int_{\mathbb{T}}\varpi(\eta,t)\log G(\gamma,\eta,t)d\eta,$$

with (the present computation is done in the periodic setting, similar analysis follows for $\mathbb{R}$)

$$\begin{aligned} G(\gamma,\eta,t) = {} & \sin^2\Big(\frac{z_1(\gamma,t)-z_1(\eta,t)}{2}\Big)\cosh^2\Big(\frac{z_2(\gamma,t)-z_2(\eta,t)}{2}\Big) \\ & + \cos^2\Big(\frac{z_1(\gamma,t)-z_1(\eta,t)}{2}\Big)\sinh^2\Big(\frac{z_2(\gamma,t)-z_2(\eta,t)}{2}\Big), \end{aligned}$$

and therefore

$$\int_{\mathbb{T}}\frac{\mu^2-\mu^1}{\kappa}BR(z,\varpi)(\gamma,t)\cdot\partial_\gamma^\perp z(\gamma,t)d\gamma = 0.$$

This shows that the condition $\rho^2 \neq \rho^1$ is crucial in order to have a sign in the normal direction of the difference of the gradient. If we consider $z_1(\gamma,t)-\gamma$ periodic, then

$$\int_{\mathbb{T}}\partial_\gamma z_1(\gamma,t)d\gamma = 2\pi.$$

In the case of a closed curve

$$\int_{\mathbb{T}}\sigma(\gamma,t)d\gamma = 0,$$

making impossible the task of prescribing a sign in the Rayleigh-Taylor condition.

Siegel, Caflish and Howison [22] proved ill-posedness in a 2-D case when this condition is not satisfied (unstable case and same densities). On the other hand, they showed global-in-time solutions when the initial data are nearly planar and the Rayleigh-Taylor condition holds initially.

From Biot-Savart law, at first expansion, the expression at infinity is of the order of $\frac{1}{|x|}\int\varpi$ for a closed curve o near planar at infinity. To obtain a velocity field in L^2 it is necessary to have $\int\varpi = 0$. In the periodic case, $z(\gamma+2\pi k,t) = z(\gamma,t)+(2\pi k,0)$, the following classical identity for complex numbers

$$\frac{1}{\pi}\Big(\frac{1}{z}+\sum_{k\geq 1}\frac{2z}{z^2-(2\pi k)^2}\Big) = \frac{1}{2\pi\tan(z/2)},$$

yields (ignoring the variable t)

$$v(x) = \frac{-1}{4\pi}\int_{-\pi}^{\pi}\varpi(\eta)\Big(\frac{\tanh(\frac{x_2-z_2(\eta)}{2})(1+\tan^2(\frac{x_1-z_1(\eta)}{2}))}{\tan^2(\frac{x_1-z_1(\eta)}{2})+\tanh^2(\frac{x_2-z_2(\eta)}{2})},\frac{\tan(\frac{x_1-z_1(\eta)}{2})(\tanh^2(\frac{x_2-z_2(\eta)}{2})-1)}{\tan^2(\frac{x_1-z_1(\eta)}{2})+\tanh^2(\frac{x_2-z_2(\eta)}{2})}\Big)d\eta,$$

for $x \neq z(\gamma,t)$. Then

$$\lim_{x_2\to\pm\infty} v(x,t) = \mp\frac{1}{4\pi}\int_{-\pi}^{\pi}\varpi(\eta)d\eta(1,0),$$

and to have the same value at infinity it is necessary again mean zero.

