Carlos E. Kenig

Lecture notes: Global Well-posedness, scattering and blow up for the energy-critical, focusing, non-linear Schrödinger and wave equations


<http://jedp.cedram.org/item?id=JEDP_2007_____A1_0>
Lecture Notes:

Global Well-posedness, scattering and blow up for the energy-critical, focusing, non-linear Schrödinger and wave equations

Carlos E. Kenig

Lecture 1

In these lectures I will discuss recent joint works with F. Merle. In them we have developed an approach to the study of non-linear critical problems of dispersive type. The issues studied are global well-posedness and scattering. The approach works for both focusing and defocusing problems, but in these lectures I will concentrate on two focusing problems. The approach proceeds in steps, some of which are general and hence apply to “all problems” and some which are specific to each particular problem. The concrete problems to be discussed here are the energy critical, focusing non-linear Schrödinger equation and wave equation. I will try to separate both kinds of arguments in the exposition. I will start out by discussing (NLS).

Consider thus the Cauchy problem for the \( \dot{H}^1 \) critical non-linear Schrödinger equation

\[
\begin{cases}
    i\partial_t u + \Delta u \pm |u|^{4/N-2} u = 0, & x \in \mathbb{R}^N, t \in \mathbb{R}, \\
    u|_{t=0} = u_0 \in \dot{H}^1.
\end{cases}
\]

The problem is called “critical” because if \( u \) is a solution and \( \lambda > 0 \), \( u_\lambda(x, t) = \frac{1}{\lambda^{N-2}} u(\frac{x}{\lambda}, \frac{t}{\lambda^2}) \) is also a solution and \( ||u_\lambda(-, 0)||_{\dot{H}^1} = ||u_0||_{\dot{H}^1}, \forall \lambda > 0 \). Here the \( - \) sign corresponds to the defocusing problem and the \( + \) sign to the focusing problem. The theory of the local Cauchy problem (Cazenave-Weissler 90, [4]) shows that if \( ||u_0||_{\dot{H}^1} \leq \delta, \delta = \delta_N > 0 \) is small (and \( N \leq 5 \)) then \( \exists! \) solution to \((CP)\) with \( u \in C(\mathbb{R}; \dot{H}^1)\), \( ||u||_{L^{2(N+2)/N-2}_{x,t}} < \infty \) (i.e. the solution scatters). As we will see later this is equivalent to \( \exists u_0^\pm \in \dot{H}^1 \) s.t. \( ||u(-, t) - e^{i\Delta} u_0^\pm||_{\dot{H}^1} \xrightarrow{t \to \pm \infty} 0 \). Also, the energy identity holds, i.e.

\[
E(u(t)) = \frac{1}{2} \int |\nabla u(x, t)|^2 \, dx \pm \frac{1}{2^*} \int |u(x, t)|^{2^*} \, dx = E(u_0).
\]

Supported in parts by NSF grant DMS-0456583.
Here $\pm$ corresponds to the defocusing, focusing cases, $\frac{1}{2'} = \frac{1}{2} - \frac{1}{N}$ is the “Sobolev conjugate” exponent. Here we see the difference between the defocusing and focusing cases. In the defocusing case, Bourgain ([3], 1998) proved that for $N = 3, 4$, $u_0$ radial, the above result holds for $\|u_0\|_{\dot{H}^1} < \infty$. Bourgain’s result was extended to $u_0$ radial, $N \geq 5$ by Tao ([30], 2004). For $N = 3$, general $u_0$ the same result was proved by Colliander-Keel-Staffilani-Takaoka-Tao ([5], 2004). This was extended to $N = 4$ by Ryckman-Visan ([25], 2005) and to $N \geq 5$ by Visan ([33], 2007). In the focusing case, these last results do not hold. In fact, a classical argument, based on the “virial identity” (Zakharov, Glassey) shows that if $\int |x|^2 |u_0(x)|^2 < \infty$ and $E(u_0) < 0$, then the solution must break-down in finite time (Glassey 77, [11]). Also,

$$W(x, t) = W(x) = \left(1 + \frac{|x|^2}{N(N-2)}\right)^{-\frac{(N-2)}{2}} \in \dot{H}^1$$

and solves the elliptic equation

$$\Delta W + |W|^{4/N-2} W = 0, \quad x \in \mathbb{R}^N,$$

and hence (NLS), but scattering does not occur, even though the solution is global in time. Our main result in this case is:

**Theorem A** (K-Merle [16], 2006). For the energy critical, focusing (NLS), $N = 3, 4, 5$, $u_0$ radial with $E(u_0) < E(W)$,

i) if $\|u_0\|_{\dot{H}^1} < \|W\|_{\dot{H}^1}$ the solution exists for all times and scatters.

ii) if $\|u_0\|_{\dot{H}^1} > \|W\|_{\dot{H}^1}$ (and $\|u_0\|_{L^2} < \infty$) then the solution breaks down in finite time.

**Remark.** The conditions $E(u_0) < E(W)$ and $\|\nabla u_0\| = \|\nabla W\|$ are incompatible (from now on, $\| \|$ is the $L^2$ norm).

I will now turn to the proof of Theorem A. We start with a quick review of the local (CP) theory.

**Theorem** (Cazenave-Weissler [4], 1990). Let $u_0 \in \dot{H}^1(\mathbb{R}^N)$, $\|u_0\|_{\dot{H}^1} \leq A$. Then, (for $3 \leq N \leq 5$) $\exists \delta = \delta(A)$ s.t. if $\|e^{it\Delta}u_0\|_{S(I)} \leq \delta$, $0 \in I$, there exists a unique solution to (CP) in $\mathbb{R}^N \times I$ s.t. $u \in C(I; \dot{H}^1)$, $\sup_{t \in I} \|u(t)\|_{\dot{H}^1} + \|\nabla u\|_{W(I)} \leq C(A)$ and $\|u\|_{S(I)} \leq 2\delta$. (Here $\|f\|_{S(I)} = \|f\|_{L^2(N+2)/N-2}^2 L^2(N+2)/N-2$ ; $\|f\|_{W(I)} = \|f\|_{L^1(N+2)/N-2 L^2(N+2)/N-2}$. Moreover, $u_0 \mapsto u \in C(I; \dot{H}^1)$ is Lipschitz.

**Sketch of the Proof.** The key ingredients are the Strichartz estimates (Strichartz 77, [28], Keel-Tao 98, [15]) ($N \geq 3$)

i) $\|\nabla e^{it\Delta} u_0\|_{W(-\infty, +\infty)} \leq C \|u_0\|_{\dot{H}^1}$

(S)

ii) $\|\nabla \int_0^t e^{i(t-t')\Delta} g(-, t') dt'\|_{W(-\infty, +\infty)} \leq C \|\nabla g\|_{L^2_t L^2(N+2)}$

iii) $\sup_{t} \|\nabla \int_0^t e^{i(t-t')\Delta} g(-, t') dt'\| \leq C \|\nabla g\|_{L^2_t L^2(N+2)}$

and the Sobolev embedding:

(Sob) $\|v\|_{L^1(N+2)/N-2 L^2(N+2)/N-2} \leq C \|\nabla v\|_{W(I)}$.

I–2
We then have (with \( f(u) = \pm |u|^{4/N-2} u \)) to solve the integral equation (Duhamel’s principle)

\[
u(t) = e^{it\Delta} u_0 + \int_0^t e^{i(t-t')\Delta} f(u) \, dt'.
\]

Let \( B_{a,b} = \{ v \in \mathbb{R}^N \times I : ||v||_{S(I)} \leq a, ||\nabla v||_{W(I)} \leq b \} \), \( \Phi u_0(v) = e^{it\Delta} u_0 + \int_0^t e^{i(t-t')\Delta} f(v) \, dt' \). We will show that we can choose \( \delta, a, b \) s.t.

\[
\Phi u_0 : B_{a,b} \rightarrow B_{a,b}
\]

and is a contraction. From this, also using (S) iii), the theorem follows. But, by (S) i), ii)

\[
||\nabla \Phi u_0(v)||_{S(I)} \leq C A + C ||\nabla f(v)||_{L_t^2 L_x^{2N/N+2}}.
\]

But \( |\nabla f(v)| \leq C |\nabla v| |v|^{4/N-2} \), so Hölder gives that this is

\[
\leq C A + C ||v||^{4/N-2}_{S(I)} ||\nabla v||_{W(I)} \leq C A + C a^{4/N-2} b.
\]

By (Sob), \( ||\Phi u_0(v)||_{S(I)} \leq \delta + C a^{4/N-2} b \). We then choose \( b = 2AC, a \) so that

\[
C a^{4/N-2} \leq \frac{1}{2}, \text{ so that } ||\nabla \Phi u_0(v)||_{W(I)} \leq b. \text{ If we now set } \delta = \frac{\delta}{2}, \text{ we obtain } ||\Phi u_0||_{S(I)} \leq a. \text{ The contraction property is similar and the Theorem follows.} \]

\[\square\]

Remark. Because of (S), (Sob), \( \exists \delta \text{ s.t. if } ||u_0||_{H^1} < \delta, \text{ the hypothesis of the Theorem holds for } I = (-\infty, +\infty). \text{ Moreover, given } u_0 \in H^1, \exists I \text{ s.t. } ||e^{it\Delta} u_0||_{S(I)} \leq \delta, \text{ so the Theorem applies on } I. \text{ Note also that if } u^{(1)}, u^{(2)} \text{ are solutions of (CP) on } I (u \in C(I; H^1), \nabla u \in W(I), \text{ the integral equation holds with } u^{(1)}(t_0) = u^{(2)}(t_0), t_0 \in I), \text{ then } u^{(1)} \equiv u^{(2)} \text{ on } I. \text{ This is because we can partition } I \text{ into } I_j \text{ s.t. } ||u^{(i)}||_{S(I_j)} \leq a, ||\nabla u^{(i)}||_{W(I_j)} \leq b : \text{ choosing } t_0 \in I_{j_0} \text{ using the uniqueness of the fixed point in } I_{j_0} \text{ and then induction on } j, \text{ our claim follows. Thus, there exists a maximal interval } I = I(u_0) = (-T_-(u_0), T_+(u_0)) \text{ where the solution } u \in C(I', H^1) \cap \{ \nabla u \in W(I') \}, \forall I' \in I, I' \neq I, \text{ is defined. We call } I \text{ the maximal interval of existence. For } t \in I \text{ we have } E(u(t)) = E(u_0). \]

Standard blow-up criterion

If \( T_+(u_0) < +\infty \), we must have \( ||u||_{S[0,T_+(u_0))] = +\infty. \text{ If not, } M = ||u||_{S[0,T_+(u_0))] < \infty. \)

For \( \varepsilon > 0, \) to be chosen, partition \( [0, T_+(u_0)) = \gamma \bigcup_{j=1} \gamma_{j} I_j \), so that \( ||u||_{S(I_j)} \leq \varepsilon. \) If \( I_j = [t_j, t_{j+1}) \), using the integral equation and the proof of the Theorem above, we have

\[
\sup_{t \in I_j} ||u(t)||_{H^1} + ||\nabla u||_{W(I_j)} \leq C ||u(t_j)||_{H^1} + C ||u||^{4/N-2}_{S(I_j)} ||\nabla u||_{W(I_j)}.
\]

If \( C \varepsilon^{4/N-2} \leq \frac{1}{2} \) we can show inductively that

\[
\sup_{t \in [0,T_+(u_0))} ||u(t)||_{H^1} + ||\nabla u||_{W([0, T_+(u_0)))} \leq C(M).
\]

Choose now \( (t_n) \uparrow T_+(u_0) \) and show, again using the integral equation, that for \( n \) large, \( ||e^{i(t-t_n)\Delta} u(t_n)||_{S(t_n, T_+(u_0))} \leq \frac{\delta}{2} \) (on \( [t_n, T_+(u_0)) \)), \( u(t) = e^{i(t-t_n)\Delta} u(t_n) + \int_{t_n}^t e^{i(t-t')\Delta} f(u) \, dt' \). But then, for same \( \varepsilon > 0 \) we have

\[
||e^{i(t-t_n)\Delta} u(t_n)||_{S(t_n, T_+(u_0)+\varepsilon_0)} \leq \delta
\]

which, by the Theorem contradicts the definition of \( T_+(u_0) \).
Scattering

If \( T_+(u_0) = +\infty \) and \( M = \|u\|_{S(0, +\infty)} < \infty \) then \( u \) scatters at \( +\infty \). In fact, by the integral equation, as before we show that 
\[
\sup_{t \in [0, +\infty)} \|u(t)\|_{\dot{H}^1} + \|\nabla u\|_{W(0, +\infty)} \leq C(M).
\]

But then, since
\[
\exists \epsilon = \epsilon_0(M, A, A') \text{ s.t. if } 0 < \epsilon \leq \epsilon_0 \text{ and } \|\nabla u\|_{L^2_xL^2_t} \leq \epsilon, \quad \|e^{it\Delta} [u_0 - \tilde{u}(0)]\|_{\delta(I)} \leq \epsilon,
\]

then \( \exists! \) solution \( u \) on \( \mathbb{R}^N \times I \), s.t.
\[
\|u\|_{\dot{H}^1} \leq C(A', A, M) \text{ and } \sup_{t \in I} \|u(t) - \tilde{u}(t)\|_{\dot{H}^1} \leq C(A, A', M)(A' + \epsilon + \epsilon')
\]

where \( \epsilon' = \epsilon^\beta \) for some \( \beta > 0 \).

In the proof it suffices to give a priori estimates for \( u \), assuming that it exists. The (CP) theory gives the rest. We will need 2 new ingredients:

\[ (F) \quad \left\| \int_0^t e^{i(t-t')\Delta} h(t') \, dt' \right\|_{L^q_xL^r_t} \leq C \|h\|_{L^q_xL^r_t} \]

(Foschi [6], 2003, Vilela [32], 2007) holds, provided
\[
\frac{1}{q} + \frac{1}{r} = \frac{N}{2} \left[ 1 - \frac{1}{r} - \frac{1}{\tilde{r}} \right] \text{ and } \frac{1}{q} < N \left( \frac{1}{2} - \frac{1}{r} \right),
\]
\[
\frac{1}{q} < N \left( \frac{1}{2} - \frac{1}{r} \right), \quad \frac{1}{q} + \frac{1}{\tilde{r}} < 1, \quad \frac{N-2}{r} < \frac{N}{\tilde{r}}, \quad \frac{N-2}{\tilde{r}} < \frac{N}{r}.
\]

Notice that \( (q, r) = \left( \frac{2(N+2)}{N-2}, \frac{2(N+2)}{(N-2)} \right) \), \( (\tilde{q}, \tilde{r}) = (2, 2N) \) verify the conditions. For \( 0 < \alpha < 1, \alpha \text{ near } 1, \) let
\[
\frac{1}{r} = \frac{N-2}{2(N+2)} + \frac{\alpha}{N} \text{ so that }
\]
\[
\|f\|_{L^2_x(2N/(N-2))} \leq C \|D^\alpha f\|_{L^q_x},
\]

I-4
Thus, if \( C \eta \) that there exists \( \alpha \) for \( q \) and with \( \eta = 2(\alpha - 1)N - 4\alpha \), so that \( \frac{1}{r'} = \frac{1}{r} + \frac{2}{(N+2)} \). Note then that, for \( \alpha \) close to 1, \((F)\) is verified. By interpolation we have \( ||D^\alpha e^{i\alpha t} u_0 - \tilde{u}(0)||_{L_t^q L_x^s} \leq C(A') \varepsilon', \varepsilon' = \varepsilon^{(1-\alpha)}. \) Moreover, by Hölder,

\[
||u||^{4/N-2} D^\alpha u ||_{L_t^q L_x^s} \leq ||u||^{4/N-2} ||D^\alpha u||_{L_t^q L_x^s}.
\]

The second ingredient is the chain rule and Leibniz rule for fractional derivatives ([19], [93]): in this case,

\[
||D^\alpha \left[ f(\tilde{u} + w) - f(\tilde{u}) \right] ||_{L_t^q L_x^s} \leq C[||\tilde{u}||_{S^{4/N-2}} + ||w||_{S^{4/N-2}}] ||D^\alpha w||_{L_t^q L_x^s} + C ||w||_{s} \left[ ||\tilde{u}||_{S^{6/N-2}} + ||w||_{S^{6/N-2}} \right] \left[ ||D^\alpha \tilde{u}||_{L_t^q L_x^s} + ||D^\alpha w||_{L_t^q L_x^s} \right].
\]

