

*Journées*

# **ÉQUATIONS AUX DÉRIVÉES PARTIELLES**

Évian, 2 juin–6 juin 2008

J. Bona, D. Lannes and J.-C. Saut  
**Asymptotic behaviors of internal waves**

*J. É. D. P.* (2008), Exposé n° V, 17 p.

<[http://jedp.cedram.org/item?id=JEDP\\_2008\\_\\_\\_\\_A5\\_0](http://jedp.cedram.org/item?id=JEDP_2008____A5_0)>

**cedram**

*Article mis en ligne dans le cadre du  
Centre de diffusion des revues académiques de mathématiques  
<http://www.cedram.org/>*

# Asymptotic behaviors of internal waves

J. Bona   D. Lannes   J.-C. Saut

## Abstract

We present here a systematic method of derivation of asymptotic models for internal waves, that is, for the propagation of waves at the interface of two fluids of different densities. Many physical regimes are investigated, depending on the physical parameters (depth of the fluids, amplitude and wavelength of the interface deformations). This systematic method allows us to recover the many models existing in the literature and to derive some new models, in particular in the case of large amplitude internal waves and two-dimensional interfaces. We also provide rigorous consistency results for these models. We refer to [5] for full details.

## 1. Introduction

This note is devoted to the study of the equations describing the interface between two layers of immiscible fluids of different densities. This is the simplest idealization for internal wave propagation (see [12] for a recent survey).

The idealized system that will be the focus of the discussion here, when it is at rest, consists of a homogeneous fluid of depth  $d_1$  and density  $\rho_1$  lying over another homogeneous fluid of depth  $d_2$  and density  $\rho_2 > \rho_1$ . The bottom on which both fluids rest is presumed to be horizontal and featureless while the top of fluid 1 is restricted by the rigid lid assumption, which is to say, the top is viewed as an impenetrable, bounding surface. We also assume that the deviation of the interface is a graph over the flat bottom, so overturning waves are not within the purview of our theory (see Figure 1 for a definition sketch).

Many models describing the motion of such internal waves have been formally derived in the literature. Weakly nonlinear models in two-dimensions have been derived by Camassa and Choi [6]. Nguyen and Dias [15] have derived and studied a Boussinesq-type system in a weakly nonlinear regime. Fully nonlinear models were obtained in the two-dimensional case by Camassa and Choi [7]. A different and systematic approach has been carried out by Craig, Guyenne and Kalisch [9] in the one-dimensional case; these authors use the Hamiltonian formulation of the Euler equations (due originally to Zakharov [19] for surface waves and to Benjamin and

Bridges [2] for internal waves) and expand the Hamiltonian with respect to the relevant small parameters.

The strategy followed here is inspired by that initiated in [3]. Namely, following the procedure introduced in [9, 11, 19], we rewrite the full system as a system of two evolution equations posed on  $\mathbb{R}^d$ , where  $d = 1$  or  $2$ . The reformulated system, which depends only upon the spatial variable on the interface, involves two non-local operators, a Dirichlet-to-Neumann operator  $G[\zeta]$ , and what we term an “interface operator”  $\mathbf{H}[\zeta]$ , defined precisely below. Of course the operator  $\mathbf{H}[\zeta]$  does not appear in the theory of surface waves. A rigorously justified asymptotic expansion of the non-local operators with respect to dimensionless small parameters is then mounted. For the considered scaling regimes, these expansions then lead to an asymptotic evolution system, which can be further analyzed. This analysis recovers most of the systems which have been introduced in the literature and also some interesting new ones, such as the Shallow Water/Shallow Water equations (25). For instance, in the so-called shallow water/shallow water regime, a non-local operator appears in the two-dimensional case whose analog is not present in any of the one-dimensional cases.

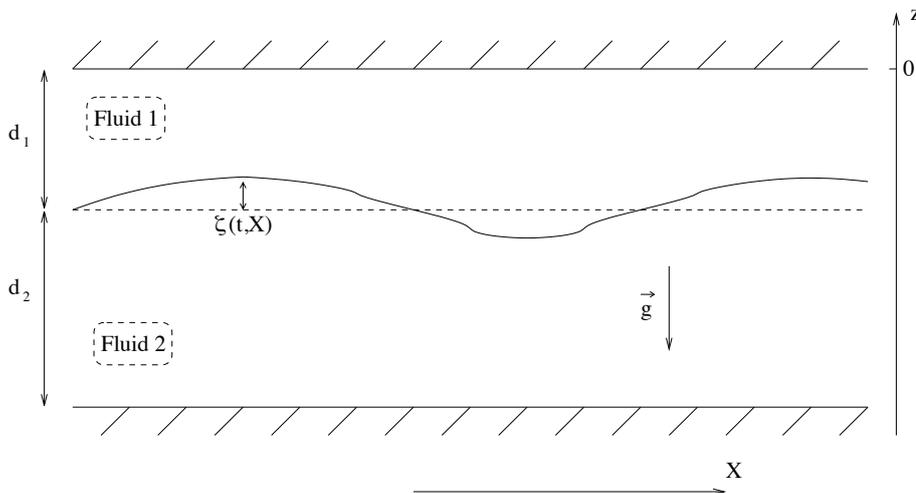
## 2. The internal waves equations

### 2.1. The two layers Euler equations

As in Figure 1, the origin of the vertical coordinate  $z$  is taken at the rigid top of the two-fluid system. Assuming each fluid is incompressible and each flow irrotational, there exists velocity potentials  $\Phi_i$  ( $i = 1, 2$ ) associated to both the upper and lower fluid layers which satisfy

$$\Delta_{X,z}\Phi_i = 0 \quad \text{in } \Omega_t^i \quad (1)$$

for all time  $t$ , where  $\Omega_t^i$  denotes the region occupied by fluid  $i$  at time  $t$ ,  $i = 1, 2$ . As above, fluid 1 refers to the upper fluid layer whilst fluid 2 is the lower layer (see again Figure 1). Assuming that the densities  $\rho_i$ ,  $i = 1, 2$ , of both fluids are constant,



we also have two Bernoulli equations, namely,

$$\partial_t \Phi_i + \frac{1}{2} |\nabla_{X,z} \Phi_i|^2 = -\frac{P}{\rho_i} - gz \quad \text{in } \Omega_i^i, \quad (2)$$

where  $g$  denotes the acceleration of gravity and  $P$  the pressure inside the fluid. These equations are complemented by two boundary conditions stating that the velocity must be horizontal at the two rigid surfaces  $\Gamma_1 := \{z = 0\}$  and  $\Gamma_2 := \{z = -d_1 - d_2\}$ , which is to say

$$\partial_z \Phi_i = 0 \quad \text{on } \Gamma_i, \quad (i = 1, 2). \quad (3)$$

Finally, as mentioned earlier, it is presumed that the interface is given as the graph of a function  $\zeta(t, X)$  which expresses the deviation of the interface from its rest position  $(X, -d_1)$  at the spatial coordinate  $X$  at time  $t$ . The interface  $\Gamma_t := \{z = -d_1 + \zeta(t, X)\}$  between the fluids is taken to be a bounding surface, or equivalently it is assumed that no fluid particle crosses the interface. This condition, written for fluid  $i$ , is classically expressed by the relation  $\partial_t \zeta = \sqrt{1 + |\nabla \zeta|^2} v_n^i$ , where  $v_n^i$  denotes the upwards normal derivative of the velocity of fluid  $i$  at the surface. Since this equation must of course be independent of which fluid is being contemplated, it follows that the normal component of the velocity is continuous at the interface. The two equations

$$\partial_t \zeta = \sqrt{1 + |\nabla \zeta|^2} \partial_n \Phi_1 \quad \text{on } \Gamma_t, \quad (4)$$

and

$$\partial_n \Phi_1 = \partial_n \Phi_2 \quad \text{on } \Gamma_t, \quad (5)$$

with

$$\partial_n := \mathbf{n} \cdot \nabla_{X,z} \quad \text{and} \quad \mathbf{n} := \frac{1}{\sqrt{1 + |\nabla \zeta|^2}} (-\nabla \zeta, 1)^T$$

follow as a consequence. A final condition is needed on the pressure to close this set of equations, namely,

$$P \text{ is continuous at the interface,} \quad (6)$$

if we neglect surface tension effects (see Remark 11 for a comment on this point).