An expression for the pressure is obtained by

$$\Delta p(x,t) = -\mathrm{div}\,(\frac{\mu(x,t)}{\kappa}v(x,t)) - \mathrm{g}\,\partial_{x_2}\rho(x,t),$$

therefore the Laplacian of the pressure has its measure on the interface $z(\gamma,t)$

$$\Delta p(x,t) = \Pi(\gamma,t)\delta(x - z(\gamma,t)),$$

where $\Pi(\gamma,t)$ is given by

$$\Pi(\gamma,t) = (\frac{\mu^2-\mu^1}{\kappa}v(z(\gamma,t),t)\cdot\partial_\gamma^\perp z(\gamma,t) + \mathrm{g}(\rho^2-\rho^1)\partial_\gamma z_1(\gamma,t)).$$

It follows that:

$$p(x,t) = -\frac{1}{2\pi}\int_{\mathbb{T}}\ln\Big(\cosh(x_2 - z_2(\gamma,t)) - \cos(x_1 - z_1(\gamma,t))\Big)\Pi(\gamma,t)d\gamma,$$

for $x \neq z(\gamma,t)$, implying the important identity

$$p^2(z(\gamma,t),t) = p^1(z(\gamma,t),t),$$

which is just a mathematical consequence of Darcy's law, making unnecessary to impose it as a physical assumption. Notice that since we are dealing with finite energy solutions the pressure is force to diverge as the vertical variable tends to infinity.

2. Local existence

There are several publications (see [1],[23] and [24] for example) where different authors have treated these problems assuming that the Rayleigh-Taylor condition is preserved during the evolution and that the operator $(I + A_\mu T)^{-1}$ is bounded by a fixed constant. Under such hypothesis the proof can be considerably simplified.

Recently in [9] it is obtained local existence in the 2D case when the fluid has different densities and viscosities. In the proof it is crucial to get control of the norm of the inverse operators $(I + A_\mu T)^{-1}$. The arguments rely upon the boundedness properties of the Hilbert transforms associated to $C^{1,\delta}$ curves, for which it is needed precise estimates obtained with arguments involving conformal mappings, the Hopf maximum principle and Harnack inequalities. Then bounds are provided in the Sobolev spaces H^k for ϖ obtaining

$$\begin{aligned}\frac{d}{dt}(\|z\|^2_{H^k}+\|\mathcal{F}(z)\|^2_{L^\infty})(t) \leq -K\int_{\mathbb{T}}\frac{\sigma(\gamma)}{|\partial_\gamma z(\gamma)|^2}\partial_\gamma^k z(\gamma)\cdot\Lambda(\partial_\gamma^k z)(\gamma)d\gamma \\ + \exp C(\|z\|^2_{H^k}+\|\mathcal{F}(z)\|^2_{L^\infty})(t),\end{aligned} \tag{2.1}$$

where $K = -\kappa/(2\pi(\mu^1+\mu^2))$, $\sigma(\gamma,t)$ is the Rayleigh-Taylor condition, the operator Λ is the square root of the Laplacian and the function $\mathcal{F}(z)$ (which measures the arc-chord condition) is defined by

$$\mathcal{F}(z)(\gamma,\eta,t) = \frac{|\eta|}{|z(\gamma,t)-z(\gamma-\eta,t)|} \qquad \forall\, \gamma,\eta \in (-\pi,\pi), \tag{2.2}$$

with

$$\mathcal{F}(z)(\gamma,0,t) = \frac{1}{|\partial_\gamma z(\gamma,t)|}.$$

When $\sigma(\gamma,t)$ is positive, there is a kind of heat equation in the above inequality but with the operator Λ in place of the Laplacian. Then, the most singular terms in the evolution equation depend on the Rayleigh-Taylor condition. In order to integrate the system we study the evolution of

$$m(t) = \min_{\gamma\in\mathbb{T}} \sigma(\gamma,t), \tag{2.3}$$

which satisfies the following bound

$$|m'(t)| \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty} + \|z\|^2_{H^k})(t).$$

Using the pointwise estimate $f\Lambda(f) \geq \frac{1}{2}\Lambda(f^2)$ in estimate (2.1), we obtain

$$\frac{d}{dt}E_{RT}(t) \leq C\exp CE_{RT}(t),$$

where E_{RT} is the energy of the system given by

$$E_{RT}(t) = \|z\|^2_{H^k}(t) + \|\mathcal{F}(z)\|^2_{L^\infty}(t) + (m(t))^{-1}.$$

Here we point out that it is completely necessary to consider the evolution of the Rayleigh-Taylor condition to obtain bona fide energy estimates.