To carry out the proof, we write \( u = \tilde{u} + w \), so that the equation for \( w \) is \( i\partial_t w + \Delta w = f(\tilde{u} + w) - f(\tilde{u}) - e, \) \( \|w\|_{t=0} = u_0 - \tilde{u}(0) \). Note that by the integral equation for \( \tilde{u} \), splitting into sub-intervals we obtain \( ||\nabla \tilde{u}||_{W(I)} \leq M = M(M, A) \), so that, by interpolation, \( ||D^\alpha \tilde{u}||_{L_t^q L_x^s} \leq M_t = M_t(M, A). \) We then split \( I = \bigcup_{j=1}^J I_j, J = J(M, A, \eta) \) so that on each \( I_j \) we have \( ||D^\alpha \tilde{u}||_{L_t^q L_x^s} \leq \eta, \eta > 0 \) to be chosen. Let \( I_j = [a_j, a_{j+1}], a_0 = 0, a_{J+1} = L. \) By the integral equation on \( I_j \)

\[
w(t) = e^{i(t-a_j)\Delta} w(a_j) + \int_{a_j}^t e^{i(t-t')\Delta} \left[ f(\tilde{u} + w) - f(\tilde{u}) \right] dt' + \int_{a_j}^t e^{i(t-t')\Delta} e(t') dt'.
\]

By \((F)\) (and \((S)\) and \((S)\)) we have

\[
||D^\alpha w||_{L_t^q L_x^s} \leq ||D^\alpha e^{i(t-a_j)\Delta} w(a_j)||_{L_t^q L_x^s} + C \varepsilon_0 + C ||D^\alpha [f(\tilde{u} + w) - f(\tilde{u})]||_{L_t^q L_x^s}
\leq \left( ||D^\alpha e^{i(t-a_j)\Delta} w(a_j)||_{L_t^q L_x^s} + C \varepsilon_0 \right) + C \eta^{4/N-2} ||D^\alpha w||_{L_t^q L_x^s} + C ||D^\alpha w||_{L_t^q L_x^s}^{(N+2)/N-2}.
\]

Thus, if \( C \eta^{4/N-2} \leq \frac{1}{3} \), we get

\[
||D^\alpha w||_{L_t^q L_x^s} \leq \frac{3}{2} \gamma_j + \hat{C} ||D^\alpha w||_{L_t^q L_x^s}^{(N+2)/N-2},
\]

where \( \gamma_j = \left[ ||D^\alpha e^{i(t-a_j)\Delta} w(a_j)||_{L_t^q L_x^s} + C \varepsilon_0 \right]. \)

Note that \( \eta \) depends only on \( N \). From this a standard continuity argument shows that there exists \( C_0 = C_0(\hat{C}) \) s.t. if \( \gamma_j \leq C_0 \), we have

\[
a) \ ||D^\alpha w||_{L_t^q L_x^s} \leq 3\gamma_j. \quad b) \ \hat{C} ||D^\alpha w||_{L_t^q L_x^s}^{(N+2)/N} \leq 3\gamma_j.
\]

I-5
Hence, \(|D^αw|_{L^q_j L^r_x} \leq 3(||D^α e^{i(t-a_j)Δ} w(a_j)||_{L^q_j L^r_x} + C \varepsilon_0||. To continue in the iteration, put \(t = a_{j+1}\) in the integral formula, apply \(e^{i(t-a_{j+1})Δ}\) to obtain:

\[
e^{i(t-a_{j+1})Δ} w(a_{j+1}) = e^{i(t-a_{j})Δ} + \int_{a_j}^{a_{j+1}} e^{i(t-t')Δ} [f(\tilde{u} + w) - f(\tilde{u})] dt' + \int_{a_j}^{a_{j+1}} e^{i(t-t')Δ} e(t') dt'.
\]

By the same argument we get:

\[
||D^α e^{i(t-a_{j+1})Δ} w(a_{j+1})||_{L^q_j L^r_x} \leq ||D^α e^{i(t-a_{j})Δ} w(a_{j})||_{L^q_j L^r_x} + C \varepsilon_0 + C \eta^{4/N-2} ||D^α w||_{L^q_j L^r_x} + \tilde{C} ||D^α w||_{L^q_j L^r_x}.
\]

Again, taking \(\eta\) small we see that \(γ_{j+1} \leq 10γ_j\) provided \(γ_j \leq C_0\). Recall that by assumption we have \(γ_0 \leq ε'_0 + C \varepsilon_0\). Iterating, \(γ_j \leq 10^j(ε'_0 + C \varepsilon_0)\), if \(γ_j \leq C_0\). If we have \(10^{j+1}(ε'_0 + C \varepsilon_0) \leq C_0\), this always holds. Repeating the argument we obtain

\[
||D^α w||_{L^q_j L^r_x} \leq 3(J + 1) 10^{(J+1)} (ε'_0 + C \varepsilon_0),
\]

for \(ε_0\) small. Hence, by Sobolev \(||w||_{S(I)} \leq C(ε'_0 + ε_0)\). The rest of the argument follows similarly.

Some useful corollaries:

**Corollary 1.** Let \(K \subset \dot{H}^1\) be s.t. \(\overline{K}\) is compact. Then \(∃T^-_K, T^+_K\) s.t. \(∀ u_0 ∈ K, T^+_T(u_0) ≥ T^-_K, T^-(u_0) ≥ T^-_K\).

Choose \(M = 1, \tilde{A} = \sup_{u_0 ∈ K} ||u_0||_{\dot{H}^1}, A = C(\tilde{A})\) as in (CP), \(ε_0 = ε_0(1, A, 1)\) as in Perturbation Theorem, \(ε_1 \leq \min(ε_0, 1)\). Cover \(K\) by balls \(B(u_{0,j}, ε_1), 1 \leq j \leq J\) (compactness of \(\overline{K}\)). Consider \(\bar{T}_j^+, \bar{T}_j^-\) s.t. \(||u_j||_{S[-\bar{T}_j^-, \bar{T}_j^+]} \leq 1\) and \(T^+ = \min_{1 \leq j \leq J} \bar{T}_j^+, T^- = \min_{1 \leq j \leq J} \bar{T}_j^-\). Then, if \(u_0 ∈ B(u_{0,j}, ε_1)\) for some \(j\), the solution exists in \([-T^-, T^+]\) by Perturbation Theorem.

**Corollary 2.** Let \(\tilde{u}_0 ∈ \dot{H}^1, ||\tilde{u}_0||_{\dot{H}^1} ≤ A, \tilde{u}\) solution in \((-T^-(\tilde{u}_0), T^+(\tilde{u}_0))\). If \(u_{0,n} → \tilde{u}_0\) in \(\dot{H}^1\), then \(T^-(-\tilde{u}_0) ≤ \lim\inf T^-(-u_{0,n}); T^+(-\tilde{u}_0) \leq \lim\inf T^+(u_{0,n})\) and \(∀ t ∈ (-T^-(\tilde{u}_0), T^+(\tilde{u}_0))\) we have \(u_n(t) → \tilde{u}(t)\).

In fact, if \(I' \subset I = (-T^-(\tilde{u}_0), T^+(\tilde{u}_0))\), \(\sup_{t ∈ I'} ||\tilde{u}(t)||_{\dot{H}^1} ≤ C(A, I'), ||\tilde{u}||_{S(I')} ≤ M\). Apply the Perturbation Theorem with \(u = u_n, u_0 = u_{0,n} on I'\). If \(ε_0 = ε_0(M, C(A, I'), 1)\) and \(n\) is so large that \(||u_{0,n} - \tilde{u}_0||_{\dot{H}^1} ≤ 1, ||e^{itA}u_{0,n} - \tilde{u}_0||_{S} ≤ ε_0\), we have \(u_n\) exists on \(I'\) and \(\sup_{t ∈ I'} ||u_n(t) - \tilde{u}(t)||_{\dot{H}^1} ≤ C(A, M)\). The claim follows.

From now on we concentrate on the focusing case,

\[
\begin{cases}
    i \frac{∂}{∂t} u + \Delta u + |u|^{4/N-2} u = 0, \\
    u|_{t=0} = u_0 ∈ \dot{H}^1.
\end{cases}
\]

We start out with a review of Glassey’s blow-up result: assume that

\[
\int |x|^2 |u_0(x)|^2 dx < ∞, \quad u_0 ∈ \dot{H}^1, \quad E(u_0) < 0, \quad I = (-T^-(u_0), T^+(u_0)).
\]
Let \( y(t) = \int |x|^2 |u(x,t)|^2 \, dx \). A calculation shows that \( y''(t) = 8 \int \left| \nabla u(x,t) \right|^2 - \left| u(x,t) \right|^2 \, dx \) (the same calculation also gives \( y(t) < \infty, \forall t \in I \)). Since \( E(u(t)) = E(u_0) < 0, \frac{1}{2} \int \left| \nabla u(t) \right|^2 - |u(t)|^2 = E(u(t)) + \left( \frac{1}{2} - \frac{1}{2} \right) \int |u(t)|^2 \leq E(u(t)) = E(u_0) < 0, y''(t) < 16 E(u_0) < 0. \) But, since \( y \geq 0 \), \( I \) cannot be infinite. The next step is to establish some variational estimates. Recall that \( W(x) = (1 + \frac{|x|^2}{N(N-2)})^{-(N-2)/2} \) is a stationary solution of \( \text{CP} \), \( E \in H^1 \) and solves the elliptic equation
\[
\Delta W + |W|^{4/N-2} W = 0.
\]
\( W \geq 0 \) and is radially decreasing. By the invariances of the equation,
\[
W_{\theta_0, \lambda_0, x_0}(x) = e^{i\theta_0} \lambda_0^{N-2/2} W(\lambda_0(x-x_0))
\]
is still a solution. Aubin and Talenti (76) gave the following variational characterization of \( W \): let \( C_N \) be the best constant in the Sobolev embedding \( ||u||_{L^{2^*}} \leq C_N ||\nabla u||_{L^2} \). Then \( ||u||_{L^{2^*}} = C_N ||\nabla u||_{L^2}, u \neq 0 \Leftrightarrow u = W_{(\theta_0, \lambda_0, x_0)} \) for some \( (\theta_0, \lambda_0, x_0) \). Note that by the elliptic equation, \( \int |\nabla W|^2 = \int |W|^2 \). Also \( C_N ||\nabla W|| = ||W||_{L^{2^*}} \) so that \( C_N ||\nabla W||^2 = \left( \int |\nabla W|^2 \right)^{(N-2)/N}. \) Hence, \( \int |\nabla W|^2 = \frac{1}{C_N^2} \). Moreover
\[
E(W) = \left( \frac{1}{2} - \frac{1}{2^*} \right) \int |\nabla W|^2 = \frac{1}{N C_N^2}.
\]

**Lemma.** Assume that \( ||\nabla v|| < ||\nabla W|| \) and that \( E(v) \leq (1 - \delta_0) E(W), \delta_0 > 0 \). Then \( \exists \overline{\delta} = \overline{\delta}(\delta_0, N) \) s.t.
\[
i) \ ||\nabla v||^2 \leq (1 - \overline{\delta}) ||\nabla W||^2
\]
\[
ii) \ f ||\nabla v||^2 - |v|^{2^*} \geq \overline{\delta} ||\nabla v||^2
\]
\[
iii) \ E(v) \geq 0.
\]

**Proof.** Let \( f(y) = \frac{1}{2} y - \frac{C_N^{2^*}}{2^{2*}} y^{2^*/2}, \overline{\gamma} = ||\nabla v||^2. \) Note that \( f(0) = 0, f(y) > 0 \) for \( y \) near 0, \( y > 0 \) and that \( f'(y) = \frac{1}{2} - \frac{C_N^{2^*}}{2^{2*}} y^{2^*/2-1} \), so that \( f'(y) = 0 \) iff \( y = y_0 = \frac{1}{C_N} = ||\nabla W||^2. \) Also, \( f(y_0) = \frac{1}{2} ||\nabla W||^2 = E(W) \). Since \( 0 \leq \overline{\gamma} < y_c, f(\overline{\gamma}) \leq (1 - \delta_0) f(y_c) \) and \( f \) is non negative and strictly increasing between 0 and \( y_c, \) and \( f'(y_c) \neq 0, \) we obtain \( 0 \leq f(\overline{\gamma}), \overline{\gamma} \leq (1 - \overline{\delta}) y_c = (1 - \overline{\delta})||\nabla W||^2. \) This shows (i), (iii). For (ii), note that
\[
\int |\nabla v|^2 - |v|^{2^*} \geq \int |\nabla v|^2 - C_N^{2^*} \left( \int |\nabla v|^2 \right)^{2^*/2}
\]
\[
= \int |\nabla v|^2 \left[ 1 - C_N^{2^*} \left( \int |\nabla v|^2 \right)^{2/(N-2)} \right]
\]
\[
\geq \int |\nabla v|^2 \left[ 1 - C_N^{2^*} (1 - \overline{\delta})^{2/(N-2)} \left( \int |\nabla W|^2 \right)^{2/(N-2)} \right]
\]
\[
= \int |\nabla v|^2 (1 - (1 - \overline{\delta})^{2/(N-2)}]
\]
which gives (iii). \( \square \)

**Remark.** If \( ||\nabla u_0|| < ||\nabla W||, E(u_0) \geq 0. \)

From this static Lemma, we obtain dynamic consequences.
Corollary (Energy trapping). Let \( u \) be a solution of (CP) with maximal interval \( I, \|\nabla u_0\|^2 < \|\nabla W\|^2, E(u_0) < E(W) \). Choose \( \delta_0 > 0 \) s.t. \( E(u_0) \leq (1 - \delta_0)E(W) \). Then, for each \( t \in I \), we have for \( \delta = \delta(\delta_0) \),

i) \( \| \nabla u(t) \|^2 \leq (1 - \delta)\|\nabla W\|^2, E(u(t)) \geq 0 \)

ii) \( \int |\nabla u(t)|^2 - |u(t)|^2 \geq \delta \int |\nabla u(t)|^2 \)

iii) (Coercivity and uniform bound)
\[ E(u(t)) \sim \| \nabla u(t) \|^2 \sim \| \nabla u_0 \|^2, \]
with comparability constants which depend only on \( \delta_0 \).

Proof. From the continuity of the flow, conservation of energy and the previous Lemma.

Remark. Let \( u_0 \in \dot{H}^1, E(u_0) < E(W) \) but \( \|\nabla u_0\|^2 > \|\nabla W\|^2 \). If \( \delta_0 \) is chosen so that \( E(u_0) \leq (1 - \delta_0)E(W) \), we can conclude, in the same way that \( \int |\nabla u(t)|^2 \geq (1 + \delta) \int |\nabla W|^2, t \in I \).

But then, notice that:
\[ \int |\nabla u(t)|^2 - |u(t)|^2 \leq 2E(u_0) - \frac{2}{(N - 2)} \int |\nabla u(t)|^2 \]
\[ \leq 2E(W) - \frac{2}{(N - 2)} \frac{1}{C_N} - \frac{2\delta}{(N - 2)C_N} = -\frac{2\delta}{(N - 2)C_N} < 0.\]

Hence, if \( \int |x|^2 |u_0(x)|^2 dx < \infty \), Glassey’s proof shows that \( I \) cannot be finite. If \( u_0 \) is radial, \( u_0 \in L^2 \), using “local virial identities” one can see that the some holds. We now turn to the next step in the proof:

Concentration - Compactness Procedure

We now turn to the proof of the positive result in Theorem A. Recall that by the coercivity-uniform bound estimate, if \( E(u_0) < E(W), \|\nabla u_0\|^2 < \|\nabla W\|^2 \), if \( \delta_0 \) is s.t. \( E(u_0) \leq (1 - \delta_0)E(W), E(u(t)) \sim \|\nabla u(t)\|^2 \sim \|\nabla u_0\|^2, t \in I \), and that if \( \|\nabla u_0\|^2 = \|\nabla W\|^2 \), we have \( E(u_0) \geq 0 \). It now follows from (CP) that if \( \|\nabla u_0\|^2 < \|\nabla W\|^2 \) and \( E(u_0) \leq \eta_0, \eta_0 \) small, then \( I = (\infty, +\infty) \) and \( u \) scatters. Hence by considering \( G = \{ E : 0 < E < E(W) \text{ if } \|\nabla u_0\|^2 < \|\nabla W\|^2 \} \) and \( E(u_0) < E, \|u\|_{S(I)} < \infty \) and \( E_c = \sup G \), we find \( E_c \) with \( \eta_0 \leq E_c \leq E(W) \) s.t. if \( \|\nabla u_0\|^2 < \|\nabla W\|^2 \) and \( E(u_0) < E_c \), then \( I = (\infty, +\infty) \) and \( u \) scatters, and \( E_c \) is optimal with this property. Theorem A is the assertion \( E_c = E(W) \). Assume then \( E_c < E(W) \) and we will reach a contradiction. Note that if \( 0 \leq E < E_c, \|\nabla u_0\|^2 < \|\nabla W\|^2 \) and \( E(u_0) < E \), then \( \|u\|_{S(I)} < \infty \), while if \( E_c < E < E(W), \exists u_0 \) s.t. \( \|\nabla u_0\|^2 < \|\nabla W\|^2, E_c \leq E(u_0) \leq E < E(W) \) and \( \|u\|_{S(I)} = +\infty \). We will use a concentration-compactness argument to deduce some consequences of this that will eventually lead to a contradiction.