## 2.2. Transformation of the Equations

In this subsection, a new set of equations is deduced from the internal-wave equations (1)-(6). Introduce the trace of the potentials  $\Phi_1$  and  $\Phi_2$  at the interface,

$$\psi_i(t, X) := \Phi_i(t, X, -d_1 + \zeta(t, X)), \quad (i = 1, 2).$$

One can evaluate Eq. (2) at the interface and use (4) and (5) to obtain a set of equations coupling  $\zeta$  to  $\psi_i$  ( $i = 1, 2$ ), namely

$$\partial_t \zeta - \sqrt{1 + |\nabla \zeta|^2} \partial_n \Phi_i = 0, \quad (7)$$

$$\rho_i \left( \partial_t \psi_i + g\zeta + \frac{1}{2} |\nabla \psi_i|^2 - \frac{(\sqrt{1 + |\nabla \zeta|^2} (\partial_n \Phi_i) + \nabla \zeta \cdot \nabla \psi_i)^2}{2(1 + |\nabla \zeta|^2)} \right) = -P, \quad (8)$$

where in (7) and (8),  $(\partial_n \Phi_i)$  and  $P$  are both evaluated at the interface  $z = -d_1 + \zeta(t, X)$ . Notice that  $\partial_n \Phi_1$  is fully determined by  $\psi_1$  since  $\Phi_1$  is uniquely given as the solution of Laplace's equation (1) in the upper fluid domain, the Neumann condition (3) on  $\Gamma_1$  and the Dirichlet condition  $\Phi_1 = \psi_1$  at the interface. Following

the formalism introduced for the study of surface water waves in [10, 11, 19], we can therefore define the Dirichlet-Neumann operator  $G[\zeta]\cdot$  by

$$G[\zeta]\psi_1 = \sqrt{1 + |\nabla\zeta|^2}(\partial_n\Phi_1)|_{z=-d_1+\zeta}. \quad (9)$$

Similarly, one remarks that  $\psi_2$  is determined up to a constant by  $\psi_1$  since  $\Phi_2$  is given (up to a constant) by the resolution of the Laplace equation (1) in the lower fluid domain, with Neumann boundary conditions (3) on  $\Gamma_2$  and  $\partial_n\Phi_2 = \partial_n\Phi_1$  at the interface (this latter being provided by (5)). It follows that  $\psi_1$  fully determines  $\nabla\psi_2$  and we may thus define the operator  $\mathbf{H}[\zeta]\cdot$  by

$$\mathbf{H}[\zeta]\psi_1 = \nabla\psi_2. \quad (10)$$

Using the continuity of the pressure at the interface expressed in (6), we may equate the left-hand sides of (8)<sub>1</sub> and (8)<sub>2</sub> using the operators  $G[\zeta]$  and  $\mathbf{H}[\zeta]$  just defined. This yields the equation

$$\partial_t(\psi_2 - \gamma\psi_1) + g(1 - \gamma)\zeta + \frac{1}{2}\left(|\mathbf{H}[\zeta]\psi_1|^2 - \gamma|\nabla\psi_1|^2\right) + \mathcal{N}(\zeta, \psi_1) = 0$$

where  $\gamma = \rho_1/\rho_2$  and

$$\mathcal{N}(\zeta, \psi_1) := \frac{\gamma\left(G[\zeta]\psi_1 + \nabla\zeta \cdot \nabla\psi_1\right)^2 - \left(G[\zeta]\psi_1 + \nabla\zeta \cdot \mathbf{H}[\zeta]\psi_1\right)^2}{2(1 + |\nabla\zeta|^2)}.$$

Taking the gradient of this equation and using (7) then gives the system of equations

$$\begin{cases} \partial_t\zeta - G[\zeta]\psi_1 = 0, \\ \partial_t(\mathbf{H}[\zeta]\psi_1 - \gamma\nabla\psi_1) + g(1 - \gamma)\nabla\zeta \\ \quad + \frac{1}{2}\nabla\left(|\mathbf{H}[\zeta]\psi_1|^2 - \gamma|\nabla\psi_1|^2\right) + \nabla\mathcal{N}(\zeta, \psi_1) = 0, \end{cases} \quad (11)$$

for  $\zeta$  and  $\psi_1$ . This is the system of equations that will be used in the next sections to derive asymptotic models.

*Remark 1.* Setting  $\rho_1 = 0$ , and thus  $\gamma = 0$ , in the above equations, one recovers the usual surface water-wave equations written in terms of  $\zeta$  and  $\psi$  as in [10, 11, 19].

### 2.3. Non-Dimensionalization of the Equations

The asymptotic behavior of (11) is more transparent when these equations are written in dimensionless variables. Denoting by  $a$  a typical amplitude of the deformation of the interface in question, and by  $\lambda$  a typical wavelength, the following dimensionless independent variables

$$\widetilde{X} := \frac{X}{\lambda}, \quad \widetilde{z} := \frac{z}{d_1}, \quad \widetilde{t} := \frac{t}{\lambda/\sqrt{gd_1}},$$

are introduced. Likewise, we define the dimensionless unknowns

$$\widetilde{\zeta} := \frac{\zeta}{a}, \quad \widetilde{\psi}_1 := \frac{\psi_1}{a\lambda\sqrt{g/d_1}},$$

as well as the dimensionless parameters

$$\gamma := \frac{\rho_1}{\rho_2}, \quad \delta := \frac{d_1}{d_2}, \quad \varepsilon := \frac{a}{d_1}, \quad \mu := \frac{d_1^2}{\lambda^2}.$$

Though they are redundant, it is also notationally convenient to introduce two other parameter's  $\varepsilon_2$  and  $\mu_2$  defined as

$$\varepsilon_2 = \frac{a}{d_2} = \varepsilon\delta, \quad \mu_2 = \frac{d_2^2}{\lambda^2} = \frac{\mu}{\delta^2}.$$

*Remark 2.* The parameters  $\varepsilon_2$  and  $\mu_2$  correspond to  $\varepsilon$  and  $\mu$  with  $d_2$  rather than  $d_1$  taken as the unit of length in the vertical direction.

The equations (11) can then be written in dimensionless variables as

$$\begin{cases} \partial_t \tilde{\zeta} - \frac{1}{\mu} G^\mu[\varepsilon \tilde{\zeta}] \tilde{\psi}_1 & = 0, \\ \partial_t (\mathbf{H}^{\mu, \delta}[\varepsilon \tilde{\zeta}] \tilde{\psi}_1 - \gamma \nabla \tilde{\psi}_1) + (1 - \gamma) \nabla \tilde{\zeta} \\ \quad + \frac{\varepsilon}{2} \nabla (|\mathbf{H}^{\mu, \delta}[\varepsilon \tilde{\zeta}] \tilde{\psi}_1|^2 - \gamma |\nabla \tilde{\psi}_1|^2) + \varepsilon \nabla \mathcal{N}^{\mu, \delta}(\varepsilon \tilde{\zeta}, \tilde{\psi}_1) & = 0, \end{cases} \quad (12)$$

where  $\mathcal{N}^{\mu, \delta}$  is defined for all pairs  $(\zeta, \psi)$  smooth enough by the formula

$$\mathcal{N}^{\mu, \delta}(\zeta, \psi) := \mu \frac{\gamma \left( \frac{1}{\mu} G^\mu[\zeta] \psi + \nabla \zeta \cdot \nabla \psi \right)^2 - \left( \frac{1}{\mu} G^\mu[\zeta] \psi + \nabla \zeta \cdot \mathbf{H}^{\mu, \delta}[\zeta] \psi \right)^2}{2(1 + \mu |\nabla \zeta|^2)},$$

and where the operators  $G^\mu$  and  $\mathbf{H}^{\mu, \delta}$  are the nondimensionalized versions of the Dirichlet-Neumann and interface operators defined in (9) and (10) (see §3.1 and §3.2 for precise definitions).

*Notation 1.* The tildes which indicate the non-dimensional quantities will be systematically dropped henceforth.

*Remark 3.* Linearizing the equations (12) around the rest state, one finds the linearized dispersion relation

$$\omega^2 = (1 - \gamma) \frac{|\mathbf{k}|}{\sqrt{\mu}} \frac{\tanh(\sqrt{\mu} |\mathbf{k}|) \tanh(\frac{\sqrt{\mu}}{\delta} |\mathbf{k}|)}{\tanh(\sqrt{\mu} |\mathbf{k}|) + \gamma \tanh(\frac{\sqrt{\mu}}{\delta} |\mathbf{k}|)}; \quad (13)$$

corresponding to plane-wave solutions  $e^{i\mathbf{k}\cdot X - i\omega t}$ . In particular, the expected instability is found when  $\gamma > 1$ , corresponding to the case wherein the heavier fluid lies over the lighter one.

## 2.4. Asymptotic regimes

Our work centers around the study of the asymptotics of the non-dimensionalized equations (12) in various physical regimes corresponding to different relationships among the dimensionless parameters  $\varepsilon$ ,  $\mu$  and  $\delta$ . Here is a summary of the different asymptotic regimes investigated in this paper.