3. Equal viscosities $\mu^1 = \mu^2$

In this section we shall examine the case where the viscosities are the same: the free boundary is given by a fluid with different densities. Despite its deceiving simplicity the operators capture the non-local and non-linear character of the system and the analysis is far from trivial. Currently we are studying the long-time behavior of the stable case for which we can show that there are initial data where the Rayleigh-Taylor breakdown in finite time (see a forthcoming paper [6]).

In order to simplify the notation, one could take $\mu/\kappa = \mathrm{g} = 1$ in Darcy's law and then apply the rotational operator to obtain the vorticity given by $\omega = -\partial_{x_1}\rho$. The Biot-Savart law yields the velocity field in terms of the density as follows:

$$u(x,t) = P.V.\int_{\mathbb{R}^2} H(x-y)\rho(y,t)dy - \frac{1}{2}\left(0,\rho(x,t)\right),$$

where the Calderon-Zygmund kernel $H(\cdot)$ is defined by

$$H(x) = \frac{1}{2\pi}\left(-2\frac{\xi_1\xi_2}{|\xi|^4}, \frac{\xi_1^2-\xi_2^2}{|\xi|^4}\right).$$

The above identity indicates the non-local structure of the equation and that the velocity is at the same level as the scalar ρ:

$$\|u\|_{L^p}(t) \leq C\|\rho\|_{L^p}(t)$$

for $1 < p < \infty$.

The density $\rho = \rho(x,t)$ satisfies the transport equation:

$$(\partial_t + u \cdot \nabla)\rho = 0$$

with $(x,t) \in \mathbb{R}^2 \times \mathbb{R}^+$ and $u = (u_1, u_2)$ the fluid velocity. This transport equation reveals that the quantity ρ moves with the fluid flow and is conserved along trajectories.

The incompressibility of the flow yields the system to be conservative in such a way that the L^p norms of ρ are constants for all time:

$$\|\rho\|_{L^p}(t) = \|\rho\|_{L^p}(0)$$

for $1 \leq p \leq \infty$. In [13] a bound of the velocity of the fluid is obtained in terms of $C^{1,\gamma}$ norms $(0 < \gamma < 1)$ of the free boundary:

$$\begin{aligned}\|u\|_{L^\infty} \leq C\Big(1 + \frac{1}{\gamma} + \frac{1}{\gamma}\ln(1 + \|\nabla f\|_{C^\gamma})\\ + \ln(1 + \|f\|_{L^\infty} + \|\nabla f\|_{L^2}\Big),\end{aligned}$$

where $0 < \gamma < 1$ and the constant $C = C(\rho^1, \rho^2)$ depends on ρ^1 and ρ^2. The estimate is based on the property that in the principal value, on the expression of u, the mean of the kernels H are zero on hemispheres. This extra cancelation was used by Bertozzi and Constantin [4] for the vortex patch problem of the 2D Euler equation to prove no formation of singularities. For this system the convected vorticity takes constant values in disjoint domains and is related with the incompressible velocity by the Biot-Savart law.

By means of Darcy's law, we can find the following formula for the difference of the gradients of the pressure in the normal direction and the strength of the vorticity:

$$\begin{aligned}\sigma(\gamma,t) &= (\rho^2 - \rho^1)\partial_\gamma z_1(\gamma,t)\\ \varpi(\gamma,t) &= -(\rho^2 - \rho^1)\partial_\gamma z_2(\gamma,t). \end{aligned} \tag{3.1}$$

Then, by choosing an appropriate term c in (1.1), the dynamics of the interface satisfies

$$z_t(\gamma,t) = \frac{\rho^2 - \rho^1}{2\pi} P.V. \int \frac{(z_1(\gamma,t) - z_1(\eta,t))}{|z(\gamma,t) - z(\eta,t)|^2}(\partial_\gamma z(\gamma,t) - \partial_\gamma z(\eta,t))d\eta. \tag{3.2}$$