Proposition 1. There exists \( u_{0,c} \in \dot{H}^1, \|\nabla u_{0,c}\|^2 < \|\nabla W\|^2, \text{with } E(u_{0,c}) = E_c(< E(W)) \) s.t. if \( u_c \) is the corresponding solution then \( \|u_c\|_{S(I)} = +\infty \).
Proposition 2. For any $u_c$ as in Proposition 1, with (say) $||u_c||_{S(I^+)} = +\infty$ ($I^+ = I \cap [0, +\infty)$), there exist $x(t) \in \mathbb{R}^N$, $\lambda(t) \in \mathbb{R}^+$, $t \in I^+$ such that

$$K = \left\{ v(x, t) : \frac{1}{\lambda(t)^{(N-2)/2}} u_c \left( \frac{x - x(t)}{\lambda(t)} \right) , t \right\}$$

has compact closure in $\dot{H}^1$.

The proof of Propositions 1 and 2 uses the coercitivity and uniform bound estimates, in conjunction with the “profile decomposition” of Keraani ([20], 2001), which describes the defect of compactness in the estimate

$$||e^{it\Delta} u_0||_S \leq C ||u_0||_{\dot{H}^1},$$

which combines Strichartz (S) (i) with Sobolev (Sob). This is based on the “improved inequality” ($N = 3$)

$$||h||_{L^6(\mathbb{R}^3)} \leq C ||\nabla h||_{L^2(\mathbb{R}^3)}^{1/3} ||\nabla h||_{B^0_{2,\infty}}^{2/3},$$

where $B^0_{2,\infty}$ is the standard Besov space (see [7]).

Theorem (Profile decomposition, Keraani 2001). Let $\{v_{0,n}\} \in \dot{H}^1$, $||v_{0,n}||_{\dot{H}^1} \leq A$, $||e^{it\Delta} v_{0,n}|| \geq \delta > 0$. Then, there exists a subsequence of $\{v_{0,n}\}$ and $\{V_{0,j}\}_{j=1}^{\infty}$ in $\dot{H}^1$ and triples $(\lambda_{j,n}; x_{j,n}; t_{j,n}) \in \mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}$, with

$$\lambda_{j,n} + \lambda_{j',n} + \frac{\lambda_{j,n}}{\lambda_{j',n}} = \frac{\lambda_{j,n}}{\lambda_{j,n}} + \frac{|t_{j,n} - t_{j',n}|}{\lambda_{j,n}} + \frac{|x_{j,n} - x_{j',n}|}{\lambda_{j,n}} \rightarrow \infty, j \neq j'$$

(the triple is orthogonal), s.t.

i) $||V_{0,1}||_{\dot{H}^1} \geq \alpha_0(A) > 0$.

ii) If $V_j^t = e^{it\Delta} V_{0,j}$, then we have, for each $J$

$$v_{0,n} = \sum_{j=1}^{J} \frac{1}{\lambda_{j,n}} V_j^t \left( \frac{x - x_{j,n}}{\lambda_{j,n}} \right) + w_n^J,$$

where $\liminf_{n \rightarrow \infty} ||e^{it\Delta} w_n^J||_S \rightarrow 0$, and for each $J \geq 1$ we have:

(iii a) $||\nabla v_{0,n}||^2 = \sum_{j=1}^{J} ||\nabla V_{0,j}||^2 + ||\nabla w_n^J||^2 + o(1)$, as $n \rightarrow \infty$

and

(iii b) $E(v_{0,n}) = \sum_{j=1}^{J} E(V_j^t(-\frac{x_{j,n}}{\lambda_{j,n}})) + E(w_n^J) + o(1)$

as $n \rightarrow \infty$.

Lecture 2

In order to apply Keraani’s Theorem to our non-linear problem, we need the notion of a “non-linear profile” : let $v_0 \in \dot{H}^1$, $v = e^{it\Delta} v_0$, $\{t_n\}$ a sequence with $\lim_{n \rightarrow \infty} t_n = \bar{t} \in [-\infty, +\infty]$. We say that $u(x, t)$ is a non linear profile associated with $(v_0, \{t_n\})$ if $\exists$ an interval $I$ with $\bar{t} \in I$ (if $\bar{t} = \pm \infty$, $I = [a, +\infty)$, $(-\infty, a]$ respectively) such that $u$ is a solution of (CP) on $I$ and $\lim_{n \rightarrow \infty} ||u(-, t_n) - v(-, t_n)||_{\dot{H}^1} = 0$. There always
exists a non-linear profile: if \( t \in (-\infty, +\infty) \) we solve (CP) with data at \( \bar{t} = v(x, t) \). If \( \bar{t} = +\infty \) (say), we solve integral equation
\[
\int_{\bar{t}}^{\infty} e^{i(t-\bar{t})\Delta} f(u) \, dt,
\]
in \( \mathbb{R}^N \times [t_{n_0}, +\infty) \), where \( n_0 \) is so large that \( \|e^{i\Delta_0}v_0\|_{S(t_{n_0}+\infty)} < \delta \). Then, \( u(t_n) - v(t_n) = \int_{t_n}^{\infty} e^{i(t-t_n)\Delta} f(u) \, dt' \), which \( \to 0 \) in \( \tilde{H}^1 \), since \( \nabla f(u) \in L^2_{(t>t_n)} L^2_{2/N+2} \). It is easy to see that if \( u^{(1)}, u^{(2)} \) are non-linear profiles associated to \( (v_0, \{t_n\}) \) in \( I \ni \bar{t} \), then \( u^{(1)} = u^{(2)} \) on \( I \). Hence, there exists a maximal interval of existence \( I \) for the non-linear profile. Clearly, near finite end points of \( I \), the \( S \) norm is infinite. These concepts are used in the following:

**Proposition 3.** Let \( \{z_{0,n}\} \in \tilde{H}^1 \), \( \|\nabla z_{0,n}\| < \|\nabla W\|^2 \) and \( E(z_{0,n}) \to E_c(<E(W)) \), \( \|e^{i\Delta} z_{0,n}\|_{S(-\infty, +\infty)} \geq \delta > 0 \). Let \( (V_{0,j})_{j=1}^{\infty} \) be as in the profile decomposition. Assume that one of

a) \( \liminf_{n \to -\infty} E(V_{1}^{\ell}(-\frac{t_{1,n}}{\lambda^2_{1,n}})) < E_c \)
or

b) \( \liminf_{n \to -\infty} E(V_{1}^{\ell}(-\frac{t_{1,n}}{\lambda^2_{1,n}})) = E_c \)

and for \( s_n = -\frac{t_{1,n}}{\lambda^2_{1,n}} \), after passing to a subsequence so that \( s_n \to \bar{s} \in [-\infty, +\infty] \)
and \( E(V_{1}^{\ell}(-s_n)) \to E_c \), and if \( U_1 \) is the non-linear profile associated to \( (V_{0,1}, \{s_n\}) \) then \( I = (-\infty, +\infty) \), \( \|U_1\|_{S(-\infty, +\infty)} < \infty \).

Then, (after passing to a subsequence) if \( z_n \) solves \( (CP) \) for \( (z_{0,n}), \|z_n\|_{S(-\infty, +\infty)} < \infty \), for \( n \) large. (In fact it is uniformly bounded in \( n \).)

We will first assume Proposition 3, use it to prove Propositions 1, 2, then prove Proposition 3.

**Proof of Proposition 1.** Find \( u_{0,n} \in \tilde{H}^1 \), \( \|\nabla u_{0,n}\|^2 < \|\nabla W\|^2 \), \( E(u_{0,n}) \to E_c \), \( \|e^{i\Delta} u_{0,n}\|_{S(-\infty, +\infty)} \geq \delta \), \( \|u_{0,n}\|_{S(I_n)} = +\infty \), \( I_n \) a maximal interval. Since \( E_c < E(W) \), for \( n \) large \( E(u_{0,n}) \leq (1-\delta_0)E(W) \). By energy trapping, \( \exists \delta_0 \) s.t. \( \|\nabla u_{0,n}(t)\|^2 < (1-\delta)||\nabla W||^2 \), \( t \in I_n \). Fix \( J \geq 1 \) and apply the profile decomposition to \( \{u_{0,n}\} \). We have
\[
\|\nabla u_{0,n}\|^2 = \sum_{j=1}^{J} \|\nabla V_{0,j}\|^2 + \|\nabla w_j\|^2 + o(1),
\]
\[
E(u_{0,n}) = \sum_{j=1}^{J} E(V_{j}^{\ell}(-\frac{t_{j,n}}{\lambda^2_{j,n}})) + E(w_j^\ell) + o(1).
\]

For \( n \) large, we have, from (1) that \( \|\nabla w_j^\ell\|^2 \leq (1-\frac{\delta}{2})||\nabla W||^2 \) and \( \|\nabla V_{0,j}\|^2 \leq (1-\frac{\delta}{2})||\nabla W||^2 \), \( 1 \leq j \leq J \). Hence, for \( n \) large \( E(w_j^\ell) \geq \bar{\theta} \), \( E(V_{j}^{\ell}(-\frac{t_{j,n}}{\lambda^2_{j,n}})) \geq \bar{\theta} \). Thus, \( E(V_{j}^{\ell}(-\frac{t_{j,n}}{\lambda^2_{j,n}})) \leq E(u_{0,n}) + o(1) \) by (1), so that \( \lim_{n \to -\infty} E(V_{j}^{\ell}(-\frac{t_{j,n}}{\lambda^2_{j,n}})) \leq E_c \). Assume first that we have strict inequality. Then Proposition 3 a) gives a contradiction for large \( n \). Thus, \( \liminf_{n \to -\infty} E(V_{1}^{\ell}(s_n)) = E_c \). Let \( U_1 \) be the non-linear profile associated to

I–10
(V_{0,1}, \{s_n\}). The first observation is that $V_{0,j} = 0$, $j > 1$. Indeed, by (\(\dagger\)) and the facts that $E(u_{0,n}) \to E_c$, $E(V_1^f(s_n)) \to E_c$ (after passing to a subsequence), we see that $E(w_n^J) \to 0$, $E(V_j^f(-\frac{t_{1,n}}{\lambda_{j,n}^2})) \to 0$, $j = 2, \ldots, J$. But then, by coercitivity, we see that

$$\sum_{j=2}^J \|\nabla V_j^f(\frac{-t_{1,n}}{\lambda_{j,n}^2})\|^2 + \|\nabla w_n^J\|^2 \to 0.$$ 

But, $\|\nabla V_j^f(-\frac{t_{1,n}}{\lambda_{j,n}^2})\|^2 = \|\nabla V_0\|^2$, establishing the claim, and in addition showing that $\|\nabla w_n^J\| \to 0$, so that

$$u_{0,n} = \frac{1}{\lambda_{1,n}^{N-2/2}} V_1^f\left(\frac{x-x_{1,n}}{\lambda_{1,n}}, s_n\right) + w_n^J.$$ 

Renormalize, setting $v_0, n = \lambda_{1,n}^{N-2/2} u_{0,n}(\lambda_{1,n}(x+x_{1,n}))$, which has the same properties as $u_{0,n}$, and so that

$$v_{0,n} = V_1^f(s_n) + \tilde{w}_n^J, \quad \|\nabla \tilde{w}_n^J\| \to 0.$$ 

Let $I_1 = \text{max} \text{int} \text{ of } U_1$. By definition of non-linear profile, $E(U_1(s_n)) = E(V_1^f(s_n)) + o(1) = E_c + o(1)$, $\|\nabla U_1(s_n)\|^2 = \|\nabla V_1^f(s_n)\|^2 + o(1) = \|\nabla V_1^f_0\|^2 + o(1) = \|\nabla u_{0,n}\|^2 + o(1) < \|\nabla W\|^2$ for $n$ large. Now fix $\hat{s} \in I_1$, so that $E(U_1(\hat{s})) = E(U_1(s_n)) \to E_c$, so that $E(U_1(\hat{s})) = E_c$. Also, $\|\nabla U_1(s_n)\|^2 < \|\nabla W\|^2$ for $n$ large, so that, by energy trapping, $\|\nabla U_1(\hat{s})\|^2 < 2 \|\nabla W\|^2$. If $\|U_1\|_{S(I_1)} < \infty$, Proposition 3 b) gives a contradiction. Hence $\|U_1\|_{S(I_1)} = +\infty$, we take $u_c = U_1$.

**Proof of Proposition 2.** (by contradiction) Let $u(x, t) = u_c(x, t)$. If not, $\exists \eta_0 > 0$, $\{t_n\}_{n=1}^{\infty}, t_n \geq 0$ s.t. $\forall \lambda_0 \in \mathbb{R}^+, x_0 \in \mathbb{R}^N$ we have (after rescaling)

$$\|\frac{1}{\lambda_0^{N-2/2}} u\left(\frac{x-x_0}{\lambda_0}, t_n\right) - u(x, t_n)\|_{H^1} \geq \eta_0, \quad n \neq n'.$$

After passing to a subsequence, $t_n \to \bar{t} \in [0, T_+(u_0)]$, so that $\bar{t} = T_+(u_0)$ by continuity of the flow. We can also assume, by (CP) that $\|e^{it\Delta} u(t_n)\|_{S(0, +\infty)} \geq \delta$. We now apply the profile decomposition to $v_0, n \to u_{0,n}$. We have $E(u(t)) = E(u_0) = E_c < E(W)$, $\|\nabla u_0\|^2 < \|\nabla W\|^2$, so that $\|\nabla u(t)\|^2 \leq (1 - \bar{\delta}) \|\nabla W\|^2, t \in I_+$. But then $\lim_{n \to \infty} E(V_1^f(\frac{-t_{1,n}}{\lambda_{1,n}^2})) \leq E_c$. If we have strict inequality, Proposition 3 a) gives a contradiction. Hence we have equality and as before $V_{0,j}, j = 2, \ldots, J$, are all 0 and $\|\nabla w_n^J\| \to 0$. Thus, we have

$$u(t_n) = \frac{1}{\lambda_{1,n}^{N-2/2}} V_1^f\left(\frac{x-x_{1,n}}{\lambda_{1,n}}, -\frac{t_{1,n}}{\lambda_{1,n}^2}\right) + w_n^J,$$

$\|w_n^J\|_{H^1} \to 0$. We next claim that $s_n = -\frac{t_{1,n}}{\lambda_{1,n}^2}$ must be bounded. In fact, if $\frac{t_{1,n}}{\lambda_{1,n}^2} \leq -C_0, C_0$ a large positive constant, for $n$ large we have $\|e^{it\Delta} w_n^J\|_{S(-\infty, +\infty)} \leq \frac{\delta}{2}$ and

$$\left\|\frac{1}{\lambda_{1,n}^{N-2/2}} V_1^f\left(\frac{x-x_{1,n}}{\lambda_{1,n}}, \frac{t-t_{1,n}}{\lambda_{1,n}^2}\right)\right\|_{S(0, +\infty)} \leq \|V_1^f\|_{S(C_0, +\infty)} \leq \frac{\delta}{2}$$

for $C_0$ large, a contradiction.
If, on the other hand $t_{1,n}/\lambda_{1,n} > C_0$, $C_0$ large positive, for $n$ large, we have
\[
\left\| \frac{1}{\lambda_{1,n}^{N-2/2}} V_1^f \left( \frac{x - 1_n, t - t_{1,n}}{\lambda_{1,n}^2} \right) \right\|_{L^\infty_{(\infty,0)}} \leq \left\| V_1^f \right\|_{L^\infty_{((-\infty,-C_0))}} \leq \frac{\delta}{2}.
\]
Thus, for $n$ large, $\| e^{t\Delta} u(t_n) \|_{L^\infty_{(\infty,0)}} \leq \delta$, so that (CP) gives $\| u \|_{L^\infty_{(-\infty,t_n)}} \leq 2\delta$. But $t_n \uparrow T_+(u_0)$, a contradiction. Hence $\| \frac{t_{1,n}}{\lambda_{1,n}} \| \leq C_0$, so that, after passing to a subsequence $\frac{t_{1,n}}{\lambda_{1,n}} \to t_0 \in (-\infty, +\infty)$. But then by (ast), $w_n^j \to 0$ gives for $n \neq n'$, both large
\[
\left\| \left( \frac{1}{(\lambda_0)^{-N-2/2}} \frac{1}{(\lambda_{1,n})^{N-2/2}} V_1^f \left( \frac{x - x_0 \lambda_0 - x_1,n}{\lambda_{1,n}}, -\frac{t_{1,n}}{\lambda_{1,n}^2} \right) \right. \right. \left. \left. - \frac{1}{(\lambda_{1,n'})^{N-2/2}} V_1^f \left( \frac{x - x_1,n'}{\lambda_{1,n'}}, -\frac{t_{1,n'}}{\lambda_{1,n'}^2} \right) \right\|_{H^1} \geq \frac{\kappa_0}{2}
\]
for all $\lambda_0, x_0$. After changing variables this gives, for all $\lambda_0, \tilde{x}_0$ that
\[
\left\| \left( \frac{\lambda_{1,n'}}{\lambda_0 \lambda_{1,n'}} \right)^{(N-2/2)} V_1^f \left( \frac{\lambda_1,n y}{\lambda_0 \lambda_{1,n'}}, x_{n,n'} - \tilde{x}_0, -\frac{t_{1,n}}{\lambda_{1,n}} \right) - V_1^f \left( y, \frac{t_{1,n'}}{\lambda_{1,n'}} \right) \right\|_{H^1} \geq \frac{\kappa_0}{2}.
\]
Choosing now $\lambda_0, \tilde{x}_0$ suitably this is a contradiction since $\frac{t_{1,n}}{\lambda_{1,n}} \to t_0, \frac{t_{1,n}}{\lambda_{1,n}} \to t_0$. 