It is convenient to organize the discussion around the parameters  $\varepsilon$  and  $\varepsilon_2 = \varepsilon\delta$  (the nonlinearity, or amplitude, parameters for the upper and lower fluids, respectively), and in terms of  $\mu$  and  $\mu_2 = \frac{\mu}{\delta^2}$  (the shallowness parameters for the upper and lower fluids) (notice that the assumptions made about  $\delta$  are therefore implicit:

- The interfacial wave is said to be of *small amplitude* for the upper fluid layer (resp. the lower layer) if  $\varepsilon \ll 1$  (resp.  $\varepsilon_2 \ll 1$ ).
- The upper (resp. lower) layer is said to be *shallow* if  $\mu \ll 1$  (resp.  $\mu_2 \ll 1$ ).

This terminology is consistent with the usual one for surface water waves (recovered by taking  $\rho_1 = 0$  and  $\delta = 1$ ). In the discussion below, the notation *regime 1/regime 2* means that the wave motion is such that the upper layer is in regime 1 (small amplitude or shallow water) and the lower one is in regime 2.

1. The small-amplitude/small-amplitude regime:  $\varepsilon \ll 1$ ,  $\varepsilon_2 \ll 1$ . This regime corresponds to interfacial deformations which are small for both the upper and lower fluid domains. Various sub-regimes are defined by making further assumptions about the size of  $\mu$  and  $\mu_2$ .
  - (a) The Full Dispersion /Full Dispersion (FD/FD) regime:  $\varepsilon \sim \varepsilon_2 \ll 1$  and  $\mu \sim \mu_2 = O(1)$  (and thus  $\delta \sim 1$ ).
  - (b) The Boussinesq / Full dispersion (B/FD) regime:  $\mu \sim \varepsilon \ll 1$ ,  $\mu_2 \sim 1$ . This configuration occurs when  $\delta^2 \sim \varepsilon$ , that is, when the lower region is much larger than the upper one.
  - (c) The Boussinesq/Boussinesq (B/B) regime:  $\mu \sim \mu_2 \sim \varepsilon \sim \varepsilon_2 \ll 1$ . In this regime, one has  $\delta \sim 1$ .
2. The Shallow Water/Shallow Water (SW/SW) regime:  $\mu \sim \mu_2 \ll 1$ . This regime, which allows relatively large interfacial amplitudes ( $\varepsilon \sim \varepsilon_2 = O(1)$ ), does not belong to the regimes singled out above. The structure of the flow is then of shallow water type in both regions.
3. The Shallow Water/Small Amplitude (SW/SA) regime:  $\mu \ll 1$  and  $\varepsilon_2 \ll 1$ . In this regime, the upper layer is shallow (but with possibly large surface deformations), and the surface deformations are small for the lower layer (but it can be deep). Various sub-regimes arise in this case also.
  - (a) The Shallow Water/Full dispersion (SW/FD) regime:  $\mu \sim \varepsilon_2^2 \ll 1$ ,  $\varepsilon \sim \mu_2 \sim 1$ .
  - (b) The Intermediate Long Waves (ILW) regime:  $\mu \sim \varepsilon^2 \sim \varepsilon_2 \ll 1$ ,  $\mu_2 \sim 1$ . In this regime, the interfacial deformations are also small for the upper fluid (which is not the case in the SW/FD regime).
  - (c) The Benjamin-Ono (BO) regime:  $\mu \sim \varepsilon^2 \ll 1$ ,  $\mu_2 = \infty$ .

The range of validity of these regimes is summarized in the following table.

	$\varepsilon = O(1)$	$\varepsilon \ll 1$
$\mu = O(1)$	Full equations	$\delta \sim 1$ : FD/FD eq'ns
$\mu \ll 1$	$\delta \sim 1$ : SW/SW eq'ns $\delta^2 \sim \mu \sim \varepsilon_2^2$ : SW/FD eq'ns	$\mu \sim \varepsilon$ and $\delta^2 \sim \varepsilon$ : B/FD eq'ns $\mu \sim \varepsilon$ and $\delta \sim 1$ : B/B eq'ns $\delta^2 \sim \mu \sim \varepsilon^2$ : ILW eq'ns $\delta = 0$ and $\mu \sim \varepsilon^2$ : BO eq'ns

*Remark 4.* The small amplitude/shallow water regime is not investigated here. It corresponds to the situation where the upper fluid domain is much larger than the lower one, which is more of an atmospheric configuration than an oceanographic case.

### 3. Asymptotic expansions of the Dirichlet-Neumann and interface operators

#### 3.1. Asymptotic expansion of the Dirichlet-Neumann operator

Let us first define the nondimensionalized Dirichlet-Neumann operator  $G^\mu[\varepsilon\zeta]$  that appears in (12). Denoting the non-dimensionalized upper fluid domain by

$$\Omega_1 = \{(X, z) \in \mathbb{R}^{d+1}, -1 + \varepsilon\zeta(X) < z < 0\}$$

and assuming that the height of this domain never vanishes,

$$\exists H_1 > 0, \quad 1 - \varepsilon\zeta \geq H_1 \quad \text{on} \quad \mathbb{R}^d, \quad (14)$$

we can state the following definition:

**Definition 1.** Let  $\zeta \in W^{2,\infty}(\mathbb{R}^d)$  be such that (14) is satisfied and let  $\psi_1 \in H^{3/2}(\mathbb{R}^d)$ . If  $\Phi_1$  is the unique solution in  $H^2(\Omega_1)$  of the boundary-value problem

$$\begin{cases} \mu\Delta\Phi_1 + \partial_z^2\Phi_1 = 0 & \text{in } \Omega_1, \\ \partial_z\Phi_1|_{z=0} = 0, \quad \Phi_1|_{z=-1+\varepsilon\zeta(X)} = \psi_1, \end{cases} \quad (15)$$

then  $G^\mu[\varepsilon\zeta]\psi_1 \in H^{1/2}(\mathbb{R}^d)$  is defined by

$$G^\mu[\varepsilon\zeta]\psi_1 = -\mu\varepsilon\nabla\zeta \cdot \nabla\Phi_1|_{z=-1+\varepsilon\zeta} + \partial_z\Phi_1|_{z=-1+\varepsilon\zeta}.$$

*Remark 5.* Another way to approach  $G^\mu$  is to define

$$G^\mu[\varepsilon\zeta]\psi_1 = \sqrt{1 + \varepsilon^2|\nabla\zeta|^2}\partial_n\Phi_1|_{z=-1+\varepsilon\zeta}$$

where  $\partial_n\Phi_1|_{z=-1+\varepsilon\zeta}$  stands for the upper conormal derivative associated to the elliptic operator  $\mu\Delta\Phi_1 + \partial_z^2\Phi_1$ .

The following lemma connects  $\zeta$  with the vertically integrated horizontal velocity via the Dirichlet-Neumann operator  $G^\mu[\varepsilon\zeta]$ . (the proof is a consequence of Green's identity).

**Lemma 1.** Let  $\zeta \in W^{2,\infty}(\mathbb{R}^d)$  be such that (14) is satisfied and let  $\psi \in H^{3/2}(\mathbb{R}^d)$  and  $\Phi_1$  be the solution of (15) with  $\psi_1 = \psi$ . If  $V^\mu$  is defined by

$$V^\mu[\varepsilon\zeta]\psi := \int_{-1+\varepsilon\zeta}^0 (\sqrt{\mu}\nabla\Phi_1)dz,$$

then one has

$$G^\mu[\varepsilon\zeta]\psi = \sqrt{\mu}\nabla \cdot (V^\mu[\varepsilon\zeta]\psi).$$

As suggested by the terminology used in §2.4, two kinds of asymptotic expansions of  $G^\mu[\varepsilon\zeta]\psi$  are needed to cover the full range of asymptotic regimes. Namely, we need *small amplitude* and *shallow-water* type expansions of  $G^\mu[\varepsilon\zeta]\psi$ . This is done in the next subsections (we in fact give, for later use, expansions of  $V^\mu[\varepsilon\zeta]\psi$ , which is a more precise result according to Lemma 1).