A wise choice of parameterizing the curve is that for which we have $\partial_\gamma z_1(\gamma,t) = 1$ (for more details see [10]). This yields the denser fluid below the less dense fluid if $\rho^2 > \rho^1$ and therefore the Rayleigh-Taylor condition holds for all time. An additional advantage is that we avoid a kind of singularity in the fluid when the interface collapses due to the fact that we can take $z(\gamma,t) = (\gamma, f(\gamma,t))$ which implies $\mathcal{F}(z)(\gamma,\eta) \leq 1$ obtaining the arc-chord condition for all time. Then the character of the interface as the graph of a function is preserved, and in [10] this fact has been used to show local-existence in the stable case $(\rho^2 > \rho^1)$, together with ill-posedness in the unstable situation ($\rho^2 < \rho^1$). From now on we will use the parameter $x \in \mathbb{R}$ due to the curve is given by a graph which satisfies the following evolution equation:

$$\begin{aligned} f_t(x,t) &= \frac{\rho^2-\rho^1}{2\pi} PV \int_{\mathbb{R}} \frac{(\partial_x f(x,t) - \partial_x f(x-\gamma,t))\gamma}{\gamma^2 + (f(x,t)-f(x-\gamma,t))^2}\, d\gamma, \\ f(x,0) &= f_0(x). \end{aligned} \tag{3.3}$$

The above equation can be linearized around the flat solution to find the following nonlocal partial differential equation

$$\begin{aligned} f_t(x,t) &= -\frac{\rho^2-\rho^1}{2} \Lambda f(x,t), \\ f(x,0) &= f_0(x), \quad x \in \mathbb{R}, \end{aligned}$$

where the operator Λ is the square root of the Laplacian. This linearization shows the parabolic character of the problem in the stable case ($\rho^2 > \rho^1$), as well as the ill-posedness in the unstable case ($\rho^2 < \rho^1$).

The nonlinear equation (3.3) is ill-posed in the unstable situation and locally well-posed in H^k ($k \geq 3$) for the stable case [10].

3.1. Maximum Principles

3.1.1. L^2 of f

The contour equation (3.3) can be written as follows:

$$f_t(x,t) = PV \int_{\mathbb{R}} \partial_x \arctan\Big(\frac{f(x,t)-f(x-\gamma,t)}{\gamma}\Big) d\gamma.$$

We multiply by f, integrate over dx, and use integration by parts (for details [7]) to observe

$$\frac{1}{2}\frac{d}{dt}\|f\|^2_{L^2}(t) = -\int_{\mathbb{R}}\int_{\mathbb{R}} \ln\sqrt{1+\Big(\frac{f(x,t)-f(\gamma,t)}{x-\gamma}\Big)^2}\, dx d\gamma, \tag{3.4}$$

and integrating in time we get

$$\|f\|^2_{L^2}(t) + \frac{\rho^2-\rho^1}{2\pi}\int_0^t ds \int_{\mathbb{R}} d\gamma \int_{\mathbb{R}} dx \,\ln\bigg(1+\Big(\frac{f(x,s)-f(\gamma,s)}{x-\gamma}\Big)^2\bigg) = \|f_0\|^2_{L^2}.$$

The above equality indicates that for large initial data, the system is not parabolic at the level of f. The inequality

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \ln\bigg(1+\Big(\frac{f(x,t)-f(\gamma,t)}{x-\gamma}\Big)^2\bigg) dx d\gamma \leq 4\pi\sqrt{2}\|f\|_{L^1}(t)$$

shows that there is no gain of derivatives for the stable case. If the initial data are positive, then $\|f\|_{L^1}(t) \leq \|f_0\|_{L^1}$ follows from [11] (see next section below), so that the dissipation is bounded in terms of the initial data with zero derivatives.