Proof of Proposition 3. Assume first that $\lim inf E(V_1^f(-\frac{t_{1,n}}{\lambda_{1,n}})) = E_c$. Fix $J \geq 1$ and note that as in the proof of Proposition 1, we have $V_{0,j} = 0$, $j > 1$, and $||\nabla w_n^j|| \to 0$. Moreover, if $w_0, n = \lambda_n^{N-2/2} z_0, n(\lambda_n(x + x_1,n))$, $\tilde{w}_n = \lambda_n^{-2/2} w_n^j(\lambda_n(x + x_1,n))$, we have $||\nabla \tilde{w}_n|| \to 0$, $v_0, n = V_1^f(s_0) + \tilde{w}_n$, with $||\nabla v_0, n||^2 < ||\nabla W||^2$, $E(v_0, n) \to E_c < E(W)$. By definition of the non-linear profile, $||\nabla (V_1^f(s_0) - U_1(s_0))|| \to 0$, so that $v_0, n = U_1(s_0) + \tilde{w}_n$, $||\nabla \tilde{w}_n|| \to 0$. From this we see that $E(U_1) = E_c < E(W)$ and so, by energy-trapping sup $||\nabla U_1(t)||^2 < ||\nabla W||^2$, since $||\nabla \tilde{w}_n|| \to 0$ the Perturbation Theorem gives this case, under assertion b). Assume next that $\lim inf E(V_1^f(-\frac{t_{1,n}}{\lambda_{1,n}})) < E_c$ and passing to a subsequence that $\lim E(V_1^f(-\frac{t_{1,n}}{\lambda_{1,n}})) < E_c$. We next show that $\lim inf E(V_1^f(-\frac{t_{1,n}}{\lambda_{1,n}})) < E_c$, $j = 2, \cdots, J$. In fact, $||\nabla z_0,n||^2 = J \sum_{j=1}^J ||\nabla V_0^j||^2 + ||\nabla w_n^j||^2 + o(1)$ and since $E(z_0,n) \to E_c < E(W)$, for $n$ large $E(z_0,n) \leq (1 - \delta_0) E(W)$. Since $||\nabla z_0,n||^2 < ||\nabla W||^2$, energy trapping gives that $||\nabla z_0,n||^2 \leq (1 - \bar{\delta})||\nabla W||^2$. Thus, for all $n$ large $E(V_1^f(-\frac{t_{1,n}}{\lambda_{1,n}})) \geq 0$, $E(w_n^j) \geq 0$. Coercitivity shows that $E(V_1^f(-s_0)) \geq C \alpha_0 = \bar{\alpha}_0 > 0$, for $n$ large. Then, $E(z_0,n) \geq \bar{\alpha}_0 + J \sum_{j=1}^J E(V_1^f(-\frac{t_{1,n}}{\lambda_{1,n}})) + o(1)$, so our claim follows from $E(z_0,n) \to E_c$. Next, note that if $U_j$ is the non-linear profile associated to $(V_0, j, -\frac{t_{1,n}}{\lambda_{1,n}})$ (after passing to a subsequence in $n$) then $U_j$ exists for all time and $||U_j||_{L^\infty_{(-\infty, \infty)}} < \infty$, $1 \leq j \leq J$. In fact, for $n$ large, $E(U_j(\frac{t_{1,n}}{\lambda_{1,n}})) \to E_c$, so $E(U_j) < E_c$ by definition of non-linear profile. Moreover, $||\nabla V_j^f(-\frac{t_{1,n}}{\lambda_{1,n}})||^2 \leq ||\nabla z_0,n||^2 + o(1) \leq (1 - \bar{\delta})||\nabla W||^2 + o(1)$, so by energy trapping we have $||\nabla U_j(t)||^2 < ||\nabla W||$, $\forall t \in I_j$. But then, by definition of $E_c$,
\[ I_j = (-\infty, +\infty), \quad ||U_j||_{S(-\infty, +\infty)} < \infty. \] Next, note that there exists \( j_0 \) such that for \( j \geq j_0 \) we have
\[ ||U_j||^2_{S(-\infty, +\infty)} \leq C||\nabla V_{0,j}||^2. \]
In fact, \( J \) fixed, choosing \( n \) large we have
\[ \sum_{j=1}^{J} ||\nabla V_{0,j}||^2 \leq ||\nabla z_{0,n}||^2 + o(1) \leq 2||\nabla W||^2. \]
Hence, for \( j \geq j_0, \quad ||\nabla V_{0,j}|| \leq \delta, \delta \) so small that \( ||e^{t\Delta} V_{0,j}||_{S(-\infty, +\infty)} \leq 2\delta, \sup \||U_j(t)||_{H^1} + ||\nabla U_j||_{L^2(-\infty, +\infty)} \leq C||V_{0,j}||_{H^1}. \] But then, \( ||U_j||_{S(-\infty, +\infty)} \leq C||V_{0,j}||_{H^1} \) as desired. Next, for \( \varepsilon > 0 \), to be chosen, define
\[ H_{n,\varepsilon 0} = \sum_{j=1}^{J_{(\varepsilon 0)}} \frac{1}{\lambda_{j,n}^{(N-2)/2}} U_j \left( \frac{x - x_{j,n}}{\lambda_{j,n}}, \frac{t - t_{j,n}}{\lambda_{j,n}^2} \right). \]
Then \( ||H_{n,\varepsilon 0}||_{S(-\infty, +\infty)} \leq C_0 \) uniformly in \( \varepsilon _0 \), for \( n \geq n(\varepsilon 0) \):
\[ ||H_{n,\varepsilon 0}||_{S(-\infty, +\infty)} \leq \int \left[ \sum_{j=1}^{J_{(\varepsilon 0)}} \frac{1}{\lambda_{j,n}^{(N-2)/2}} U_j \left( \frac{x - x_{j,n}}{\lambda_{j,n}}, \frac{t - t_{j,n}}{\lambda_{j,n}^2} \right) \right]^{2(N+2)/(N-2)} \]
\[ \leq \sum_{j=1}^{J_{(\varepsilon 0)}} \int \left| \frac{1}{\lambda_{j,n}^{(N-2)/2}} U_j \left( \frac{x - x_{j,n}}{\lambda_{j,n}}, \frac{t - t_{j,n}}{\lambda_{j,n}^2} \right) \right|^{2(N+2)/(N-2)} \]
\[ + C_{j_{(\varepsilon 0)}} \sum_{j \neq j'} \int \left| \frac{1}{\lambda_{j,n}^{(N-2)/2}} U_j \left( \frac{x - x_{j,n}}{\lambda_{j,n}}, \frac{t - t_{j,n}}{\lambda_{j,n}^2} \right) \right|^{(N+6)/(N-2)} = I + II. \]
For \( n \) large, \( II \to 0 \) by orthogonality of \((\lambda_{j,n}, x_{j,n}, t_{j,n})\). Thus, for \( n \) large \( II \leq I \). But
\[ I \leq \sum_{j=1}^{j_0} ||U_j||_{S(-\infty, +\infty)}^{2(N+2)/(N-2)} + \sum_{j=j_0+1}^{J_{(\varepsilon 0)}} ||U_j||_{S(-\infty, +\infty)}^{2(N+2)/(N-2)} \]
\[ \leq \sum_{j=1}^{j_0} ||U_j||_{S(-\infty, +\infty)}^{2(N+2)/(N-2)} + C \sum_{j=j_0+1}^{J_{(\varepsilon 0)}} ||\nabla V_{0,j}||^{2(N+2)/(N-2)} \]
\[ \leq \sum_{j=1}^{j_0} ||U_j||_{S(-\infty, +\infty)}^{2(N+2)/(N-2)} + C \sup_{j > j_0} ||\nabla V_{0,j}||^{(2(N+2)/(N-2))} \sum_{j > j_0} ||\nabla V_{0,j}||^2 \leq \frac{C_0}{2}. \]
Define now \( R_{n,\varepsilon 0} = ||H_{n,\varepsilon 0}||_{4/(N-2)}^{4/(N-2)} H_{n,\varepsilon 0} - \sum_{j=1}^{J_{(\varepsilon 0)}} ||U_{j,n}||^{4/(N-2)} U_{j,n} \), where
\[ U_{j,n}(x,t) = \frac{1}{\lambda_{j,n}^{(N-2)/2}} U_j \left( \frac{x - x_{j,n}}{\lambda_{j,n}}, \frac{t - t_{j,n}}{\lambda_{j,n}^2} \right). \]
We then have \( ||\nabla R_{n,\varepsilon 0}||_{L^2_{t,x} L^2_{t,x}} \to 0 \) as \( n \to +\infty \). This uses orthogonality, \( ||U_j||_{S(-\infty, +\infty)} < \infty, \quad ||\nabla U_j||_{L^2(-\infty, +\infty)} < \infty \). Let now \( \tilde{u} = H_{n,\varepsilon 0}, \quad e = R_{n,\varepsilon 0} \). Choose now \( J_{(\varepsilon 0)} \) so large that for \( n \) large \( ||e^{t\Delta} \tilde{u}_{j_{(\varepsilon 0)}}||_{S(-\infty, +\infty)} \leq \frac{\varepsilon _0}{2} \). Note that by the profile decomposition and the definition of non-linear profile, we have, for \( n \) large
\[ z_{0,n} = H_{n,\varepsilon 0}(0) + \tilde{w}_{j_{(\varepsilon 0)}} \]
where \( |e^{it\Delta} \hat{w}_n^{(\epsilon_0)}|_S \leq \epsilon_0 \). Also, arguments as above show also that \( \sup_{t} ||\nabla H_{n,\epsilon_0}(t)|| \leq \tilde{C}_0 \) uniformly in \( \epsilon_0 \), for \( n \) large and \( ||\nabla \hat{w}_n^{(\epsilon_0)}|| \leq 2||\nabla W|| \). Now choose \( \epsilon_0 < \epsilon_0(C_0, \tilde{C}_0, 2||\nabla W||) \) as in Perturbation Theorem, and \( n \) so large that

\[
||\nabla R_{n,\epsilon_0}||_{L^2_{\omega} S^{N-2} L^2} \leq \epsilon_0.
\]

Then the Perturbation Theorem gives us Proposition 3 a).

An important Corollary of the above arguments is (Keraani [20], 2001, Bahouri-Gérard [1],1999).

**Lemma.** There exists a function \( g : (0, E_c] \to [0, +\infty) \), \( g \downarrow \) s.t. \( \forall u_0 \) with \( ||\nabla u_0||^2 < ||\nabla W||^2 \) and \( E(u_0) \leq E_c - \eta \), then \( ||u||_{S(\infty, +\infty)} \leq g(\eta) \).

**Proof.** If not, \( \exists \eta_0 > 0 \) and a sequence \( u_{0,n} \) s.t. \( ||\nabla u_{0,n}||^2 < ||\nabla W||^2 \), \( E(u_{0,n}) \leq E_c - \eta_0 \) and \( ||u_{n}||_{S(\infty, +\infty)} \to +\infty \). For \( n \) large we must have \( ||e^{it\Delta} u_{0,n}||_{S(\infty, +\infty)} \geq \delta \). But if we now apply the proof of Proposition 3, case a), we reach a contradiction. \( \square \)

**Remark.** In the profile decomposition, if all the \( v_{0,n} \) are radial the \( V_{0,j} \) can be chosen radial and \( x_{n,j} = 0 \). We can then repeat our procedure restricted to radial function and conclude the analogs of Propositions 1, 2 with \( u_c \) radial, \( x(t) \equiv 0 \).

**Remark.** Because of the continuity of \( u(t) \), \( t \in I \) in \( \hat{H}^1 \), in Proposition 2 we can construct \( \lambda(t), x(t) \) continuous in \( [0, T_+(u_0)) \), with \( \lambda(t) > 0 \) for each \( t \in [0, T_+(u_0)) \).

To see this, first one can construct piecewise constant (with small jump) \( \lambda_1(t) \), \( x_1(t) \) so that the corresponding set \( K_1 \) is contained in \( \tilde{K}_1 \) = \{ solution of (CP) with initial data in \( K, t \in [0, t_0] \) \}. It is clear that \( \tilde{K}_1 \) is compact. We can then construct continuous \( \lambda_2(t), x_2(t) \) s.t. \( K_2 \) is contained in the precompact set \{ \( \lambda_0^{(N-2)/2} w((x - x_0)\lambda_0^{-1}), w \in \tilde{K}_1, \frac{1}{2} \leq \lambda_0 \leq 2, |x_0| \leq 1 \} \).

We now turn to further properties of critical elements.

**Lemma.** Let \( u_c \) be as in Proposition 2, with \( T_+(u_0) < \infty \). (After scaling we can assume \( T_+(u_0) = 1 \). Then \( \exists C_0 = C_0(K) > 0 \) s.t. \( \lambda(t) \geq C_0(K)^{1/2}/(1-t)^{1/2} \).

**Proof.** Consider \( t_n \uparrow 1, u_{0,n} = \frac{1}{\lambda(t_n)^{N-2}} u\left(\frac{\lambda(t_n)}{\lambda(t_n)^{N-2}}, t_n\right) \). We know that \( \exists C_0 = C_0(\tilde{K}) \) s.t. \( T_+(u_{0,n}) \geq C_0 \). Note that \( u(x, t_n) = \lambda(t_n)^{N-2} u_{0,n}(\lambda(t_n) x + x(t_n)) \), hence by uniqueness in (CP), for \( t_n + t < T_+(u_0) = 1 \), we have

\[
  u(x, t+n) = \lambda(t_n)^{N-2} u_{0,n}(\lambda(t_n) x + x(t_n), \lambda^2(t_n)t).
\]

Hence, \( t_n + t \leq 1 \) for all \( 0 < \lambda^2(t_n)t \leq C_0 \). With \( t = \frac{C_0}{\lambda^2(t_n)} \), we get \( t_n + \frac{C_0}{\lambda^2(t_n)} \leq 1 \) or \( \lambda^2(t_n) \geq \frac{C_0}{1-t_n} \) as desired. \( \square \)

**Lemma.** Let \( u_c \) be a critical element as in Proposition 2, with \( T_+(u_0) = +\infty \). Then, there is a (possibly different) critical element \( v_c \), with a corresponding \( \hat{\lambda} \), and \( A_0 > 0 \), with \( \hat{\lambda}(t) \geq A_0 > 0 \), for \( t \in [0, T_+(v_c)) \).