### 3.1.1. Asymptotic Expansion of $V^\mu[\varepsilon\zeta]\cdot$ when $\varepsilon \ll 1$ (small amplitude)

When  $\varepsilon \ll 1$ , the approach to obtaining an asymptotic expansion of  $V^\mu[\varepsilon\zeta]\psi$  is to make a Taylor expansion in terms of the interface deformation around the rest state, *viz.*

$$V^\mu[\varepsilon\zeta]\psi = V^\mu[0]\psi + \varepsilon(d_0(V^\mu[\cdot])\zeta)\psi + \dots$$

**Proposition 1.** *Let  $s > d/2$  and  $\zeta \in H^{s+3/2}(\mathbb{R}^d)$  be such that (14) is satisfied. Then for  $\psi$  such that  $\nabla\psi \in H^{s+1/2}(\mathbb{R}^d)$ , the inequality*

$$\begin{aligned} & \left| V^\mu[\varepsilon\zeta]\psi - \left[ \mathcal{T}_{0,\mu}\nabla\psi + \varepsilon\sqrt{\mu}(-\zeta + \mathcal{T}_{1,\mu}[\zeta])\nabla\psi \right] \right|_{H^s} \\ & \leq \varepsilon^2 C\left(\frac{1}{H_1}, \varepsilon\sqrt{\mu}, |\zeta|_{H^{s+3/2}}, |\nabla\psi|_{H^{s+1/2}}\right), \end{aligned}$$

holds for all  $\varepsilon \in [0, 1]$  and  $\mu > 0$ , where  $\mathcal{T}_{0,\mu} = \frac{\tanh(\sqrt{\mu}|D|)}{|D|}$ ,  $\mathcal{T}_{1,\mu}[\zeta] = -\nabla\mathcal{T}_{0,\mu}(\zeta\mathcal{T}_{0,\mu}\nabla^T)$ , and  $V^\mu[\varepsilon\zeta]\psi$  is as defined in Lemma 1 (so that  $G^\mu[\varepsilon\zeta]\psi = \sqrt{\mu}\nabla \cdot V^\mu[\varepsilon\zeta]\psi$ ).

*Proof.* The key point in the proof is an *explicit* formula of the derivative of the mapping  $\zeta \mapsto V^\mu[\varepsilon\zeta]\psi$ , which generalizes the formula obtained in [14] for the shape derivative of Dirichlet-Neumann operators: for all  $\zeta, \zeta' \in H^s(\mathbb{R}^d)$  ( $s > d/2$ ) the derivative of  $V^\mu[\varepsilon\cdot]\psi$  at  $\zeta$  in the direction  $\zeta'$  is given by the formula (see Lemma 2 of [5] for a proof)

$$d_\zeta(V^\mu[\varepsilon\cdot]\psi)\zeta' = -\varepsilon V^\mu[\varepsilon\zeta](\zeta' Z^\mu[\varepsilon\zeta]\psi) - \varepsilon\zeta' \left( \sqrt{\mu}\nabla\psi - \varepsilon\sqrt{\mu}\nabla\zeta Z^\mu[\varepsilon\zeta]\psi \right),$$

where  $Z^\mu[\varepsilon\zeta]\psi := \frac{1}{1+\varepsilon^2\mu|\zeta|^2}(G^\mu[\varepsilon\zeta]\psi + \varepsilon\mu\nabla\zeta \cdot \nabla\psi)$ .

A second order Taylor expansion then reveals that

$$V^\mu[\varepsilon\zeta]\psi = V^\mu[0]\psi + d_0(V^\mu[\varepsilon\cdot]\psi)\zeta + \int_0^1 (1-z)d_{z\zeta}^2(V^\mu[\varepsilon\cdot]\psi)(\zeta, \zeta)dz,$$

which, together with the above formula for the derivative of  $V^\mu[\cdot]$ , yields

$$\begin{aligned} & V^\mu[\varepsilon\zeta]\psi = V^\mu[0]\psi \\ & -\varepsilon V^\mu[0](\zeta G^\mu[0]\psi) - \varepsilon\sqrt{\mu}\zeta\nabla\psi + \int_0^1 (1-z)d_{z\zeta}^2(V^\mu[\varepsilon\cdot]\psi)(\zeta, \zeta)dz. \end{aligned}$$

By a simple Fourier analysis, one gets that  $G^\mu[0]\psi = -\sqrt{\mu}|D|\tanh(\sqrt{\mu}|D|)\psi$  and  $V^\mu[0]\psi = \frac{\tanh(\sqrt{\mu}|D|)}{|D|}\nabla\psi$ . It remains therefore to control the residual integral term in the expansion; this can be done as in Proposition 3.3 of [1].  $\square$

### 3.1.2. Asymptotic Expansion of $V^\mu[\varepsilon\zeta]\cdot$ for Large-Amplitude Waves and Shallow Water ( $\varepsilon = O(1)$ and $\mu \ll 1$ )

For larger amplitude waves, the expansion of the Dirichlet-Neumann operator  $G^\mu[\varepsilon\zeta]\psi$  (and also of  $V^\mu[\varepsilon\zeta]\psi$ ) around the rest state no longer provides an accurate approximation. However, if  $\mu \ll 1$  (shallow water regime for the upper fluid), it is possible to obtain an expansion of  $V^\mu[\varepsilon\zeta]\psi$  with respect to  $\mu$  which is uniform with respect to  $\varepsilon \in [0, 1]$ .

**Proposition 2.** *Let  $s > d/2$  and  $\zeta \in H^{s+3/2}(\mathbb{R}^d)$ . Then for all  $\mu \in (0, 1)$  and  $\psi$  such that  $\nabla\psi \in H^{s+5/2}(\mathbb{R}^d)$ , one has*

$$\left| \sqrt{\mu}V^\mu[\varepsilon\zeta]\psi - \mu(1 - \varepsilon\zeta)\nabla\psi \right|_{H^s} \leq \mu^2 C(|\zeta|_{H^{s+3/2}}, |\nabla\psi|_{H^{s+5/2}}),$$

uniformly with respect to  $\varepsilon \in [0, 1]$ , where  $V^\mu[\varepsilon\zeta]\psi$  is as defined in Lemma 1 (so that  $G^\mu[\varepsilon\zeta]\psi = \sqrt{\mu}\nabla \cdot V^\mu[\varepsilon\zeta]\psi$ ).

*Remark 6.* As in Prop. 3.8 of [1], one can carry out the expansion explicitly to second order in  $\mu$ , thereby obtaining

$$\sqrt{\mu}V^\mu[\varepsilon\zeta]\psi = \mu(1 - \varepsilon\zeta)\nabla\psi + \frac{\mu^2}{3}\Delta\nabla\psi + O(\mu^3, \varepsilon\mu^2).$$

*Proof.* This follows from well known results on the Dirichlet-Neumann operator in the case of one single fluid layer (e.g. Proposition 3.8 of [1]).  $\square$

### 3.2. Asymptotic expansion of the interface operator

We first define here the dimensionless operator  $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\cdot$  that appears in (12). Denoting the non-dimensionalized lower fluid domain by

$$\Omega_2 = \{(X, z) \in \mathbb{R}^{d+1}, -1 - 1/\delta < z < -1 + \varepsilon\zeta(X)\},$$

and assuming that the height of this domain never vanishes,

$$\exists H_2 > 0, \quad 1 + \varepsilon\delta\zeta \geq H_2 \quad \text{on} \quad \mathbb{R}^d, \quad (16)$$

we can state the following definition:

**Definition 2.** Let  $\zeta \in W^{2,\infty}(\mathbb{R}^d)$  be such that (14) and (16) are satisfied, and suppose that  $\psi_1 \in H^{3/2}(\mathbb{R}^d)$  is given. If the function  $\Phi_2$  is the unique solution (up to a constant) of the boundary-value problem

$$\begin{cases} \mu\Delta\Phi_2 + \partial_z^2\Phi_2 = 0 & \text{in } \Omega_2, \\ \partial_z\Phi_2|_{z=-1-1/\delta} = 0, & \partial_n\Phi_2|_{z=-1+\varepsilon\zeta(X)} = \frac{1}{(1+\varepsilon^2|\nabla\zeta|^2)^{1/2}}G^\mu[\varepsilon\zeta]\psi_1, \end{cases} \quad (17)$$

then the operator  $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\cdot$  is defined on  $\psi_1$  by

$$\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1 = \nabla(\Phi_2|_{z=-1+\varepsilon\zeta}) \in H^{1/2}(\mathbb{R}^d).$$

*Remark 7.* In the statement above,  $\partial_n\Phi_2|_{z=-1+\varepsilon\zeta}$  stands here for the upwards conormal derivative associated to the elliptic operator  $\mu\Delta\Phi_2 + \partial_z^2\Phi_2$ ,

$$\sqrt{1 + \varepsilon^2|\nabla\zeta|^2}\partial_n\Phi_2|_{z=-1+\varepsilon\zeta} = -\mu\varepsilon\nabla\zeta \cdot \nabla\Phi_2|_{z=-1+\varepsilon\zeta} + \partial_z\Phi_2|_{z=-1+\varepsilon\zeta}.$$

The Neumann boundary condition of (17) at the interface can also be stated as  $\partial_n\Phi_2|_{z=-1+\varepsilon\zeta} = \partial_n\Phi_1|_{z=-1+\varepsilon\zeta}$ .