3.1.2. L^∞ of f

For $f_0 \in H^k$ with $k \geq 3$, we prove in [10] that there exists a time $T > 0$ such that the unique solution $f(x,t)$ to (3.3) belongs to $C^1([0,T]; H^k)$. In particular we have $f(x,t) \in C^1([0,T] \times \Omega)$, hence Rademacher theorem shows that the functions

$$M(t) = \max_x f(x,t),$$

and

$$m(t) = \min_x f(x,t),$$

are differentiable at almost every t. In the non-periodic case, we also notice that by Riemann-Lebesgue lemma there always exists a point $x_t \in \mathbb{R}$ where

$$|f(x_t,t)| = \max_x |f(x,t)|,$$

since $f(\cdot,t) \in H^s$ with $s > 1/2$ implies that $f(x,t)$ tends to 0 when $|x| \to \infty$. First, we suppose that this point x_t satisfies that $0 < f(x_t,t) = M(t)$ (a similar argument can be used for $m(t) = f(x_t,t) < 0$). Then by an appropriate integration by parts follows that

$$M'(t) = -\frac{\rho^2 - \rho^1}{2\pi} PV \int_{\mathbb{R}} \frac{M(t) - f(x_t - \gamma, t)}{\gamma^2 + (M(t) - f(x_t - \gamma, t))^2} d\gamma \leq 0,$$

$$m'(t) = -\frac{\rho^2 - \rho^1}{2\pi} PV \int_{\mathbb{R}} \frac{m(t) - f(x_t - \gamma, t)}{\alpha^2 + (m(t) - f(x_t - \gamma, t))^2} d\gamma \geq 0,$$

for almost every t.

Furthermore the stable system gives a maximum principle $\|f\|_{L^\infty}(t) \leq \|f\|_{L^\infty}(0)$, see [11]; decay rates are obtained for the periodic case as:

$$\|f\|_{L^\infty}(t) \leq \|f_0\|_{L^\infty} e^{-Ct},$$

and also for the case on the real line (flat at infinity) as:

$$\|f\|_{L^\infty}(t) \leq \frac{\|f_0\|_{L^\infty}}{1 + Ct}.$$

Numerical solutions performed in [12] further indicate a regularizing effect. The decay of the slope and the curvature is stronger than the rate of decay of the maximum of the difference between f and its mean value. Thus, the irregular regions in the graph are rapidly smoothed and the flat regions are smoothly bent.

3.1.3. L^∞ of $\partial_x f$

It is shown analytically in [11] that, if the initial data satisfy $\|\partial_x f_0\|_{L^\infty} < 1$, then there is a maximum principle that shows that this derivative remains in absolute value smaller than 1. The proof follows from the following equivalent system

$$f_t(x,t) = \frac{\rho^2 - \rho^1}{2\pi} PV \int_{\mathbb{R}} \frac{\partial_x f(x,t)(x-\alpha) - (f(x,t) - f(\gamma,t))}{(x-\alpha)^2 + (f(x,t) - f(\gamma,t))^2} d\gamma.$$

Taking one derivative in this formula, we have

$$\partial_x f_t(x) = N_1(x) + N_2(x), \tag{3.5}$$

with

$$N_1(x) = \frac{\rho^2 - \rho^1}{2\pi} PV \int_{\mathbb{R}} \frac{\partial_x^2 f(x)(x-\gamma)}{(x-\alpha)^2 + (f(x) - f(\gamma))^2} d\gamma,$$

$$N_2(x) = -\frac{\rho^2-\rho^1}{2\pi} PV \int_{\mathbb{R}} \frac{\partial_x f(x) - \triangle_\gamma f(x)}{(x-\alpha)^2} Q(x,\gamma) d\gamma,$$

where

$$Q(x,\gamma) = 2\,\frac{1+\partial_x f(x) \triangle_\gamma f(x)}{(1+(\triangle_\gamma f(x))^2)^2},$$