**Proof.** Recall that \( E(u_c) = E_c \geq \gamma_0 \). By a previous remark, \( \exists t_n, t_n \uparrow +\infty \) s.t. \( \lambda(t_n) \to 0 \), or the result holds for \( u_c \). After possibly redefining \( \{t_n\} \), we can assume that \( \lambda(t_n) \leq \inf_{[0,t_n]} \lambda(t) \). By compactness of \( \tilde{K}, w_{0,n}(x) = \frac{1}{\lambda(t_n)^{N-2}} u_c\left(\frac{\lambda(t_n)}{\lambda(t_n)^{N-2}}, t_n\right) \to w_0 \).
in $\hat{H}^1$. Hence, $E(w_0) = E_c \geq \eta_0 > 0$. Moreover, $||\nabla w_0||^2 < ||\nabla W||^2$ by the corresponding property of $u_c$ and energy trapping ($E_c < E(W)$). Let $w(x, \tau), \tau \in (-T_-(w_0), 0]$ be the corresponding solution of (CP). If $T_-(w_0) < \infty$, we let $v_\tau(x, t) = \bar{w}(x, -t)$ and Proposition 2, and the previous lemma, give the result. If $T_-(w_0) = +\infty$, let $w_n(x, \tau)$ be the solution of (CP) with data $w_{0,n}, \tau \in (-T_-(w_{0,n}), 0]$. By semicontinuity we have $\liminf T_-(w_{0,n}) = +\infty$ and for every $\tau \in (-\infty, 0], w_n(x, \tau) \rightarrow w(x, \tau)$ in $\hat{H}^1$. By uniqueness in (CP), for $0 \leq t_n + \frac{\tau}{\lambda(t_n)}$, we have $w_n(x, \tau) = \frac{1}{\lambda(t_n)(N-2)/2} u_c\left(\frac{x - x(t_n)}{\lambda(t_n)}, t_n + \frac{\tau}{\lambda(t_n)^2}\right)$. Define now $\tau_n = -\lambda(t_n)^2 t_n$. Note that $\lim \inf_n (\tau_n) = \lim \inf_n (\lambda(t_n)^2 t_n) = +\infty$. In fact, if $\tau_n \rightarrow -\tau_0 < \infty$, $w_n(x, -\tau_n) = \frac{1}{\lambda(t_n)(N-2)/2} u_c\left(\frac{x - x(t_n)}{\lambda(t_n)}, 0\right)$ would converge to $w(x, -\tau_0)$ in $\hat{H}^1$, with $\lambda(t_n) \rightarrow 0$, a contradiction to $E(w_0) \neq 0$, so $w_0 \neq 0$. Hence, for all $\tau \in (-\infty, 0], n$ large we have $0 \leq t_n + \frac{\tau}{\lambda(t_n)} \leq t_n$. Note also that we must have $||w||_{S(-\infty, 0)} = +\infty$. Otherwise, by The Perturbation Theorem we would have, for $n$ large, $T_-(w_{0,n}) = +\infty, ||w_n||_{S(0, \infty)} \leq M$, which contradicts $||u_c||_{S(0, +\infty)} = +\infty$. Fix $\tau \in (-\infty, 0], n$ so large that $t_n + \frac{\tau}{\lambda(t_n)} \geq 0$ and $\lambda(t_n + \frac{\tau}{\lambda(t_n)})$ is defined. Then,

$$\frac{1}{\lambda(t_n + \frac{\tau}{\lambda(t_n)})^{N-2}/2} u_c\left(\frac{x - x(t_n)}{\lambda(t_n + \frac{\tau}{\lambda(t_n)}), t_n + \frac{\tau}{\lambda(t_n)}}, \frac{\lambda(t_n + \frac{\tau}{\lambda(t_n)})}{\lambda(t_n)}\right) \in K,$$

with

$$\lambda(t_n + \frac{\tau}{\lambda(t_n)}) \geq 1, \quad x(t_n) = x(t_n + \frac{\tau}{\lambda(t_n)}), \quad \lambda(t_n + \frac{\tau}{\lambda(t_n)}) = \lambda(t_n),$$

Since $\frac{1}{\lambda(t_n)^{N-2}/2} \nabla u_c\left(\frac{x - x(t_n)}{\lambda(t_n)}, t_n\right) = \nabla \bar{v} \text{ in } L^2$ with either $\lambda_n \rightarrow 0$ or $\infty$ or $|x_n| \rightarrow \infty$ implies that $\bar{v} \equiv 0$, we can assume, after passing to a subsequence that $\lambda_n(\tau) \rightarrow \tilde{\lambda}(\tau), 1 \leq \tilde{\lambda}(\tau) < \infty \tilde{x}_n(\tau) \rightarrow \tilde{x}(\tau) \in \mathbb{R}^N$. But then, $\frac{1}{\lambda(t_n)^{N-2}/2} w\left(\frac{x - \tilde{x}_n(\tau)}{\tilde{\lambda}(\tau)}, \tau\right) \in \hat{K}$ as desired. $
$

We now conclude the proof of Theorem A, by establishing:

**Theorem (Rigidity Theorem).** Let $u_0 \in H^1, E(u_0) < E(W), ||\nabla u_0||^2 < ||\nabla W||^2$. Let $u$ be the corresponding solution of (CP) with maximal interval

$$I = (-T_-(u_0), T_+(u_0)).$$

Assume $\exists \lambda(t) > 0$, defined for $t \in [0, T_+(u_0))$ s.t.

$$K = \left\{v(x, t) = \frac{1}{\lambda(t)^{N-2}/2} u\left(\frac{x}{\lambda(t)}, t\right), t \in [0, T_+(u_0))\right\}$$

has compact closure in $\hat{H}^1$. Assume that if $T_+(u_0) < \infty, \lambda(t) \geq C_0(K) t^{1/2}$ and if $T_+(u_0) = +\infty, \lambda(t) \geq A_0 > 0$. Then we must have $T_+(u_0) = +\infty, u_0 \equiv 0$.

**Proof. Case 1 :** $T_+(u_0) = +\infty$ so that $\lambda(t) \rightarrow +\infty$ as $t \rightarrow T_+(u_0)$. Fix $\varphi$ radial, $\varphi \in C_0^\infty, \varphi \equiv 1$ on $|x| \leq 1, \text{supp} \varphi \subset \{|x| < 2\}$ set $\varphi_R(x) = \varphi\left(\frac{x}{R}\right)$.

Define $y_R(t) = \int |u(x, t)|^2 \varphi_R(x) dx, t \in [0, T_+).$ A classical computation shows that

$$y'_R(t) = 2 \text{Im} \int \bar{v} \nabla u \nabla \varphi_R.$$
Note that $\nabla \varphi_R = \frac{1}{R} \nabla \varphi(\frac{x}{R})$ is supported in $R \leq |x| \leq 2R$. Then,

$$|y_R'(t)| \leq \tilde{C}_N \left( \int |\nabla u|^2 \right)^{1/2} \left( \int \frac{|u|^2}{|x|^2} \right)^{1/2} \leq \tilde{C}_N \left( \int |\nabla u|^2 \right) \leq \tilde{C}_N \|\nabla W\|
$$

where we have used Hardy’s inequality and energy trapping. Next, we will show that, for all $R > 0$,

$$\lim_{t \uparrow T_+ (u_0)} \int |u(x, t)|^2 \, dx = 0.
$$

In fact, $u(x, t) = \lambda(t)^{N-2} v(\lambda(t) x, t)$, where $v$ is compact. Then,

$$\int_{|x| < R} |u(x, t)|^2 \, dx = \lambda(t)^{-2} \int_{|y| < R\lambda(t)} |v(y, t)|^2 \, dy
$$

$$= \lambda(t)^{-2} \int_{|y| < \varepsilon R\lambda(t)} |v(y, t)|^2 \, dy + \lambda(t)^{-2} \int_{\varepsilon R\lambda(t) \leq |y| \leq R\lambda(t)} |v(y, t)|^2 \, dy
$$

$$= A + B,
$$

where $\varepsilon > 0$ is at our disposal. By Hölder, we have

$$A \leq \lambda(t)^{-2} (\varepsilon R\lambda(t))^{N-2/N} \|v(t)\|_{L^{2^*}}^2 \leq C \varepsilon^2 R^2 \|\nabla W\|^2
$$

which, for fixed $R$ is small with $\varepsilon$

$$B \leq \lambda(t)^{-2} (R\lambda(t))^{N-2/N} \|v(t)\|_{L^{2^*}(|y| \geq \varepsilon R\lambda(t))} \xrightarrow{t \uparrow T_+} 0
$$

by the compactness of $v$, since $\lambda(t) \uparrow +\infty$.

Now, using that $|y_R'(t)| \leq C$ and the fundamental theorem of calculus, we have

$$y_R(0) \leq \lim_{t \uparrow T_+} y_R(t) + C T_+ (u_0) = C T_+ (u_0).
$$

Letting $R \to \infty$, we conclude that $u_0 \in L^2$. Fix now $\varepsilon > 0$ and choose $\alpha$ so small that

$$\int_{T_+(u_0) - \alpha}^{T_+(u_0)} |y_R'| \leq C \alpha \leq \frac{\varepsilon}{2}
$$

for all $R > 0$. By invariance of the $L^2$ norm (and this is a fundamental point here), we have:

$$\|u_0\|_2^2 = \|u(T_+(u_0) - \alpha)\|_2^2.
$$

For $\alpha$ fixed as above, choose $R$ so large that

$$\|u(T_+(u_0) - \alpha)\|_2^2 \leq \|u(T_+(u_0) - \alpha)\|_{L^2(|x| < R)}^2 + \frac{\varepsilon}{2}.
$$

We then have

$$\|u_0\|_2^2 \leq y_R(T_+(u_0) - \alpha) + \frac{\varepsilon}{2} \leq \lim_{t \uparrow T_+} - \int_{T_+(u_0) - \alpha}^{T_+(u_0)} y_R' + \frac{\varepsilon}{2} \leq \varepsilon.
$$

Since this is true for each $\varepsilon > 0$, $\|u_0\|_2 = 0$, which contradicts $T_+ < \infty$. This ends the proof in Case 1. \qed

I–16
Lecture 3

To conclude the proof of Theorem A, we need to treat the

Proof. Case 2 : \( T_+(u_0) = +\infty, \lambda(t) \geq A_0 > 0 \).

Note first that the compactness of \( \tilde{K} \), together with \( \lambda(t) \geq A_0 > 0 \) gives that \( \forall \varepsilon > 0, \exists R(\varepsilon) > 0 \) s.t. \( \forall t \in [0, \infty) \), we have

\[
\int_{|x| > R(\varepsilon)} |\nabla u|^2 + |u|^2 + \frac{|u|^2}{|x|^2} \leq \varepsilon.
\]

In fact, since \( u(x, t) = \lambda(t)^{-2/3} v(\lambda(t)x, t) \) a change of variables shows that the integral equals

\[
\int_{|y| > R(\varepsilon)(\lambda)} |\nabla v|^2 + |v|^2 + \frac{|v|^2}{|y|^2} \leq \int_{|y| > A_0 R(\varepsilon)} \leq \varepsilon
\]

for \( R(\varepsilon) \) large by the compactness of \( \tilde{K} \).

To continue with the proof, pick \( \delta_0 \) s.t. \( E(u_0) < (1 - \delta_0)E(W) \). Then, \( \exists R_0 > 0 \) s.t. for \( R > R_0 \),\( t \in [0, \infty) \) we have (if \( ||\nabla u_0|| \neq 0 \))

\[
\int_{|x| < R} |\nabla u|^2 - |u|^2 \geq C_{\delta_0}||\nabla u_0||^2.
\]

In fact, by our coercitivity estimate we have, for all \( t \in [0, \infty) \), \( \int |\nabla u|^2 - |u|^2 \geq C_{\delta_0}||\nabla u_0||^2 \), but, by the first claim, we can make the tails smaller than \( \frac{C_{\delta_0}}{2}||\nabla u_0||^2 \).

Next, choose \( \psi \in C_0^\infty \), radial, with \( \psi(x) = |x|^2 \) for \( |x| \leq 1 \), \( \psi(x) \equiv 0 \) for \( |x| \geq 2 \). Define

\[
z_R(t) = \int |u(x, t)|^2 R^2 \psi\left(\frac{x}{R}\right) dx.
\]

The computations that we used in Glassey’s blow-up proof to yield the “virial identity” now give:

\[
z'_R(t) = 2R \text{Im} \int \bar{u} \nabla u \nabla \psi\left(\frac{x}{R}\right),
\]

\[
z''_R(t) = 4 \sum_{\ell,j} \text{Re} \int \partial_{x_{\ell}} u(x) \partial_{x_{j}} \psi\left(\frac{x}{R}\right) \cdot \partial_{x_{\ell}} u \partial_{x_{j}} \bar{u} - \frac{1}{R^2} \int \Delta^2 \psi\left(\frac{x}{R}\right) |u|^2 - \frac{4}{n} \int \Delta \psi\left(\frac{x}{R}\right) |u|^2.
\]

From these formulas, we deduce:

\[
|z'_R(t)| \leq C R \int_{|x| \leq 2R} |u| |\nabla u| \leq C R^2 \left( \int_{|x| \leq 2R} \frac{|u|^2}{|x|^2} \right)^{1/2} \left( \int |\nabla u|^2 \right)^{1/2}
\]

\[
\leq C R^2 \int |\nabla u|^2 \leq C_{\delta_0} R^2 ||\nabla W||^2.
\]

On the other hand,

\[
z''_R(t) \geq 8 \left[ \int_{|x| \leq R} |\nabla u|^2 - |u|^2 \right] - \tilde{C}_N \left[ \int_{R \leq |x| \leq 2R} |\nabla u|^2 + \frac{|u|^2}{|x|^2} + |u|^2 \right],
\]

which, for \( R \) large is bounded below by \( \tilde{C}_{\delta_0,N} ||\nabla u_0||^2 \). Integrating in \( t \), we obtain

\[
z_R(t) - z_R(0) \geq \tilde{C}_{\delta_0,N} t ||\nabla u_0||^2,
\]

I-17
In the defocusing case, for $N = 3, 4, 5$, this approach (in a simplified form since the variational estimates are not needed) provides an alternative proof of the result of Bourgain, Tao for radial functions in the defocusing case.

**Remark.** In the defocusing case, for $N = 3, 4, 5$, this approach (in a simplified form since the variational estimates are not needed) provides an alternative proof of the result of Bourgain, Tao for radial functions in the defocusing case.

**Corollary (focusing case).** $u_0 \in \dot{H}^1$, radial, $E(u_0) < E(W)$, $||\nabla u_0||^2 < ||\nabla W||^2$, $N = 3, 4, 5$. Then $I = (−\infty, +\infty)$, $||u||_{S(−\infty, +\infty)} < \infty$, $\exists u_0^\pm \in \dot{H}^1$ s.t. $||u(t) - e^{it\Delta}u_0^\pm||_{H^1} \xrightarrow{t \to \pm\infty} 0$. Also, if $E(u_0) \leq (1 - \delta_0)E(W)$, $||u||_{S(−\infty, +\infty)} \leq g(\delta_0)$.

**Remark.** The result admits the following strengthening: if $u_0 \in \dot{H}^1$ is s.t. $\forall t \in (−T_-(u_0), T_+(u_0))$ we have $||\nabla u(t)||^2 \leq ||\nabla W||^2 - \delta_0$, for some $\delta_0 > 0$, then $I = (−\infty, +\infty)$ and $||u||_{S(−\infty, +\infty)} < \infty$. For a detailed proof, see the arguments in [18].

This remark and our Theorem A have consequences for the concentration of finite time blow-up solutions (see [17] for the details of the proof):

**Corollary.** Let $u_0 \in \dot{H}^1$ be radial (no size restriction). Assume $T_+(u_0) < \infty$ and $\sup_{t \in [0, T_+(u_0))] ||\nabla u(t)|| < \infty$ (type II blow-up). Then, for all $R > 0$ we have:

$$\limsup_{t \to T_+(u_0)} \int_{|x| \leq R} |\nabla u(t)|^2 \geq \int |\nabla W|^2,$$

$N = 3, 4, 5$.

**Remark.** For $N \geq 4$, $u_0$ radial, $T_+(u_0) < \infty$, $u$ not a finite blow-up solution of type II, one can show that if $\int |\nabla u(t_n)|^2 \to +\infty$, then $\forall R > 0$, $\int_{|x| \leq R} |\nabla u(t_n)|^2 \to +\infty$. For $N = 3$ this is likely false, in light of examples like those of P. Raphael [24] for $N = 2$, which should give a radial solution, blowing-up on a sphere.

We now turn our attention to the non-linear wave equation (NLW).

$$\begin{cases}
\partial_t^2 u - \Delta u = \pm |u|^{4/N-2}u, & x \in \mathbb{R}^N, \ t \in \mathbb{R} \\
u|_{t=0} = u_0 \in \dot{H}^1 \\
\partial_t u|_{t=0} = u_1 \in L^2.
\end{cases}$$

Here the $-$ sign corresponds to the defocusing case, the $+$ sign to the focusing case. The problem is energy critical because if $u(x, t)$ is a solution, $\lambda > 0$, then $u_\lambda(x, t) = \lambda^{1/N} u(\lambda^{1/N} x, \lambda^{-1} t)$ is also a solution and the norm in $\dot{H}^1 \times L^2$ of the initial data remains unchanged.