*Remark 8.* Of course, the solvability of (17) requires the condition  $\int_\Gamma \partial_n\Phi_2 d\Gamma = 0$  (where  $d\Gamma = \sqrt{1 + \varepsilon^2|\nabla\zeta|^2}dX$  is the Lebesgue measure on the surface  $\Gamma = \{z = -1 + \varepsilon\zeta\}$ ). This is automatically satisfied thanks to the definition of  $G^\mu[\varepsilon\zeta]\psi_1$ . Indeed, applying Green's identity to (15), one obtains

$$\int_\Gamma \partial_n\Phi_2 d\Gamma = \int_\Gamma \partial_n\Phi_1 d\Gamma = - \int_{\Omega_1} (\mu\Delta\Phi_1 + \partial_z^2\Phi_1) = 0.$$

The boundary-value problem (17) plays a key role in the analysis of the operator  $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\cdot$ . The analysis of this problem is easier if we first transform it into a variable-coefficient, boundary-value problem on the flat strip  $\mathcal{S} := \mathbb{R}^d \times (-1, 0)$  using the diffeomorphism

$$\sigma : \begin{array}{ccc} \mathcal{S} & \rightarrow & \Omega_2 \\ (X, z) & \mapsto & \sigma(X, z) := (X, (1 + \varepsilon\delta)\frac{z}{\delta} + (-1 + \varepsilon\zeta)). \end{array}$$

As shown in Proposition 2.7 of [14] (see also §2.2 of [1]),  $\Phi_2$  solves (17) if and only if  $\underline{\Phi}_2 := \Phi_2 \circ \sigma$  solves

$$\begin{cases} \nabla_{X,z}^{\mu_2} \cdot Q^{\mu_2}[\varepsilon_2 \zeta] \nabla_{X,z}^{\mu_2} \underline{\Phi}_2 = 0 & \text{in } \mathcal{S}, \\ \partial_n \underline{\Phi}_2|_{z=0} = \frac{1}{\delta} G^\mu[\varepsilon \zeta] \psi_1, & \partial_n \underline{\Phi}_2|_{z=-1} = 0, \end{cases} \quad (18)$$

with

$$Q^{\mu_2}[\varepsilon_2 \zeta] = \begin{pmatrix} (1 + \varepsilon_2 \zeta) I_{d \times d} & -\sqrt{\mu_2} \varepsilon_2 (z+1) \nabla \zeta \\ -\sqrt{\mu_2} \varepsilon_2 (z+1) \nabla \zeta^T & \frac{1 + \mu_2 \varepsilon_2^2 (z+1)^2 |\nabla \zeta|^2}{1 + \varepsilon_2 \zeta} \end{pmatrix},$$

and where, as before,  $\varepsilon_2 = \varepsilon \delta$ ,  $\mu_2 = \frac{\mu}{\delta^2}$ , and  $\nabla_{X,z}^{\mu_2} = (\sqrt{\mu_2} \nabla, \partial_z)^T$ .

*Remark 9.* As always in the present exposition,  $\partial_n \underline{\Phi}_2$  stands for the *upward* conormal derivative associated to the elliptic operator involved in the boundary-value problem,

$$\partial_n \underline{\Phi}_2|_{z=0 \text{ or } z=-1} = \mathbf{e}_z \cdot Q^{\mu_2}[\varepsilon_2 \zeta] \nabla_{X,z}^{\mu_2} \underline{\Phi}_2|_{z=0 \text{ or } z=-1},$$

where  $\mathbf{e}_z$  is the upward-pointing unit vector along the vertical axis.

An asymptotic expansion of

$$\mathbf{H}^{\mu,\delta}[\varepsilon \zeta] \psi_1 = \nabla(\underline{\Phi}_2|_{z=0}), \quad (19)$$

is obtained by finding an approximation  $\underline{\Phi}_{app}$  to the solution of (18) and then using the formal relationship  $\mathbf{H}^{\mu,\delta}[\varepsilon \zeta] \psi_1 \sim \nabla(\underline{\Phi}_{app}|_{z=0})$ . This procedure is justified in the following proposition. To state the result, it is useful to have in place the spaces

$$H^{s,k}(\mathcal{S}) = \{f \in \mathcal{D}'(\overline{\mathcal{S}}) : \|f\|_{H^{s,k}} < \infty\}$$

for  $s \in \mathbb{R}$  and  $k \in \mathbb{N}$ , where  $\|f\|_{H^{s,k}} = \sum_{j=0}^k \|\Lambda^{s-j} \partial_z^j f\|$ .

**Proposition 3.** *Let  $s_0 > d/2$ ,  $s \geq s_0 + 1/2$ , and  $\zeta \in H^{s+3/2}(\mathbb{R}^d)$  be such that (14) and (16) are satisfied (the interface does not touch the horizontal boundaries). If  $\mathbf{h} \in H^{s+1/2,1}(\mathcal{S})^{d+1}$  and  $V \in H^{s+1}(\mathbb{R}^d)^d$  are given, then the boundary-value problem*

$$\begin{cases} \nabla_{X,z}^{\mu_2} \cdot Q^{\mu_2}[\varepsilon_2 \zeta] \nabla_{X,z}^{\mu_2} u = \nabla_{X,z}^{\mu_2} \cdot \mathbf{h} & \text{in } \mathcal{S}, \\ \partial_n u|_{z=0} = \sqrt{\mu_2} \nabla \cdot V + \mathbf{e}_z \cdot \mathbf{h}|_{z=0}, & \partial_n u|_{z=-1} = \mathbf{e}_z \cdot \mathbf{h}|_{z=-1} \end{cases} \quad (20)$$

*admits (up to a constant) a unique solution  $u$ . Moreover, the solution  $u$  obeys the inequality*

$$\left| \nabla u|_{z=0} \right|_{H^s} \leq \frac{1}{\sqrt{\mu_2}} C\left(\frac{1}{H_2}, \varepsilon_2^{\max}, \mu_2^{\max}, |\zeta|_{H^{s+3/2}}\right) \left( \|\mathbf{h}\|_{H^{s+1/2,1}} + |V|_{H^{s+1}} \right),$$

*uniformly with respect to  $\varepsilon_2 \in [0, \varepsilon_2^{\max}]$  and  $\mu_2 \in (0, \mu_2^{\max})$ .*

*Remark 10.* Suppose we take  $\mathbf{h} = 0$  and  $V = V^\mu[\varepsilon \zeta] \psi$  in Proposition 3. By Lemma 1, one has  $\nabla u|_{z=0} = \mathbf{H}^{\mu,\delta}[\varepsilon \zeta] \psi$ . and the Proposition thus provides an estimate of the operator norm of  $\mathbf{H}^{\mu,\delta}[\varepsilon \zeta]$ .

*Proof.* The main lines of the proof are:

1. Check the coercivity of  $Q^{\mu_2}[\varepsilon_2 \zeta]$
2. Derive estimates on  $\nabla_{X,z}^{\mu_2} u$  in  $H^{r,1}$  ( $r \geq 0$ ) by elliptic estimates
3. Use the trace theorem to control  $|\nabla u|_{z=0}|_{H^s} \lesssim \|u\|_{H^{s+1/2,1}} \lesssim \frac{1}{\mu_2} \|\nabla_{X,z}^{\mu_2} u\|_{H^{s+1/2,1}}$  and use Step 2

□

The remaining task is therefore to find an approximation  $\underline{\Phi}_{app}$  to the solution of (18) in all the different asymptotic regimes considered here. As for the Dirichlet-Neumann operator, different techniques must be used in the shallow-water and small amplitude regimes. The situation is made more complicated here because the interface operator couples both fluid domains and that the dichotomy small amplitude/shallow water must be investigated for each fluid. These configurations are addressed in the following subsections.