and

$$\triangle_\gamma f(x) = \frac{f(x)-f(\gamma)}{x-\gamma}.$$

Next, we set

$$M(t) = \|\partial_x f\|_{L^\infty}(t),$$

then $M(t) = \max_x \partial_x f(x,t) = \partial_x f(x_t,t)$ where x_t is the trajectory of the maximum. Similar conclusions are obtained for $m(t) = \min_x \partial_x f(x,t)$. Using the Rademacher theorem as in the previous section, we have that $M'(t) = \partial_x f_t(x_t,t)$ and $\partial_x^2 f(x_t,t) = 0$. Therefore by taking $x = x_t$ in (3.5) yields

$$M'(t) = N_2(x_t),$$

since $N_1(x_t) = 0$. The inequality

$$|\triangle_\gamma f(x_t)| \leq M(t),$$

shows that for $M(t) < 1$ the integral $N_2(x_t) \leq 0$, and therefore $M'(t) \leq 0$. If $M(0) < 1$, using the theorem of local existence, we have that for short time $M(t) < 1$ which implies $M'(t) \leq 0$ for almost every t. Consequently we obtain $M(t) < 1$. In the case of $m(t)$ we find $m(t) > 1$.

From the expression (3.5) we can also find some initial data such that the norm $||\partial_x f||_{L^\infty}$ is an increasing function for small enough t. In order to prove it we introduce in the formula for $N_2(x)$ the following function:

$$g(x) = \begin{cases} Mx & x \in [0,\alpha_1) \\ -M\frac{\alpha_1+\epsilon\alpha_2}{\alpha_2-\alpha_1}(x-\alpha_1) + M\alpha_1 & x \in [\alpha_1,\alpha_2) \\ -\epsilon M x & x \in [\alpha_2,\alpha_3) \\ M\frac{\epsilon\alpha_3}{\alpha_4-\alpha_3}(x-\alpha_3) - \epsilon M\alpha_3 & x \in [\alpha_3,\alpha_4) \\ 0 & x \in [\alpha_4,\infty) \end{cases}, \tag{3.6}$$

where $g(x) = -g(-x)$ for $x < 0$, $M > 0$, $0 < \varepsilon < 1$ and $0 < \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4$ satisfy

$$\frac{\alpha_1+\varepsilon\alpha_2}{\alpha_2-\alpha_1} < 1 \tag{3.7}$$

and

$$\frac{\varepsilon\alpha_3}{\alpha_4-\alpha_3} < 1. \tag{3.8}$$

Thus $g(x)$ is an odd Lipschitz function such that $||\partial_x g||_{L^\infty} = (\partial_x g)(0) = M$.

Therefore we have that

$$N_2(0) = -\frac{\rho^2-\rho^1}{2\pi} 4 \int_0^\infty \frac{M-\frac{g(x)}{x}}{x^2} \frac{1+M\frac{g(x)}{x}}{(1+(\frac{g(x)}{x})^2)^2} dx.$$

Computing the integral yields

$$\int_0^\infty \frac{M-\frac{g(x)}{x}}{x^2} \frac{1+M\frac{g(x)}{x}}{(1+(\frac{g(x)}{x})^2)^2} dx \tag{3.9}$$

$$
\begin{aligned}
&= \frac{(\alpha_2-\alpha_1)(1+\varepsilon)M}{2\alpha_1\alpha_2(1+M^2\varepsilon^2)} - \frac{(\alpha_3-\alpha_2)M(1+\varepsilon)(M^2\varepsilon-1)}{2\alpha_2\alpha_3(1+M^2\varepsilon^2)^2}\\
&\qquad + \frac{(\alpha_4-\alpha_3)M(2-(M^2-1)\varepsilon)}{2\alpha_3\alpha_4(1+M^2\varepsilon^2)} + \frac{M}{\alpha_4}\\
&= \frac{1+\varepsilon}{M}\left(\frac{\alpha_2-\alpha_1}{2\alpha_1\alpha_2(M^{-2}+\varepsilon^2)} - \frac{(\alpha_3-\alpha_2)(\varepsilon-M^{-2})}{2\alpha_2\alpha_3(M^{-2}+\varepsilon^2)^2}\right)\\
&\quad + \frac{M}{\alpha_4}\left(\frac{(\alpha_4-\alpha_3)(2M^{-2}-\varepsilon(1+M^{-2}))}{2\alpha_3(M^{-2}+\varepsilon^2)} + 1\right).
\end{aligned}
$$