The defocusing case has been studied for many years, going back to work of Struwe (radial)[29], Grillakis (general)[12], Shatah-Struwe [27, 26], Bahouri-Shatah [2], Kapitansky [14], Bahouri-Gérard [1], Ginibre-Velo [10], Ginibre-Soffer-Velo [9], etc. (mid to late 80’s, mid 90’s). The energy here is

$$E((u_0, u_1)) = \frac{1}{2} \int |\nabla u_0|^2 + \frac{1}{2} \int (u_1)^2 + \frac{1}{2^*} \int |u_0|^{2^*},$$

which is constant in time, with $-$ in the focusing case, and $+$ in the defocusing case, $\frac{1}{2^*} = \frac{1}{2} - \frac{1}{N}$. In the defocusing case, Shatah-Struwe and Bahouri-Shatah showed that for any data $(u_0, u_1) \in \dot{H}^1 \times L^2$ we have global well-posedness and scattering in the energy space. In the focusing case, this does not hold. In 1974, H. Levine [22]
showed (by obstruction) that if \((u_0, u_1) \in \dot{H}^1 \times L^2\), \(u_0 \in L^2\), \(E((u_0, u_1)) < 0\), there is always break-down in finite time. Very recently (2007) Krieger-Schlag-Tataru [21] have constructed explicit radial examples which break-down in finite time. Also, 
\[ W(x) = (1 + \frac{|x|^2}{N(N-2)})^{-(N-2)/2} \] solves the elliptic equation \(\Delta W + |W|^{4/N-2}W = 0\), \(W \in \dot{H}^1\), is radial and so \(W(x, t) = W(x)\) solves (CP) with data \((W, 0)\), globally in time, but does not scatter. We now turn to our study in the focusing case.

**Theorem B.** Assume that \(E((u_0, u_1)) < E((W, 0))\).

i) If \(\int |\nabla u_0|^2 < \int |\nabla W|^2\), we have g.w.p., scattering.

ii) If \(\int |\nabla u_0|^2 > \int |\nabla W|^2\), there is break-down in finite time.

The condition \(\int |\nabla u_0|^2 = \int |\nabla W|^2\) is not compatible with \(E((u_0, u_1)) < E((W, 0))\). Note that no radial assumption is made in the Theorem, which has been proved for \(3 \leq N \leq 5\).

The general scheme of the proof follows the approach we described for (NLS). To describe the proof, I will start out by a review of the local Cauchy problem. Consider first the linear wave equation

\[
\begin{align*}
\partial_t^2 w - \Delta w &= h \text{ in } \mathbb{R}^N \times \mathbb{R} \\
|w|_{t=0} &= w_0 \in \dot{H}^1(\mathbb{R}^N) \\
\partial_t w|_{t=0} &= w_1 \in L^2(\mathbb{R}^N)
\end{align*}
\]

whose solution is given by

\[
w(x, t) = \cos(t\sqrt{-\Delta})w_0 + (-\Delta)^{-1/2} \sin(t\sqrt{-\Delta})w_1 + \int_0^t \frac{\sin((t-t')\sqrt{-\Delta})}{\sqrt{-\Delta}} h(t') \, dt'.
\]

Let \(S(t)(w_0, w_1) = \cos(t\sqrt{-\Delta})w_0 + (-\Delta)^{-1/2} \sin(t\sqrt{-\Delta})w_1\). The relevant Strichartz estimates for us are:

\[
\sup_t \left\| (w(t), \partial_t w(t)) \right\|_{\dot{H}^1 \times L^2} + \int_0^t \left\| D^{1/2} w \right\|_{L^2_{t} L^2_{x} (2(N+1)/N-1, 2L^2_{t} L^2_{x} (N+1)/N-1)} dt + \int_0^t \left\| D^{1/2} h \right\|_{L^2_{t} L^2_{x} (2(N+1)/N-1, 2L^2_{t} L^2_{x} (N+1)/N-1)} dt + \int_0^t \left\| \partial_t D^{1/2} w \right\|_{L^2_{t} L^2_{x} (2(N-1)/N, 2L^2_{t} L^2_{x} (2(N-1)/N, 2L^2_{t} L^2_{x} (2(N-1)/N-1)) dt + \int_0^t \left\| w \right\|_{L^2_{t} L^2_{x} (2(N-2)/N, 2L^2_{t} L^2_{x} (2(N-2)/N-2, 2L^2_{t} L^2_{x} (2(N-2)/N-2)) dt \\
\leq C \left\{ \left\| (w, w_1) \right\|_{\dot{H}^1 \times L^2} + \left\| D^{1/2} h \right\|_{L^2_{t} L^2_{x} (2(N+1)/N+3, 2L^2_{t} L^2_{x} (2(N+1)/N+3)} \right\}
\]

We then define \(\|S(t)\| = \left\| L^2_{t} L^2_{x} (2(N+1)/N-2, 2L^2_{t} L^2_{x} (N+1)/N-2)\right\|

and \(\|W(t)\| = \left\| L^2_{t} L^2_{x} (2(N+1)/N-1, 2L^2_{t} L^2_{x} (N+1)/N-1)\right\|

We also need the Leibniz and chain rules for fractional derivatives ([19], 1993) in the following form: if \(F'(0) = F''(0) = 0, F \in C^2\) and for all \(a, b\) we have \(|F'(a+b)| \leq C\{|F'(a)| + |F'(b)|\}\) and \(|F''(a+b)| \leq C\{|F''(a)| + |F''(b)|\}\), we have, for \(0 < \alpha < 1\):

\[
\|D^\alpha F(u)\|_{L^p_x} \leq C\|F'(u)\|_{L^{p_1}_x} \|D^\alpha u\|_{L^{p_2}_x}, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2},
\]

I–19
\[ \|D^\alpha(F(u) - F(v))\|_{L^2} \leq C \left[ \|F'(u)\|_{L^p} + \|F'(v)\|_{L^p} \right] \|D^\alpha(u - v)\|_{L^p}^2 \\
+ C \left[ \|F''(u)\|_{L^2} + \|F''(v)\|_{L^2} \right] \cdot \left[ \|D^\alpha u\|_{L^2} + \|D^\alpha v\|_{L^2} \right] \cdot \|u - v\|_{L^2}, \]

\[ \frac{1}{p} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}. \]

Using these estimates and the argument in the study of (CP) for (NLS), one obtains (see also [26]):

**Theorem.** If \((u_0, u_1) \in \dot{H}^1 \times L^2, \|S(I)(u_0, u_1)\|_{\dot{H}^1 \times L^2} \leq A, 0 \in I, \exists \delta(A) > 0 \) s.t. if
\[ \|S(t)(u_0, u_1)\|_{S(I)} \leq \delta, \exists! \text{ solution of (CP) on } \mathbb{R}^N \times I, \text{ with } (u, \partial_t u) \in C(I; \dot{H}^1 \times L^2), \]
\[ \|D^{1/2}u\|_{W(I)} + \|\partial_t D^{1/2}u\|_{W(I)} < \infty, \|u\|_{S(I)} \leq 2\delta, \|u\|_{L^p_{1/2} L^{2N-2}_{1/2} L^{2N-2} L^{2(N+1)/(N-2)}} < \infty, \]
and we have Lipschitz continuity dependence on the data (3 \(\leq N \leq 5\)).

**Corollary.** \(\exists \tilde{\delta} > 0 \) s.t. if \(\|S(I)(u_0, u_1)\|_{\dot{H}^1 \times L^2} \leq \tilde{\delta} \), the hypothesis is verified for \(I = (-\infty, +\infty)\). Moreover, given \((u_0, u_1) \in \dot{H}^1 \times L^2, \exists I \ni 0 \) s.t. the hypothesis is verified on \(I\).

We say that \(u\) solves (CP) for \((u_0, u_1)\) on \(I \ni 0\) if \((u_1, \partial_t u) \in C(I; \dot{H}^1 \times L^2), \)
\(D^{1/2}u \in W(I), u \in S(I), (u, \partial_t u)_{t=0} = (u_0, u_1)\) and \(u\) solves the appropriate integral equation. It is easy to obtain uniqueness and one can then define a maximal interval of existence \(I = (-T_u(u_0, u_1), T_u(u_0, u_1))\). One also has the standard blow-up criterion: if \(T_u(u_0, u_1) < \infty\), then \(\|u\|_{S(0,T_u(u_0, u_1))} = +\infty\). Also, if \(T_u(u_0, u_1) = +\infty\) and \(\|u\|_{S(0, +\infty)} < \infty\), \(u\) scatters at \(+\infty\), i.e. \(\exists u_0^+, u_1^+ \in \dot{H}^1 \times L^2 \) s.t. \(\|(u(t), \partial_t(u)) - S(t)(u_0^+, u_1^+)\|_{\dot{H}^1 \times L^2} \to 0\). Note that for \(t \in I\), we have \(E((u(t), \partial_t(u(t)))) = E((u_0, u_1))\). It turns out that there is another very important conservation law in the energy space. This will be crucial for us, in order to be able to treat non-radial data. It says that, for \(t \in I\), we have
\[
\int \nabla u(x, t) \cdot \partial_t u(x, t) \, dx = \int \nabla u_0 \cdot u_1
\]
(conservation of momentum).

Finally, we mention that Foschi’s estimates [6] also hold for the wave equation. One can then prove the analogue of the Perturbation Theorem for (NLS), for (NLW) and all its corollaries.

We conclude these remarks on (CP) by mentioning the finite speed of propagation property. Recall that if \(R(t)\) is the forward fundamental solution for the linear wave equation, we can write the solution of the linear Cauchy problem (for \(T > 0\)) as
\[
w(t) = \partial_t R(t) \ast w_0 + R(t) \ast w_1 - \int_0^t R(t-s) \ast h(s) \, ds.
\]
The finite speed of propagation states that
\[
supp R(-t) \subset \overline{B}(0, t), supp \partial_t R(t) \subset \overline{B}(0, t).
\]
Thus, if
\[
supp w_0 \subset \mathcal{C}B(x_0, a), supp w_1 \subset \mathcal{C}B(x_0, a), supp h \subset \mathcal{C}\left[ \bigcup_{0 \leq t \leq a} \overline{B}(x_0, a-t) \times t \right],
\]

I–20
$w \equiv 0$ on $\bigcup_{0 \leq t \leq a} [B(x_0, t) \times t]$. This has consequences for solutions of (NLW). If $(u_0, u_1)$, $(u'_0, u'_1)$ are data s.t. $(u_0, u_1) = (u'_0, u'_1)$ on $\overline{B(x_0, a)}$, then, the corresponding solutions agree on $\bigcup_{0 \leq t \leq a} [B(x_0, t) \times (t)] \cap \mathbb{R}^N \times (I \cap I')$. This is because, for each $u$, if we define $u^{(n+1)}(x, t) = S(t)(u_0, u_1) + \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}} f(u^{(n)}(s)) \, ds$, $u^0(x, t) = S(t)(u_0, u_1)$, we have $u^n \to u$, $u^n \to u'$ and they agree on the required set by induction. Typical applications of this are: $\text{supp} u_0 \subset B(0, b)$, $\text{supp} u_1 \subset B(0, b)$, then $u(x, t) \equiv 0$ on $\{(x, t) : |x| > b + t, t \geq 0, t \in I\}$. Similar statements hold for $t < 0$. Thus, one can approximate solutions by regular, compactly supported solutions. The next step is to obtain energy trapping, coercivity and uniform bounds, by variational arguments, as in the case of (NLS). Recall that $\mathcal{W}_{\theta_0, \lambda_0, x_0}(x) = e^{i\theta_0} \lambda_0^{(N-2)/2} \mathcal{W}(\lambda_0(x-x_0))$ and that Aubin-Talenti showed that if $C_N$ is the best constant in the Sobolev embedding ($||u||_{L^2} \leq C_N||\nabla u||$) then $||u||_{L^2} = C_N||\nabla u||$, $u \neq 0$ $\Leftrightarrow$ $u = \mathcal{W}_{\theta_0, \lambda_0, x_0}$. Moreover, we showed that $||\nabla \mathcal{W}||^2 = \frac{1}{C_N^2}$, and if $\mathcal{E}(W) = \frac{1}{2}||\nabla \mathcal{W}||^2 - \frac{1}{2}||W||_{L^2}^2$, $\mathcal{E}(W) = \frac{1}{N}C_N^2$. Using our $t-$independent variational estimates we obtain:

**Energy trapping.** — If $u$ is a solution of (NLW), with max int $I$, $(u, \partial_t u)|_{t=0} = (u_0, u_1) \in H^1 \times L^2$ and for $\delta_0 > 0$, $E((u_0, u_1)) \leq (1 - \delta_0) E((W, 0))$, $||\nabla u_0||^2 < ||\nabla \mathcal{W}||^2$, then $\forall t \in I$ we have: $\exists \bar{\delta} = \bar{\delta}(\delta_0)$ s.t.

i) $||\nabla u(t)||^2 \leq (1 - \bar{\delta})||\nabla \mathcal{W}||^2$

ii) $f ||\nabla u(t)||^2 - |u(t)|^2^* \geq \bar{\delta} f ||\nabla u(t)||^2$

iii) $\mathcal{E}(u(t)) \geq 0$ (and hence $E((u_t, \partial_t u)) \geq 0$)

iv) $E((u(t), \partial_t u(t))) \simeq ||(u(t), \partial_t u(t))||_{H^1 \times L^2}^2 \simeq ||(u_0, u_1)||_{H^1 \times L^2}^2$, with comparability constants depending on $\delta_0$.

Also, as in the case of (NLS) we have: if $E((u_0, u_1)) \leq (1 - \delta_0) E((W, 0))$; $||\nabla u_0||^2 \geq ||\nabla \mathcal{W}||^2$, then, for $t \in I$ we have $||\nabla u(t)||^2 \geq (1 + \bar{\delta})||\nabla \mathcal{W}||^2$.

We next turn to the proof of ii) in Theorem B. We will show it in the case when $||u_0||_{L^2} < \infty$. The general case follows by using, in addition, localization and finite speed of propagation. We know that, in the situation of ii),

$$\int |\nabla u(t)|^2 \geq (1 + \bar{\delta}) \int |\nabla \mathcal{W}|^2, \quad t \in I$$

$$E((W, 0)) \geq E((u(t), \partial_t u(t))) + \bar{\delta}_0.$$ 

Then,

$$\frac{1}{2} \int |u(t)|^2^* \geq \frac{1}{2} \int (\partial_t u(t))^2 + \frac{1}{2} \int |\nabla u(t)|^2 - E((W, 0)) + \bar{\delta}_0$$

or

$$\int |u(t)|^2^* \geq \frac{N}{N - 2} \int (\partial_t u(t))^2 + \frac{N}{N - 2} \int |\nabla u(t)|^2 - 2^* E((W, 0)) + 2^* \bar{\delta}_0.$$
Let \( y(t) = \int |u(t)|^2 \), so that \( y'(t) = 2 \int u(t) \partial_t u(t) \), \( y''(t) = 2 \int ((\partial_t u)^2 - |\nabla u(t)|^2 + |u|^2(t)) \). Then,

\[
y''(t) \geq 2 \int (\partial_t u)^2 + \frac{2N}{N-2} \int (\partial_t u) - 2^* E((W,0)) + \frac{\tilde{\delta}_0}{N-2} \int |\nabla u(t)|^2 - 2 \int |\nabla u(t)|^2 = \frac{4(N-1)}{(N-2)} \int (\partial_t u)^2 + \frac{4}{(N-2)} \int |\nabla u|^2 - \frac{4}{(N-2)} \int |\nabla W|^2 + \tilde{\delta}_0 \geq \frac{4(N-1)}{(N-2)} \int (\partial_t u)^2 + \tilde{\delta}_0.
\]

If \( I \cap [0, +\infty) = [0, +\infty) \), \( \exists t_0 > 0 \) s.t. \( y'(t_0) > 0 \) and \( y'(t) > 0 \), \( \forall t > t_0 \). For \( t > t_0 \) we have:

\[
y''(t) y(t) \geq \frac{4(N-1)}{(N-2)} \int (\partial_t u)^2 \int u^2 \geq \left( \frac{N-1}{N-2} \right) y'(t)^2
\]
or

\[
\frac{y''(t)}{y'(t)} \geq \left( \frac{N-1}{N-2} \right) \frac{y'(t)}{y(t)}
\]
or

\[
y'(t) \geq C_0 y(t)^{(N-1)/N-2}, \quad t > t_0.
\]

But since \( \frac{(N-1)}{N-2} > 1 \) this leads to finite time blow-up, a contradiction.