### 3.2.1. The Small-Amplitude/Small-Amplitude Regime: $\varepsilon \ll 1$ , $\varepsilon_2 \ll 1$

In this regime, it is assumed that the interface deformations are of small amplitude for both the upper and lower fluids. The asymptotic expansion of the operator  $H^{\mu,\delta}[\varepsilon\zeta]$  is thus made in terms of  $\varepsilon$  and  $\varepsilon_2 = \varepsilon\delta$ . We construct an approximate solution  $\underline{\Phi}_{app}$  to (18) under the form

$$\underline{\Phi}_{app} = \Phi^{(0)} + \varepsilon_2 \Phi^{(1)}$$

(this form exploits the “small amplitude” assumption for the lower fluid). We may write from the expression for  $Q^{\mu_2}[\varepsilon_2\zeta]$ ,

$$\nabla_{X,z}^{\mu_2} \cdot Q^{\mu_2}[\varepsilon_2\zeta] \nabla_{X,z}^{\mu_2} = \Delta_{X,z}^{\mu_2} + \varepsilon_2 \nabla_{X,z}^{\mu_2} \cdot Q_1 \nabla_{X,z}^{\mu_2} + \varepsilon_2^2 \nabla_{X,z}^{\mu_2} \cdot Q_2 \nabla_{X,z}^{\mu_2},$$

where  $\Delta_{X,z}^{\mu_2} = \sqrt{\mu_2} \Delta + \partial_z^2$  and explicit formulas can be easily derived for  $Q_1$  and  $Q_2$ . At leading order, the elliptic operator of (18) thus reduces to  $\Delta_{X,z}^{\mu_2}$ , which correspond to the case of a flat domain. In particular, since  $\mu_2$  is not assumed to be small here, this leading operator keeps the full nonlocal effects of the usual Laplace operator in a domain of  $\mathbb{R}^{d+1}$  and  $\Phi^{(0)}$  depends nonlocally on  $\psi_1$ . After some tedious computations, one finds

$$\Phi^{(0)}(X, z) = - \frac{\cosh(\sqrt{\mu_2}(z+1)|D|) \tanh(\sqrt{\mu}|D|)}{\cosh(\sqrt{\mu_2}|D|) \tanh(\sqrt{\mu_2}|D|)} \psi_1$$

(the “small amplitude assumption” for the upper fluid has been implicitly used here through the use of Prop. 1 to approximate the Neumann condition at the interface of (18)). At leading order, one has therefore

$$\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1 \sim \nabla \Phi_{|z=0}^{(0)} \sim - \frac{\tanh(\sqrt{\mu}|D|)}{\tanh(\sqrt{\mu_2}|D|)} \nabla \psi_1.$$

The next order term of the expansion is needed to take into account the interface deformation. The computations are performed in [5] and a precise meaning to the symbol  $\sim$  is also given (see §2.2.1) but we do not give the details here. We just note that the formula can be simplified under additional smallness assumptions on  $\mu$  and  $\mu_2$ , namely, in the

$$\text{Boussinesq/Boussinesq regime:} \quad \mu \sim \varepsilon \text{ and } \mu_2 \sim \varepsilon_2. \quad (21)$$

Indeed, one then obtains

$$\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1 \sim -\delta \nabla \psi_1 - \frac{\delta}{3} \mu \left(1 - \frac{1}{\delta^2}\right) \Delta \nabla \psi_1 + \varepsilon_2 (1 + \delta) \Pi(\zeta \nabla \psi_1),$$

where  $\Pi = - \frac{\nabla \nabla^T}{|D|^2}$ .

### 3.2.2. The Shallow-Water/Shallow-Water Regime: $\mu \ll 1$ , $\mu_2 \ll 1$

In this regime, large amplitude waves are allowed for the upper fluid ( $\varepsilon = O(1)$ ) and for the lower fluid ( $\varepsilon_2 = O(1)$ ). Assuming that  $\mu \ll 1$  and  $\mu_2 \ll 1$  raises the prospect of making asymptotic expansions of shallow-water type, in terms of  $\mu$  and  $\mu_2$ . As before, the plan is to formally construct an approximate solution  $\underline{\Phi}_{app}$  to (18) having the form

$$\underline{\Phi}_{app} = \Phi^{(0)} + \mu_2 \Phi^{(1)}.$$

(such a form exploits the assumption that  $\mu_2$  is small). From the expression for  $Q^{\mu_2}[\varepsilon_2 \zeta]$ , we may write

$$\nabla_{X,z}^{\mu_2} \cdot Q^{\mu_2}[\varepsilon_2 \zeta] \nabla_{X,z}^{\mu_2} = \frac{1}{h_2} \partial_z^2 + \mu_2 \nabla_{X,z} \cdot Q_1 \nabla_{X,z},$$

with  $h_2 = 1 + \varepsilon_2 \zeta$  and where an explicit formula can be derived for  $Q_1$ . At leading order, the elliptic operator of (18) thus reduces to  $\frac{1}{h_2} \partial_z^2$ , which amounts to discard the horizontal derivatives of the original Laplace operator. Consequently, the non-local effects of the Laplace operators disappear in this regime (but new, unexpected enough, nonlocal effects appear, as shown below).

Using Proposition 2 (and thus the assumption that  $\mu$  is small) to approximate the Neumann condition at the interface of (18), one readily checks that  $\Phi^{(0)}$  and  $\Phi^{(1)}$  must solve

$$\begin{cases} \partial_z^2 \Phi^{(0)} = 0, \\ \partial_z \Phi^{(0)}|_{z=0} = 0, \quad \partial_z \Phi^{(0)}|_{z=-1} = 0, \end{cases}$$

which is obviously solved by any  $\Phi^{(0)}(X, z) = \Phi^{(0)}(X)$  independent of  $z$ , and

$$\begin{cases} \partial_z^2 \Phi^{(1)} = -h_2^2 \Delta \Phi^{(0)}, \\ \partial_z \Phi^{(1)}|_{z=0} = h_2 (\varepsilon_2 \nabla \zeta \cdot \nabla \Phi^{(0)} + \delta \nabla \cdot (h_1 \nabla \psi_1)), \quad \partial_z \Phi^{(1)}|_{z=-1} = 0, \end{cases}$$

where we have used the fact that  $\Phi^{(0)}$  does not depend on  $z$ . Solving this second order ordinary differential equation in the variable  $z$  with the boundary condition at  $z = 0$  yields (up to a function independent of  $z$  which we take equal to 0 for the sake of simplicity),

$$\Phi^{(1)} = -\frac{z^2}{2} h_2^2 \Delta \Phi^{(0)} + z (\partial_z \Phi_1|_{z=0}).$$

Matching the boundary condition at  $z = -1$  leads to the restriction

$$\nabla \cdot (h_2 \nabla \Phi^{(0)}) = -\delta \nabla \cdot (h_1 \nabla \psi_1),$$

and we thus deduce the following asymptotic expansion of the interface operator:

$$\mathbf{H}^{\mu, \delta}[\varepsilon \zeta] \psi_1 \sim \nabla(\underline{\Phi}|_{z=0}) \sim -\delta (I + \Pi(\varepsilon_2 \zeta \Pi \cdot))^{-1} \Pi(h_1 \nabla \psi_1), \quad (22)$$

where  $\Pi = -\frac{\nabla \nabla^T}{|D|^2}$  is the orthogonal projector onto the gradient vector fields of  $L^2(\mathbb{R}^d)^d$  defined earlier (and  $h_1 = 1 - \varepsilon \zeta$ ,  $h_2 = 1 + \varepsilon \delta \zeta$ ).

### 3.2.3. The Shallow-Water/Small-Amplitude Regime: $\mu \ll 1$ , $\varepsilon_2 \ll 1$

It is now presumed that both  $\mu$  and  $\varepsilon_2$  are small, but no such restriction is laid upon  $\varepsilon$  nor  $\mu_2$ . So, this regime is not a subcase of the regimes investigated in Sections 3.2.1 and 3.2.2. We construct an approximate solution  $\underline{\Phi}_{app}$  to (18) by using two kinds of expansions. The small amplitude assumption for the lower fluid is taken into account

by looking for  $\Phi_{app}$  under the form (3.2.1). The shallow water assumption for the upper fluid is taken into account by approximating the Neumann condition at the interface of (18) using Proposition 1. We do not give the details of the computations here (see §2.2.3 of [5]).

## 4. Asymptotic models for internal waves

### 4.1. The general procedure

It is established that the internal-wave equations (12) are *consistent* with the asymptotic models for  $(\zeta, \mathbf{v})$  derived in this paper in the following precise sense.

**Definition 3.** The internal wave equations (12) are *consistent* with a system  $S$  of  $d+1$  equations for  $\zeta$  and  $\mathbf{v}$  if for all sufficiently smooth solutions  $(\zeta, \psi_1)$  of (12) such that (14) and (16) are satisfied, the pair  $(\zeta, \mathbf{v})$ , with

$$\mathbf{v} = \mathbf{H}^{\mu, \delta}[\varepsilon\zeta]\psi_1 - \gamma\nabla\psi_1, \quad (23)$$

solves  $S$  up to a small residual called the *precision* of the asymptotic model.