Choosing $\alpha_4 = \mu\alpha_3$, for μ and M large enough we have that the second term on right hand side of the previous expression is negative. Taking the limit $M \to \infty$ we obtain that there exist $M > 1$, $\varepsilon < 1$, α_1, α_2, α_3 and α_4 satisfying the inequalities (3.7) and (3.8) such that the integral (3.9) is negative. Approximating this function for a smooth one we have the initial data that we are looking for.

3.2. Global existence for small initial data

Briefly in this section we state several family of global solutions of (3.3) for the stable case with small initial data with respect to a fixed norm:

3.2.1. Analyticity for small initial data

Let $x \in \mathbb{R}$ and

$$\|f\|_a = \sum |\hat{f}(k)| e^{a|k|}.$$

For $a > 0$, if $\|f\|_a < \infty$, then the function f can be extended analytically on the strip $|\Im z| < a$. Furthermore

$$\|\partial_x f\|_a \le C\frac{\|f\|_b}{b-a},$$

for $b > a$. Then

Theorem 1. *Let $f_0(x)$ be a function such that $\int_{\mathbb{T}} f_0(x)\,dx = 0$, $\|\partial_x f_0\|_0 \le \varepsilon$ for ε small enough and*

$$\|\partial_x^2 f_0\|_{b(t)} \le \varepsilon e^{b(t)}(1 + |b(t)|^{\gamma-1}), \tag{3.10}$$

with $0 < \gamma < 1$, $b(t) = a - (\rho^2 - \rho^1)t/2$, $\rho^2 > \rho^1$ and $a \le (\rho^2 - \rho^1)t/2$. Then, there exists a unique solution of (3.3) with $f(x,0) = f_0(x)$ and $\rho^2 > \rho^1$ satisfying

$$\|\partial_x f\|_a(t) \le C(\varepsilon)\exp((2\sigma a - (\rho^2-\rho^1)t)/4),$$

and

$$\|\partial_x^2 f\|_a(t) \le C(\varepsilon)(1 + |\sigma a - \frac{\rho^2-\rho^1}{2}t|^{\gamma-1})\exp((2\sigma a - (\rho^2-\rho^1)t)/4),$$

for $a \le \frac{\rho^2-\rho^1}{2\sigma}t$, $\sigma = 1+\delta$ and $0 < \delta < 1$.

Note that the initial data are not necessarily smooth, $\|f\|_{H^s}$ can be ∞ for $s > 3/2$. The condition (3.10) can be satisfied for example if $\|\Lambda^{1+\gamma} f_0\|_0 < \varepsilon$ and $\hat{f}_0(0) = \hat{f}_0(1) = \hat{f}_0(-1) = 0$ since

$$\|\partial_x^2 f_0\|_{b(t)} \le e^{b(t)}\|\Lambda^{1+\gamma} f_0\|_0 \max_{k\ge 2} |k|^{1-\gamma} e^{b(t)(|k|-1)}.$$

In order to prove the theorem, we use the Cauchy-Kowalewski method (see [18] and [19]) in a similar way as Caflisch and Orellana [5] and Siegel, Caflisch and Howison [22].

3.2.2. Classical solutions for initial data smaller than $\frac{1}{5}$

In [7] it is shown global existence of unique $C([0,T];H^3(\mathbb{R}))$ solutions if initially the norm of f_0 is controlled as $\|f_0\|_1 < c_0$ where in this case

$$\|f_0\|_1 = \int_{\mathbb{R}} d\xi \ |\xi||\hat{f}_0(\xi)|.$$

The key point here, in comparison to previous work [8, 22, 10, 15], is that the constant c_0 can be easily explicitly computed. We have checked numerically that c_0 is not that small; it is greater than $1/5$.