We now turn to the proof of i) in Theorem B. We repeat the “concentration-compactness” procedure, replacing Keraani’s work with the work of Bahouri-Gérard ([1], 1999) on high frequency approximation to solutions of the linear wave equation. We then obtain \( E_c \), with \( 0 < \eta_0 \leq E_c \leq E((W,0)) \) with the property that if \( E((u_0, u_1)) < E_c \), \( ||\nabla u_0||^2 < ||\nabla W||^2 \), we have \( I = (-\infty, +\infty), \|u\|_{S(-\infty, +\infty)} < \infty \) and \( E_c \) is optimal with this property. i) is the assertion \( E_c = E((W,0)) \). If not, \( E_c < E((W,0)) \), which will lead to a contradiction. Exactly as in the (NLS) case we have:

**Proposition 1.** \( \exists (u_{0,c}, u_{1,c}) \in \mathcal{H}^1 \times L^2, \) with \( ||\nabla u_{0,c}||^2 < ||\nabla W||^2, E((u_{0,c}, u_{1,c})) = E_c \) and s.t. for the solution \( u_c \) of (CP), with max int \( I \), we have \( ||u_c||_{S(I)} = +\infty \).

**Proposition 2.** For any \( u_c \) as in Proposition 1, s.t. (say) \( ||u_c||_{S(I)} = +\infty \), \( \exists x(t) \in \mathbb{R}^N, \lambda(t) \in \mathbb{R}^+, t \in I^+ \) s.t.

\[
K = \left\{ v(x, t) = \left( \frac{1}{\lambda(t)^{N-2}/2} u_c \left( \frac{x - x(t)}{\lambda(t)}, t \right), \frac{1}{\lambda(t)^{N-2}/2} \partial_t u_c \left( \frac{x - x(t)}{\lambda(t)}, t \right) \right) \right\}
\]

has compact closure in \( \mathcal{H}^1 \times L^2 \).

**Remark.** \( x(t), \lambda(t) \) can be taken continuous. Moreover, if \( T_+ < \infty, \lambda(t) \geq \frac{C_0(K)}{(T_+-t)} \) (same proof as (NLS)). Also, if \( T_+ = +\infty \), by possibly changing \( u_c \), we can find one for which \( \lambda(t) \geq A_0 > 0 \).

One can also show:

**Lemma.** \( \exists g : (0, E_c] \to [0, \infty), g \downarrow \) s.t. \( \forall (u_0, u_1) \) with \( E((u_0, u_1)) \leq E_c - \eta, ||\nabla u_0||^2 < ||\nabla W||^2 \), we have \( ||u||_{S(-\infty, +\infty)} \leq g(\eta) \).

I-22
To proceed further, we need specific features of the problem. We now will develop some further properties of critical elements, specific to (NLW). We start out with some further consequences of the finite speed of propagation.

**Lemma.** Let \((u_0, u_1) \in \dot{H}^1 \times L^2\), \(\|(u_0, u_1)\|_{\dot{H}^1 \times L^2} \leq A\). If for some \(M > 0\), \(\varepsilon > 0\), \(0 < \varepsilon < \varepsilon_0 = \varepsilon_0(A)\), we have:

\[
\int_{|x| \geq M} |\nabla u_0|^2 + |u_1|^2 + \frac{|u_0|^2}{|x|^2} \leq \varepsilon ,
\]

then for \(0 < t < T_+(u_0, u_1)\) we have

\[
\int_{|x| \geq \frac{3}{2} M + t} |\nabla u(t)|^2 + |\partial_t u(t)|^2 + |u(t)|^2 + \frac{|u(t)|^2}{|x|^2} \leq C \varepsilon.
\]

**Proof.** Choose \(\psi_M \equiv 1\), \(|x| \geq \frac{3}{2} M\), \(\psi_M \equiv 0\), \(|x| \leq M\), \(\nabla \psi_M \leq \frac{C}{M}\). Let \(u_{0, M} = \psi_M u_0\), \(u_{1, M} = \psi_M u_1\). Because of our assumption, \(\|(u_{0, M}, u_{1, M})\|_{\dot{H}^1 \times L^2} \leq C \varepsilon\). If \(\varepsilon_0\) is so small that \(C \varepsilon_0 < \delta\), then \(u_{M, t}\) solves (CP) in

\[
I = (-\infty, +\infty) \text{ and } \sup_{t \in (-\infty, +\infty)} \|(u_{M, t}, \partial_t u_{M, t})\|_{\dot{H}^1 \times L^2} < 2C \varepsilon.
\]

But by finite speed, \(u_M = u\) for \(|x| \geq \frac{3}{2} M + t, t > 0, t \in I\).

**Lemma.** Let \(u_c\) be a critical element as in Proposition 2, with \(T_+((u_0, u_1)) < \infty\). (Assume without loss of generality that \(T_+((u_0, u_1)) = 1\). Then, \(\exists \bar{x} \in \mathbb{R}^N\) s.t.

\[
supp u_c(-, t), \partial_t u_c(-, t) \subset B(\bar{x}, 1 - t), \quad 0 < t < 1.
\]

**Proof.** We first show, for each \(t, 0 < t < 1\), that there is a ball \(B_{1-t}\), of radius \((1-t)\) s.t. \(\sup \nabla u, \sup \partial_t u \subset B_{1-t}\). If not, for a fixed \(t\), \(\exists \varepsilon_0 > 0, \eta_0 > 0\) s.t. \(\forall x_0 \in \mathbb{R}^N\) we have

\[
\int_{|x-x_0| \geq (1+\eta_0)(1-t)} |\nabla u(t)|^2 + (\partial_t u(t))^2 \geq \varepsilon_0 > 0.
\]

Choose a sequence \(t_n \uparrow 1\). Recall that \(\lambda(t_n) \geq \frac{C_0(K)}{1-t_n}\). We claim that, given \(R_0 > 0, M > 0\), \(\eta_0 > 0\), for \(n\) large we have

\[
\int_{|x+\frac{x(t_n)}{\lambda(t_n)}| \geq R_0} |\nabla u(x, t_n)|^2 + |\partial_t u(x, t_n)|^2 + \frac{|u(x, t_n)|^2}{|x|^2} \leq \frac{\varepsilon_0}{M}.
\]

Indeed, let

\[
\bar{v}(x,t) = \frac{1}{\lambda(t)^{N/2}}(\nabla u(x-x(t), t), \partial_t u(x-x(t), t))\text{which is compact in }L^2(\mathbb{R}^N)^{N+1}.
\]

Then

\[
\int_{|x+\frac{x(t_n)}{\lambda(t_n)}| \geq R_0} |\nabla u(x, t_n)|^2 + |\partial_t u(x, t_n)|^2 = \int_{|y| \geq \lambda(t_n)R_0} |\bar{v}(y, t_n)|^2
\]

and the claim follows from the compactness of \(K, \lambda(t_n) \uparrow +\infty\). (The proof for the term \(\frac{|u(x, t_n)|^2}{|x|^2}\) follows from a similar argument). From this claim and the previous Lemma, used backward in time, we conclude that \(\forall t \in [0, t_n]\) we have

\[
\int_{|x+\frac{x(t_n)}{\lambda(t_n)}| \geq \frac{3}{2} R_0 + (t_n-t)} |\nabla u(x, t)|^2 + |\partial_t u(x, t)|^2 \leq \varepsilon_0.
\]

I-23
But if $R_0$ is so small that $(1 + \eta_0)(1 - t) \geq \frac{3}{2} R_0 + (t_n - t)$, we reach a contradiction, proving the claim.

The next step is to show that $\frac{\lambda(t_n)}{\lambda(t)} \leq M$, $0 \leq t < 1$. If not, $\exists t_n \uparrow 1$ s.t. $\frac{\lambda(t_n)}{\lambda(t)} \uparrow +\infty$. Fix a ball $B(x_0, 1)$ s.t. $\text{supp} \nabla u_0, u_1 \subset B(x_0, 1)$. For a fixed $R_0 > 0$, $\varepsilon_0 > 0$ given, the previous argument shows that, for $n$ large,

$$\int_{|x + \frac{x(t_n)}{\lambda(t_n)}| \geq \frac{3}{2} R_0 + (t_n)} |\nabla u_0|^2 + |u_1|^2 \leq \varepsilon_0.$$

But, if $\frac{\lambda(t_n)}{\lambda(t)} \to +\infty$, $B(x_0, 1) \subset \{x + \frac{x(t_n)}{\lambda(t_n)} | \geq \frac{3}{2} R_0 + t_n\}$, for $n$ large, so that $\nabla u_0, u_1$ are identically 0, contradicting $T_+ = 1$. Let now $t_n \uparrow 1$ and choose a subsequence s.t. $\frac{x(t_n)}{\lambda(t_n)} \to \bar{x}$. The same argument shows that for $0 < t < t_n$, $n$ large we have

$$\int_{|x + \frac{x(t_n)}{\lambda(t_n)}| \geq \frac{3}{2} R_0 + (t_n - t)} |\nabla u(x, t)|^2 + |\partial_t u(x, t)|^2 \leq \varepsilon_0.$$

Letting $n \to \infty$ we obtain

$$\int_{|x - \bar{x}| \geq \frac{3}{2} R_0 + (1 - t)} |\nabla u(x, t)|^2 + |\partial_t u(x, t)|^2 \leq \varepsilon,$$

so that $\text{supp} \nabla u(-, t), \partial_t u(-, t) \subset B(\bar{x}, 1 - t)$. If $\frac{\lambda(t_n)}{\lambda(t)} \to \bar{x}, \frac{\lambda(t_n)}{\lambda(t_n)} \to \bar{x}'$, $x \neq x'$ and we choose $1 - t$ so small that $(1 - t) < |x - x'|$, we must have $\nabla u(-, t), \partial_t u(-, t) \equiv 0$, which contradicts coercivity, $T_+ = 1$. □

**Remark.** After translation we can take $\bar{x} = 0$.

**Lecture 4**

We next turn to a fundamental result that is crucial in the treatment of non-radial solutions.

**Theorem** (Orthogonality for critical elements). Let $(u_{0,e}, u_{1,e})$ be as in Propositions 1,2, $\lambda(t), x(t)$ continuous, $\lambda(t) > 0$. Assume that either $T_+((u_{0,e}, u_{1,e})) < +\infty$ or $T_+((u_{0,e}, u_{1,e})) = +\infty$, $\lambda(t) \geq A_0 > 0$. Then

$$\int \nabla u_{0,e} \cdot u_{1,e} = 0.$$

Note that in the radial case this is automatic. We first sketch the proof in the case $T_+ < \infty$. We need a further linear estimate.

**Lemma** (Trace Theorem). Let

$$\begin{cases}
\partial_t^2 w - \Delta w = h \in L^1_t L^2_x, \\
w|_{t=0} = w_0 \in H^1, \\
\partial_t w|_{t=0} = w_1 \in L^2.
\end{cases}$$

I-24
Then, for $|\alpha| \leq \frac{1}{2}$, we have:

$$\sup_{t} \left\| \nabla w \left( \frac{x_1 - \alpha t}{\sqrt{1 - \alpha^2}}, x', \frac{t - \alpha x_1}{\sqrt{1 - \alpha^2}} \right) \right\|_{L^2(dx, dx')} + \sup_{t} \left\| \partial_t w \left( \frac{x_1 - \alpha t}{\sqrt{1 - \alpha^2}}, x', \frac{t - \alpha x_1}{\sqrt{1 - \alpha^2}} \right) \right\|_{L^2(dx, dx')} \leq C \left\{ \|w_0\|_{H^1} + \|w_1\|_{L^2} + ||h||_{L^1 L^2} \right\}$$

Proof. It suffices to consider $v(x, t) = U(t)f$, where $\hat{v}(\xi, t) = e^{it|\xi|} \hat{f}(\xi)$ and prove

$$\sup_{t} \left\| v \left( \frac{x_1 - \alpha t}{\sqrt{1 - \alpha^2}}, x', \frac{t - \alpha x_1}{\sqrt{1 - \alpha^2}} \right) \right\|_{L^2(dx, dx')} \leq C \|f\|_{L^2}.$$ 

Note that

$$v(x, t) = \int e^{ix_1 \xi_1} e^{ix' \xi'} e^{it\sqrt{1 + |\xi|^2}} \hat{f}(\xi_1, \xi') d\xi_1 d\xi',$$

so that

$$v \left( \frac{x_1 - \alpha t}{\sqrt{1 - \alpha^2}}, x', \frac{t - \alpha x_1}{\sqrt{1 - \alpha^2}} \right) = \int e^{ix_1 (\xi - \alpha \xi_1)/\sqrt{1 - \alpha^2}} e^{-iat \xi_1/\sqrt{1 - \alpha^2}} e^{-iat \xi_1/\sqrt{1 - \alpha^2}} e^{it\sqrt{1 - \alpha^2} \hat{g}_t(\xi)} d\xi_1 d\xi' \cdot \hat{g}_t(\xi),$$

where $\hat{g}_t(\xi) = e^{-iat \xi_1/\sqrt{1 - \alpha^2}} \hat{f}(\xi)$, so that $\|g_t\|_{L^2} = \|f\|_{L^2}$. If we now let $\eta_1 = \frac{\xi_1 - \alpha \xi}{\sqrt{1 - \alpha^2}}$, $\eta' = \xi'$ and compute $\frac{d\eta}{d\xi} = \frac{(1 - \alpha \xi_1)^2}{\sqrt{1 - \alpha^2}} \simeq 1$ for $|\alpha| \leq \frac{1}{2}$, we see that the estimate follows from Plancherel. \hfill \Box

If $u$ is a solution of (CP) with maximal interval $I, I' \subseteq I$, recall that $u \in L_{I'}^{(N + 2)/N - 2} L_{x'}^{(N + 2)/N - 2}$, so that $|u|^{1/N - u} u \in L_{I'}^1 L_{x'}^2$. Hence, the conclusion of the previous lemma holds, provided the integrations are restricted to $\left( \frac{x_1 - \alpha t}{\sqrt{1 - \alpha^2}}, x', \frac{t - \alpha x_1}{\sqrt{1 - \alpha^2}} \right) \in \mathbb{R}^N \times I'$.

Idea of the proof of Theorem 5 when $T_+((u_0, u_1)) = 1$. Assume that

$$\int \partial x_1(u_{0,c}) \cdot u_{1,c} = \gamma > 0.$$ 

Recall that $\text{supp } u_c, \partial_c u_c \subset B(0, 1 - t), 0 < 0 < 1$. For convenience, set $u(x, t) = u_c(x, 1 + t), -1 \leq t < 0$, supported in $B(0, |t|)$. For $0 < \alpha < \frac{1}{4}$ we consider the Lorentz transformation

$$z_\alpha(x_1, \bar{x}, t) = u \left( \frac{x_1 - \alpha t}{\sqrt{1 - \alpha^2}}, x', \frac{t - \alpha x_1}{\sqrt{1 - \alpha^2}} \right)$$

and fix our attention on $-\frac{1}{2} \leq t < 0$. In that region, the Lemma above and the remark following it, together with the support property of $u$, show that $z_\alpha$ is in the energy space and solves our equation. An easy calculation shows that $\text{supp } z_\alpha(-t) \subset B(0, |t|)$, $z_\alpha \neq 0$, so that $T_+ = 0$ is the final time of existence for $z_\alpha$. A long calculation shows that

$$\lim_{\alpha \downarrow 0} \frac{E(z_\alpha(-\frac{1}{2}), \partial_t z_\alpha(-\frac{1}{2})) - E((u_{0,c}, u_{1,c}))}{\alpha} = -\gamma$$

I–25
and that, for some $t_0 \in [-\frac{1}{2}, -\frac{1}{4}]$, \( f |\nabla z_\alpha(t_0)|^2 < f |\nabla W|^2 \), for \( \alpha \) small. But, since \( E((u_{0,c}, u_{1,c})) = E_c \), for \( \alpha \) small this contradicts the definition of \( E_c \), since the final time of existence is finite.