*Remark 11.* It is worth emphasis that above definition does not require the well-posedness of the internal wave equations (12). Indeed, these can be subject to Kelvin-Helmholtz type instabilities (see for instance [13]), although one might expect a “stability of the instability” result even in the face of such instabilities (see [8]). *Consistency* is only concerned with the properties of smooth solutions to the system (which do exist in the classical configuration of the Kelvin-Helmholtz problem, even when instabilities manifest themselves; see e.g. [18, 17]). In fact, the two-layer water-wave system is known to be well-posed in Sobolev spaces in the presence of surface tension [13]. In consequence, one could simply add a small amount of surface tension at the interface between the two homogeneous layers to put oneself in a well-posed situation. The resulting analysis would be exactly the same and would, in fact, lead to the same asymptotic models. (Such an approach is used in [16] for the Benjamin-Ono equation). As the resulting model systems do not change, such a regularization has been eschewed here.

In the present paper, we have refrained from pursuing the analysis to the point of obtaining convergence results for the asymptotic systems to the full internal waves system. Such a program has been fully achieved in the case of surface waves in [1]. What is needed to complete the circle of ideas in the internal wave case is a stability analysis of the asymptotic models derived here (that is, an estimation of the remainders which comprise the difference between the Euler system and the models). Together with consistency, a straightforward analysis would then provide a convergence result to the full Euler system, assuming that the large time existence results obtained by Alvarez-Samaniego and Lannes in [1] for the surface wave system are valid for the internal waves system. The latter point is far from obvious; indeed, it is even false in absence of surface tension (see Remark 11 above) and even in presence of surface tension, it is still an open problem to prove that the solution exists on a time interval that is physically relevant (cf [16] for the rigorous derivation of the Benjamin-Ono equation for the two-fluid system in the presence of surface tension, but on a time interval very small if the surface tension is small).

## 4.2. The Boussinesq-Boussinesq model

In this regime, the nonlinear and dispersive effects are of the same size for both fluids; the systems of equations that are derived from the internal waves equations (12) in this situation are the following *three-parameter family* of Boussinesq/Boussinesq systems, *viz.*

$$\begin{cases} \left(1 - \mu b \Delta\right) \partial_t \zeta + \frac{1}{\gamma + \delta} \nabla \cdot \mathbf{v}_\beta + \varepsilon \frac{\delta^2 - \gamma}{(\gamma + \delta)^2} \nabla \cdot (\zeta \mathbf{v}_\beta) + \mu a \nabla \cdot \Delta \mathbf{v}_\beta = 0 \\ \left(1 - \mu d \Delta\right) \partial_t \mathbf{v}_\beta + (1 - \gamma) \nabla \zeta + \frac{\varepsilon}{2} \frac{\delta^2 - \gamma}{(\delta + \gamma)^2} \nabla |\mathbf{v}_\beta|^2 + \mu c \Delta \nabla \zeta = 0, \end{cases} \quad (24)$$

where  $\mathbf{v}_\beta = (1 - \mu \beta \Delta)^{-1} \mathbf{v}$ , and where the coefficients  $a, b, c, d$  are provided in the statement of the next theorem.

**Theorem 1.** *Let  $0 < c^{\min} < c^{\max}$ ,  $0 < \delta^{\min} < \delta^{\max}$ , and set*

$$\begin{aligned} a &= \frac{(1 - \alpha_1)(1 + \gamma\delta) - 3\delta\beta(\gamma + \delta)}{3\delta(\gamma + \delta)^2}, & b &= \alpha_1 \frac{1 + \gamma\delta}{3\delta(\gamma + \delta)}, \\ c &= \beta\alpha_2, & d &= \beta(1 - \alpha_2), \end{aligned}$$

*with  $\alpha_1 \geq 0$ ,  $\beta \geq 0$  and  $\alpha_2 \leq 1$ . With this specification of the parameters, The internal wave equations (12) are consistent with the Boussinesq/Boussinesq equations (24) in the sense of Definition 3, with a precision  $O(\varepsilon^2)$ , and uniformly with respect to  $\varepsilon \in [0, 1]$ ,  $\mu \in (0, 1)$  and  $\delta \in [\delta^{\min}, \delta^{\max}]$  such that  $c^{\min} < \frac{\varepsilon}{\mu} < c^{\max}$ .*

*Remark 12.* Taking  $\gamma = 0$  and  $\delta = 1$  in the Boussinesq/Boussinesq equations (24), reduces them to the system

$$\begin{cases} \left(1 - \mu \frac{\alpha_1}{3} \Delta\right) \partial_t \zeta + \nabla \cdot ((1 + \varepsilon \zeta) \mathbf{v}) + \mu \frac{1 - \alpha_1 - 3\beta}{3} \nabla \cdot \Delta \mathbf{v} = 0 \\ \left(1 - \mu \beta (1 - \alpha_2) \Delta\right) \partial_t \mathbf{v} + \nabla \zeta + \frac{\varepsilon}{2} \nabla |\mathbf{v}|^2 + \mu \beta \alpha_2 \Delta \nabla \zeta = 0, \end{cases}$$

which is exactly the family of formally equivalent Boussinesq systems derived in [4, 3].

*Remark 13.* The dispersion relation associated to (24) is

$$\omega^2 = |\mathbf{k}|^2 \frac{\left(\frac{1}{\gamma + \delta} - \mu a |\mathbf{k}|^2\right) (1 - \gamma - \mu c |\mathbf{k}|^2)}{(1 + \mu b |\mathbf{k}|^2) (1 + \mu d |\mathbf{k}|^2)}.$$

It follows that (24) is linearly well-posed when  $a, c \leq 0$  and  $b, d \geq 0$ . The system corresponding to  $\alpha_1 = \alpha_2 = \beta = 0$  is ill-posed (one can check that  $a = \frac{1 + \gamma\delta}{3\delta(\gamma + \delta)^2} > 0$ ). This system corresponds to a Hamiltonian system derived in [9] (see their formula (5.10)). The present, three-parameter family of systems allows one to circumvent the problem of ill-posedness without the need of taking into account higher-order terms in the expansion, as in [9]).

*Proof.* The proof is again made based on various possibilities for the parameters in the problem. For this regime, we have that  $\varepsilon \sim \mu \sim \varepsilon_2 \sim \mu_2$  as  $\varepsilon \rightarrow 0$ .

**Step 1.** The case  $\alpha_1 = 0$ ,  $\beta = 0$ ,  $\alpha_2 = 0$ . Using Remark 6 and (21) one checks immediately that

$$\nabla \psi_1 = -\frac{1}{\gamma + \delta} \left[ 1 + \mu \frac{1}{3\delta} \frac{1 - \delta^2}{\gamma + \delta} \Delta + \varepsilon_2 \frac{1 + \delta}{\gamma + \delta} \Pi(\zeta \cdot) \right] \mathbf{v} + O(\varepsilon^2)$$

(the nonlocal operator  $\Pi$  does not appear in the final equations because of the identity  $\nabla \cdot \Pi V = \nabla \cdot V$  for all  $V \in H^1(\mathbb{R}^d)^d$ ).

**Step 2.** The case  $\alpha_1 \geq 0$ ,  $\beta = 0$ ,  $\alpha_2 = 0$ . To use the BBM-trick, remark that for all  $\alpha_1 \geq 0$ ,

$$\nabla \cdot \mathbf{v} = (1 - \alpha_1) \nabla \cdot \mathbf{v} - \alpha_1(\gamma + \delta) \partial_t \zeta + O(\varepsilon).$$

Substitute this relation into the third-derivative term of the first equation of the system derived in Step 1.

**Step 3.** The case  $\alpha_1 \geq 0$ ,  $\beta \geq 0$ ,  $\alpha_2 = 0$ . It suffices to replace  $\mathbf{v}$  by  $(1 - \mu\beta\Delta)\mathbf{v}_\beta$  in the system of equations derived in Step 2.