This norm allows us to use Fourier techniques for small initial data that give rise to a global existence result for classical solutions.

Theorem 2. *Suppose that initially $f_0 \in H^3(\mathbb{R})$ and $\|f_0\|_1 < c_0$, where c_0 is a constant such that*

$$2\sum_{n\geq 1}(2n+1)^{2+\delta}c_0^{2n} \leq 1$$

for $0 < \delta < 1/2$. Then there is a unique solution f of (3.3) that satisfies $f \in C([0,T];H^3(\mathbb{R}))$ for any $T > 0$.

The limit case $\delta = 0$

$$2\sum_{n\geq 1}(2n+1)^{2}c_0^{2n} \leq 1$$

is satisfied if

$$0 \leq c_0 \leq \frac{1}{3}\sqrt{7 - \frac{14\times 5^{2/3}}{\sqrt[3]{9\sqrt{39}-38}} + 2\sqrt[3]{5(9\sqrt{39}-38)}} \approx 0.2199617648835399.$$

In particular,

$$2\sum_{n\geq 1}(2n+1)^{2.1}c_0^{2n} < 1,$$

if say $c_0 \leq 1/5$.

3.2.3. Weak solutions for initial data smaller than 1

In [7] it is also shown global in time existence of Lipschitz continuous solutions in the stable case. We define a weak solution of (3.3) if it satisfies:

$$\begin{aligned}&\int_0^T dt \ \int_{\mathbb{R}} dx \ \eta_t(x,t)f(x,t) + \int_{\mathbb{R}} dx \ \eta(x,0)f_0(x) \\ &= \int_0^T dt \ \int_{\mathbb{R}} dx \ \eta_x(x,t)\frac{\rho^2-\rho^1}{2\pi}PV\int_{\mathbb{R}} d\gamma \ \arctan\left(\frac{f(x,t)-f(\gamma,t)}{x-\gamma}\right).\end{aligned} \tag{3.11}$$

This equality holds $\forall \eta \in C_c^\infty([0,T)\times\mathbb{R})$. Then

Theorem 3. *Suppose that* $\|f_0\|_{L^\infty} < \infty$ *and* $\|\partial_x f_0\|_{L^\infty} < 1$*.Then there exists a global in time weak solution of* (3.11) *that satisfies*

$$f(x,t) \in C([0,T] \times \mathbb{R}) \cap L^\infty([0,T]; W^{1,\infty}(\mathbb{R})).$$

In particular f *is Lipschitz continuous.*

It is clear, however, because of the condition $f \in L^\infty(\mathbb{R})$, the nonlinear term in (3.11) has to be understood as a principal value for the integral of two functions, one in $\mathcal{H}^1$ and the other in BMO.

There are several results of global existence for small initial data (small compared to 1 or $\epsilon \ll 1$) in several norms (more regular than Lipschitz) [8, 22, 15, 10] taking advantage of the parabolic character of the equation for small initial data. Here we show that we just need $\|\partial_x f_0\|_{L^\infty} < 1$, therefore

$$\left| \frac{f_0(x) - f_0(\gamma)}{x - \gamma} \right| < 1.$$

Notice that considering the first order term in the Taylor series of $\ln(1 + y^2)$ for $|y| < 1$, then the identity (3.4) gains half of a derivative which grants strong compactness properties in comparison to the log conservation law.

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Instituto de Ciencias Matemáticas, Consejo Superior de Investigaciones Científicas, Serrano 123, 28006 Madrid, Spain
castro@icmat.es

Instituto de Ciencias Matemáticas, Consejo Superior de Investigaciones Científicas, Serrano 123, 28006 Madrid, Spain
dcg@icmat.es

Department of Mathematics, University of Chicago, 5734 University Avenue, Chicago, IL 60637
fgancedo@math.uchicago.edu