**Step 4.** The case  $\alpha_1 \geq 0$ ,  $\beta \geq 0$ ,  $\alpha_2 \leq 1$ . We use once again the BBM trick. From the second equation in the system derived in Step 3, one obtains that for all  $\alpha_2 \leq 1$ ,

$$\partial_t \mathbf{v}_\beta = (1 - \alpha_2) \partial_t \mathbf{v}_\beta - \alpha_2(1 - \gamma) \nabla \zeta + O(\varepsilon).$$

If this relationship is substituted into the system derived in Step 3, the result follows.  $\square$

### 4.3. The Shallow water/Shallow water model

Contrary to the regimes investigated above, large amplitude interfacial deformations are allowed for both fluids, as  $\varepsilon \sim \varepsilon_2 = O(1)$ . As in the previous section, an asymptotic model can be derived from (12) by replacing the operators  $G^\mu[\varepsilon\zeta]$  and  $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]$  by their asymptotic expansions, provided by Proposition 2 and (22) in the present regime. The following theorem shows that the internal wave equations are consistent in this regime with the Shallow water/Shallow water system,

$$\begin{cases} \partial_t \zeta + \nabla \cdot (h_1 \mathfrak{R}[\varepsilon\zeta] \mathbf{v}) = 0, \\ \partial_t \mathbf{v} + (1 - \gamma) \nabla \zeta + \frac{\varepsilon}{2} \nabla \left( |\mathbf{v} - \gamma \mathfrak{R}[\varepsilon\zeta] \mathbf{v}|^2 - \gamma |\mathfrak{R}[\varepsilon\zeta] \mathbf{v}|^2 \right) = 0, \end{cases} \quad (25)$$

where  $h_1 = 1 - \varepsilon\zeta$ ,  $h_2 = 1 + \varepsilon\delta\zeta$ , and the operator  $\mathfrak{R}$  is defined by (recalling that  $\Pi = -\frac{\nabla \nabla^T}{|D|^2}$ )

$$\mathfrak{R}[\varepsilon\zeta] \mathbf{v} = \frac{1}{\gamma + \delta} \left( 1 - \Pi \left( \frac{1 - \gamma}{\gamma + \delta} \varepsilon \delta \zeta \Pi \cdot \right) \right)^{-1} \Pi(h_2 \mathbf{v}).$$

**Theorem 2.** *Let  $0 < \delta^{\min} < \delta^{\max} \leq (1 - \delta(1 - H_1))^{-1}$ . The internal waves equations (12) are consistent with the SW/SW equations (25) in the sense of Definition 3, with a precision  $O(\mu)$ , and uniformly with respect to  $\varepsilon \in [0, 1]$ ,  $\mu \in (0, 1)$  and  $\delta \in [\delta^{\min}, \delta^{\max}]$ .*

*Remark 14.* Taking  $\gamma = 0$  and  $\delta = 1$  in the SW/SW equations (25) yields the usual shallow water equations for surface water waves.

*Remark 15.* In the one-dimensional case  $d = 1$ , one has

$$\mathfrak{R}[\varepsilon\zeta] \mathbf{v} = \frac{h_2}{\delta h_1 + \gamma h_2} \mathbf{v}$$

and the equations (25) take the simpler form

$$\begin{cases} \partial_t \zeta + \partial_x \left( \frac{h_1 h_2}{\delta h_1 + \gamma h_2} \mathbf{v} \right) = 0, \\ \partial_t \mathbf{v} + (1 - \gamma) \partial_x \zeta + \frac{\varepsilon}{2} \partial_x \left( \frac{(\delta h_1)^2 - \gamma h_2^2}{(\delta h_1 + \gamma h_2)^2} |\mathbf{v}|^2 \right) = 0, \end{cases}$$

which coincides of course with the system (5.26) of [9]. The presence of the nonlocal operator  $\mathfrak{R}$ , which does not seem to have been noticed before, appears to be a purely two dimensional effect.

*Proof.* First remark that with the range of parameters considered in the theorem, one has  $\mu \sim \mu_2$  as  $\mu \rightarrow 0$  while  $\varepsilon \sim \varepsilon_2 = O(1)$ .

By the definition (23) of  $\mathbf{v}$  and using Proposition 2 and (22), one deduces from (12) that

$$\begin{cases} \partial_t \zeta - \nabla \cdot ((1 - \varepsilon \zeta) \nabla \psi_1) = O(\mu), \\ \partial_t \mathbf{v} + (1 - \gamma) \nabla \zeta + \frac{\varepsilon}{2} \nabla (|\mathbf{H}^{\mu, \delta}[\varepsilon \zeta] \psi_1|^2 - \gamma |\nabla \psi_1|^2) = O(\mu). \end{cases} \quad (26)$$

Recall now that  $\mathbf{H}^{\mu, \delta}[\varepsilon \zeta] \psi_1 = \mathbf{v} + \gamma \nabla \psi_1$ ; using this relation together with (22), one can get

$$\nabla \psi_1 = -\mathfrak{R}[\varepsilon \zeta] \mathbf{v} + O(\mu)$$

and consequently,

$$\begin{aligned} \mathbf{H}^{\mu, \delta}[\varepsilon \zeta] \psi_1 &= \mathbf{v} + \gamma \nabla \psi_1 \\ &= \mathbf{v} - \gamma \mathfrak{R}[\varepsilon \zeta] \mathbf{v} + O(\mu). \end{aligned}$$

Replacing  $\nabla \psi_1$  and  $\mathbf{H}^{\mu, \delta}[\varepsilon \zeta] \psi_1$  by these two expressions in (26) yields the result.  $\square$

## References

- [1] B. ALVAREZ-SAMANIEGO AND D. LANNES, *Large time existence for 3D water-waves and asymptotics*, Invent. math. **171** (2008) 485-541.
- [2] T.B. BENJAMIN AND T.J. BRIDGES, *Reappraisal of the KelvinHelmholtz problem. Part 1. Hamiltonian structure*, J. Fluid Mech. **333** (1997) 301-325.
- [3] J. L. BONA, T. COLIN AND D. LANNES, *Long wave approximations for water-waves*, Arch. Rational Mech. Anal. **178** (2005) 373-410.
- [4] J. L. BONA, M. CHEN AND J.-C. SAUT, *Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media. Part I. Derivation and linear theory*, J. Nonlinear Sci. **12** (2002) 283-318.
- [5] J. BONA, D. LANNES AND J.-C. SAUT, *Asymptotic models for internal waves*, J. Math. Pures Appl. **89** (2008) 538-566.
- [6] W. CHOI AND R. CAMASSA, *Weakly nonlinear internal waves in a two-fluid system*, J. Fluid Mech. **313** (1996) 83-103.
- [7] W. CHOI AND R. CAMASSA, *Fully nonlinear internal waves in a two-fluid system*, J. Fluid Mech. **396** (1999) 1-36.
- [8] P. CONSTANTIN, *On the Euler equations of incompressible fluids*, Bull. AMS **44**, 4, (2007), 603-621.
- [9] W. CRAIG, P. GUYENNE, AND H. KALISCH, *Hamiltonian long-wave expansions for free surfaces and interfaces*, Comm. Pure. Appl. Math. **58** (2005) 1587-1641.

- [10] W. CRAIG, U. SCHANZ AND C. SULEM, *The modulational regime of three-dimensional water waves and the Davey-Stewartson system*, Ann. Inst. H. Poincaré, Anal. Non Linéaire **14** (1997) 615-667.
- [11] W. CRAIG, C. SULEM AND P.-L. SULEM, *Nonlinear modulation of gravity waves: a rigorous approach*, Nonlinearity **5** (1992) 497-522.
- [12] K.R. HELFRICH AND W.K. MELVILLE, *Long nonlinear internal waves*, Annual Review of Fluid Mechanics **38** (2006) 395-425.
- [13] T. IGUCHI, N. TANAKA AND A. TANI, *On the two-phase free boundary problem for two-dimensional water waves*, Math. Ann. **309** (1997) 199-223.
- [14] D. LANNES, *Well-posedness of the water-waves equations*, J. Amer. Math. Soc. **18** (2005) 605-654.
- [15] H.Y. NGUYEN AND F. DIAS, *A Boussinesq system for two-way propagation of interfacial waves*, preprint , (2007).
- [16] K. OHI AND T. IGUCHI, *A two-phase problem for capillary-gravity waves and the Benjamin-Ono equation*, Discrete Contin. Dyn. Syst. **23** (2009) 1205-1240.
- [17] C. SULEM AND P.-L. SULEM, *Finite time analyticity for the two- and three-dimensional Rayleigh-Taylor instability*, Trans. American Math. Soc. **287** (1985) 127-160.
- [18] C. SULEM, P.-L. SULEM, C. BARDOS AND U. FRISCH, *Finite time analyticity for the two- and three-dimensional Kelvin-Helmholtz instability*, Comm. Math. Phys. **80** (1981) 485-516.
- [19] V.E. ZAKHAROV, *Stability of periodic waves of finite amplitude on the surface of a deep fluid*, J. Appl. Mech. Tech. Phys. **2** (1968) 190-194.

DEPARTMENT OF MATHEMATICS, STATISTICS AND COMPUTER SCIENCE,  
 UNIVERSITY OF ILLINOIS AT CHICAGO, CHICAGO, IL 60607, USA.  
 bona@math.uic.edu

ECOLE NORMALE SUPÉRIEURE, DMA ET CNRS UMR 8553, 45, RUE D'ULM,  
 75005 PARIS, FRANCE.  
 David.Lannes@ens.fr

UNIVERSITÉ DE PARIS-SUD ET CNRS UMR 8628, BÂT. 425, 91405 ORSAY  
 CEDEX, FRANCE.  
 jean-claude.saut@math.u-psud.fr