Comments on the proof of Theorem 5 when \( T_+ = +\infty \) (\( \lambda(t) \geq A_0 > 0 \)). The finiteness of the energy of \( z_\alpha \) is now unclear, because of the lack of the support property. We then do a renormalization. We first rescale \( u_c \) and consider \( u_R(x,t) = R^{(N-2)/2} u_c(Rx, Rt) \) for \( R \) large, and for \( \alpha \) small

\[
z_{\alpha,R}(x_1, \bar{x}, t) = u_R \left( \frac{x_1 - \alpha t}{\sqrt{1 - \alpha^2} \bar{x}}, \frac{t - \alpha x_1}{\sqrt{1 - \alpha^2}} \right).
\]

We assume, as before, that \( \int \partial_x u_{0,c} \cdot u_{1,c} = \gamma > 0 \). We then prove (by integration in \( t_0 \in (1, 2) \)) that if \( h(t_0) = \theta(x) z_{\alpha,R}(x_1, \bar{x}, t_0) \), with \( \theta \) a cut-off function, for some \( \alpha_1 \) small and all \( R \) sufficiently large, we have, for some \( t_0 \in (1, 2) \) that

\[
E((h(-, t_0), \partial_t h(-, t_0)) < E_c - \frac{1}{2} \gamma \alpha_1
\]

and

\[
\int |\nabla h(t_0)|^2 < \int |\nabla W|^2.
\]

We then let \( v \) be the solution of \((CP)\), with data \( h(-, t_0) \) at \( t = t_0 \). By our properties of critical elements, we know that \( ||v||_{S(-\infty, +\infty)} \leq g(\frac{1}{2} \gamma \alpha_1) \), for all \( R \) large. But, since \( ||u_c||_{S(0, +\infty)} = +\infty \), we have that \( ||u_R||_{L^2(N+1)/2, N-2} \to +\infty \), by rescaling. But, by finite speed of propagation, we have that \( v = z_{\alpha,R} \) on a large set, and after a change of variables to undo \( \alpha_1 \), we reach a contradiction. The details of the argument are lengthy.

To finish the proof of Theorem B, we are reduced to:

**Theorem** (Rigidity Theorem). Assume that \( E((u_0, u_1)) < E((W, 0)) \), \( f |\nabla u_0|^2 < f |\nabla W|^2 \), \( u \) the solution of \((CP)\) with \( I_+ = [0, T_+] \). Assume that

a) \( \int \nabla u_0 u_1 = 0 \).

b) \( \exists x(t), \lambda(t) > 0 \), \( t \in [0, T_+] \) s.t.

\[
K = \left\{ v(x, t) = \left( \frac{1}{\lambda(t)^{N-2/2}} u \left( \frac{x - x(t)}{\lambda(t) t} \right), \frac{1}{\lambda(t)^{N/2}} \partial_t u \left( \frac{x - x(t)}{\lambda(t) t} \right) \right) \right\}
\]

has compact closure in \( \dot{H}^1 \times L^2 \).

c) \( x(t), \lambda(t) \) are continuous, if \( T_+ = 1 \) (scaling) \( \lambda(t) \geq \frac{C_0(K)}{1-t} \), \( \text{supp} u, \partial_t u \subset B(0, 1-t) \), when \( T_+ = +\infty \), \( x(0) = 0 \), \( \lambda(0) = 1 \), \( \lambda(t) \geq A_0 > 0 \).

Then, \( T_+ = 1 \) cannot happen and if \( T_+ = +\infty \), \( (u_0, u_1) = (0, 0) \).

Clearly the rigidity theorem gives us the contradiction with establishes Theorem B, i).
Proof of the Rigidity Theorem

**Case 1**: $T_+ = +\infty$, $\lambda(t) \geq A_0 > 0$, $x(0) = 0$, $\lambda(0) = 1$, $x(t), \lambda(t)$ continuous. Assume $(u_0, u_1) \neq (0, 0)$. By our variational estimates we have

$$
\sup_{t > 0} ||(\nabla u, \partial_t u)(t)||_{L^2} \leq CE,
$$

where $E((u_0, u_1)) = E > 0$. We also have

$$
\int |\nabla u(t)|^2 - |u(t)|^2^\star \geq C\tilde{\delta} \int |\nabla u(t)|^2, \quad t > 0
$$

$$
\frac{1}{2} \int (\partial_t u(t))^2 + \frac{1}{2} \int |\nabla u(t)|^2 - |u(t)|^2^\star \geq C\tilde{\delta} E
$$

where $\tilde{\delta} = \tilde{\delta}(\delta_0)$, $E((u_0, u_1)) \leq (1 - \delta_0) E((W, 0))$. A change of variables, the compactness of $K$ and $\lambda(t) \geq A_0 < 0$ now give: given $\varepsilon > 0$, $\exists R_0(\varepsilon)$ s.t. for all $0 \leq t < \infty$, we have

$$
\int_{|x + \frac{\pi(t)}{\lambda(t)}| \geq R_0(\varepsilon)} |\partial_t u|^2 + |\nabla u|^2 + \frac{|u|^2}{|x|^2} + |u|^2^\star \leq \varepsilon E.
$$

We next need some algebraic identities:

**Lemma.** Let $r(R) = r(t, R) = \int_{|x| \geq R} \{|\nabla u|^2 + |\partial_t u|^2 + |u|^2^\star + \frac{|u|^2}{|x|^2}\} dx$. We have, if $\phi \in C_0^\infty(B_2)$, $\phi \equiv 1$ on $|x| \leq 1$, $\phi_R(x) = \phi(\frac{x}{R})$, $\psi_R(x) = x \phi(\frac{x}{R})$:

1. $\partial_t \left( \int \psi_R \nabla u \cdot \partial_t u \right) = -\frac{N}{2} \int (\partial_t u)^2 + \frac{N}{2} \int |\nabla u|^2 - |u|^2^\star + O(r(R))$

2. $\partial_t \left( \int \phi_R u \partial_t u \right) = \int (\partial_t u)^2 - \int |\nabla u|^2 + \int |u|^2^\star + O(r(R))$

3. $\partial_t \left( \int \psi_R \left\{ \frac{1}{2} |\nabla u|^2 + \frac{1}{2} (\partial_t u)^2 - \frac{1}{2} |u|^2^\star \right\} \right) = -\int \nabla u \partial_t u + O(r(R))$

The proof of the case $T_+ = +\infty$ is based on 2 Lemmas.

**Lemma 1.** $\exists \varepsilon_0 > 0$, $C_0 > 0$ s.t. if $0 < \varepsilon < \varepsilon_0$, $\exists R_0(\varepsilon)$ s.t. if $R > 2R_0(\varepsilon)$, $\exists t_0 = t_0(R, \varepsilon)$ with $0 < t_0 \leq C R s.t. \forall 0 < t < t_0$ we have $|\frac{\pi(t)}{\lambda(t)}| < R - R_0(\varepsilon)$ and $|\frac{\pi(t_0)}{\lambda(t_0)}| = R - R_0(\varepsilon)$.

**Remark.** In the radial case, $x(t) \equiv 0$, so a contradiction follows directly from Lemma 10. This is the analogue of the virial identity proof for NLS. In the non-radial case we also need:

**Lemma 2.** $\exists \varepsilon_0 > 0$, $R_1(\varepsilon) > 0$, $C_0 > 0$ s.t. if $R > R_1(\varepsilon)$, for $0 < \varepsilon < \varepsilon_0$, we have $t_0(R, \varepsilon) \geq \frac{C_0}{\varepsilon}$.

From Lemma 1 and Lemma 2 we have: for $0 < \varepsilon < \varepsilon_0$, $R > 2R_0(\varepsilon)$, $t_0(R, \varepsilon) \leq C R$, while for $0 < \varepsilon < \varepsilon_2$, $R > R_1(\varepsilon)$, $t_0(R, \varepsilon) \geq \frac{C_0}{\varepsilon} R$. This is a contradiction for $\varepsilon$ small.

**Proof of Lemma 1.** If not, since $x(0) = 0$, $\lambda(0) = 1$, both $x(t), \lambda(t)$ continuous, we have $\forall 0 < t < C R (C$ large) that $|\frac{\pi(t)}{\lambda(t)}| < R - R_0(\varepsilon)$. Let $z_R(t) = f \psi_R \nabla \cdot \partial_t u + (\frac{N}{2} - \alpha) f \phi_R u \partial_t u$. Then, $z_R(t) = -\alpha (\partial_t u)^2 - (1 - \alpha)(|\nabla u|^2 - |u|^2^\star) + O(r(R))$. 

I-27
But, for $|x| > R$, $0 < t < C R$ we have $|x + \frac{x(t)}{\lambda(t)}| \geq R_0(\varepsilon)$, so that $|r(R)| \leq \tilde{C} \varepsilon E$ and so, for $\varepsilon$ small, $\alpha = \frac{1}{2}$, $z'_R(t) \leq -\frac{\tilde{C} E}{2}$. Also, $|z_R(t)| \leq C_1 R E$. Integrating in $t$, we obtain: $\frac{C R \tilde{C} E}{2} \leq 2 C_1 R E$, a contradiction for $C$ large.

Proof of Lemma 2. For $0 \leq t \leq t_0$, set

$$y_R(t) = \int \psi_R \left( \frac{1}{2} (\partial_t u)^2 + \frac{1}{2} |\nabla u|^2 - \frac{1}{2^*} |u|^{2^*} \right).$$

We have for $|x| > R$, $|x + \frac{x(t)}{\lambda(t)}| > R_0(\varepsilon)$ so that, since $\int \nabla u \cdot u = 0$, $y'_R(t) = O(r(R))$ and hence

$$|y_R(t_0) - y_R(0)| \leq \tilde{C} \varepsilon E t_0.$$

But,

$$|y_R(0)| \leq \tilde{C} R_0(\varepsilon) E + O(R r(R_0(\varepsilon))) \leq \tilde{C} E \{ R_0(\varepsilon) + \varepsilon R \}$$

and

$$|y_R(t_0)| \geq \int_{|x + \frac{x(t)}{\lambda(t)}| \leq R_0(\varepsilon)} x \left[ \frac{1}{2} (\partial_t u)^2 + \frac{1}{2} |\nabla u|^2 - \frac{1}{2^*} |u|^{2^*} \right] - M R \varepsilon E.$$

The integral equals

$$\frac{-x(t_0)}{\lambda(t_0)} \int_{|x + \frac{x(t_0)}{\lambda(t_0)}| \leq R_0(\varepsilon)} \frac{1}{2} (\partial_t u)^2(t_0) + \frac{1}{2} |\nabla u|^2(t_0) - \frac{1}{2^*} |u|^{2^*}(t_0)$$

$$- \int_{|x + \frac{x(t_0)}{\lambda(t_0)}| \leq R_0(\varepsilon)} \left( x + \frac{x(t_0)}{\lambda(t_0)} \right) \left\{ \right\}$$

$$= \frac{-x(t_0)}{\lambda(t_0)} \int \left\{ \frac{1}{2} (\partial_t u)^2 + \frac{1}{2} |\nabla u|^2 - \frac{1}{2^*} |u|^{2^*} \right\}$$

$$+ \frac{x(t_0)}{\lambda(t_0)} \int_{|x + \frac{x(t_0)}{\lambda(t_0)}| \leq R_0(\varepsilon)} \left\{ \right\}$$

$$+ \int_{|x + \frac{x(t_0)}{\lambda(t_0)}| \leq R_0(\varepsilon)} \left( x + \frac{x(t_0)}{\lambda(t_0)} \right) \left\{ \right\}.$$
We next turn to:

Case 2: \( T_+ = 1, \lambda(t) \geq \frac{C_0(K)}{1-t} \), supp \( u \), \( \partial_t u \subset B(0, 1-t) \). In this case we cannot use the conservation of the \( L^2 \) norm as in the case of (NLS) and a new approach is needed.

The first step is:

Lemma 3. \( \lambda(t) \leq \frac{C_1(K)}{1-t} \).

Proof. If not, \( \exists t_n \uparrow 1 \) s.t. \( \lambda(t_n)(1-t_n) \uparrow +\infty \). Let \( z(t) = \int x \cdot \nabla u \cdot \partial_t u + \left( \frac{N}{2} - \alpha \right) \int u \partial_t u, \) \( 0 < \alpha < 1 \). This is defined for \( 0 < t < 1 \) and

\[
z'(t) = -\alpha \int (\partial_t u)^2 - (1-\alpha) \int |\nabla u|^2 - |u|^2.
\]

By our variational estimates \( E((u_0, u_1)) = E > 0 \) and \( \sup_{0 < t < 1} ||(\nabla u, \partial_t u)(t)|| \leq C E \). Also, \( z'(t) \leq -C_0 E, 0 < t < 1 \). From the support properties, we easily see that \( \lim_{t \uparrow 1} z(t) = 0 \), so that, integrating in \( t \), \( z(t) \geq C_0 E(1-t) \). We will show that \( \int_{t_n}^{(1-t_n)} \rightarrow 0 \), yielding a contradiction. We know that \( \int \nabla u \partial_t u = 0 \), \( 0 < t < 1 \). Hence,

\[
\frac{z(t_n)}{(1-t_n)} = \frac{\int (x + \frac{x(t_n)}{\lambda(t_n)}) \nabla u \cdot \partial_t u}{(1-t_n)} + \left( \frac{N}{2} - \alpha \right) \int \frac{u \partial_t u}{(1-t_n)}.
\]

Note that, for \( \varepsilon > 0 \) given, we have

\[
\int |x + \frac{x(t_n)}{\lambda(t_n)}| |\nabla u(t_n)|| |\partial_t u(t_n)| \leq C \varepsilon E(1-t_n)
\]

and similarly for \( \int |u(t_n)|| \partial_t u(t_n)| \). Next we show that \( |x + \frac{x(t_n)}{\lambda(t_n)}| \leq 2(1-t_n) \). If not, \( B(\frac{-x(t_n)}{\lambda(t_n)}, (1-t_n)) \cap B(0, 1-t_n) = \emptyset \), so that \( \int_{B(\frac{-x(t_n)}{\lambda(t_n)}, (1-t_n))} |\nabla u(t_n)|^2 = 0 \), while

\[
\int_{|x + \frac{x(t_n)}{\lambda(t_n)}| \geq (1-t_n)} |\nabla u(x, t_n)|^2 \, dx = \int_{|y| \geq \lambda(t_n)(1-t_n)} |\nabla u\left(\frac{y-x(t_n)}{\lambda(t_n)}, t_n\right)|^2 \frac{dy}{\lambda(t_n)^N} \rightarrow 0
\]

by \( \lambda(t_n)(1-t_n) \rightarrow +\infty \), compactness of \( \tilde{K} \). Arguing similarly for \( \partial_t u(t_n) \), we obtain that \( E((u(t_n), \partial_t u(t_n))) \rightarrow 0 \), a contradiction. But,

\[
\frac{1}{(1-t_n)} \int_{|x + \frac{x(t_n)}{\lambda(t_n)}| \geq (1-t_n)} |x + \frac{x(t_n)}{\lambda(t_n)}| |\nabla u(x, t_n)|| |\partial_t u(x, t_n)|
\]

\[
\leq 3 \int_{|x + \frac{x(t_n)}{\lambda(t_n)}| \geq (1-t_n)} |\nabla u(x, t_n)|| |\partial_t u(x, t_n)|
\]

\[
\leq \frac{3}{\lambda(t_n)^N} \int_{|y| \geq (1-t_n)\lambda(t_n)} |\nabla u\left(\frac{y-x(t_n)}{\lambda(t_n)}, t_n\right)|| |\partial_t u\left(\frac{y-x(t_n)}{\lambda(t_n)}, t_n\right)| \frac{dy}{n-\infty} \rightarrow 0,
\]

by compactness of \( \tilde{K} \), \( (\lambda(t_n)(1-t_n)) \rightarrow 0 \). Arguing similarly (using Hardy) for \( \int \frac{u \partial_t u}{(1-t_n)} \), we conclude the proof.

\[
\square
\]

Proposition. In this case we have \( (T_+ = 1) \supp u, \partial_t u \subset B(0, 1-t) \) and \( K = ((1-t)^{N/2})(\nabla u((1-t)x, t), \partial_t u((1-t)x, t)) \) is compact in \( L^2(\mathbb{R}^N)^{N+1} \).

I-29
Proof. \( \{ \bar{v}(x,t) = (1 - t)^{N/2}(\nabla u((1 - t)(x - x(t)), t), \partial_t u((1 - t)(x - x(t)), t) \} \) has compact closure since \( C_0(K) \leq (1 - t)\lambda(t) \leq C_1(K) \) and if \( K \) is compact, \( K_1 = \{ \lambda^{N/2} v(\lambda x) : \bar{v} \in K, C_0 \leq \lambda \leq C_1 \} \) has \( K_1 \) compact. Let now \( \bar{v}(x,t) = (1 - t)^{N/2}(\nabla u((1 - t)(x - x(t), t), \partial_t u((1 - t)(x - x(t), t), t) \), so that \( \bar{v}(x,t) = \bar{v}(x + x(t), t) \). Since \( \text{supp} \bar{v}(\cdot,t) \subset \{ x : |x - x(t)| \leq 1 \} \) and \( E > 0 \), the fact that \( \{ \bar{v}(\cdot,t) \} \) is compact \( \Rightarrow |x(t)| \leq C \). But, if \( K_2 = \{ \bar{v}(x + x_0,t) : |x_0| \leq C \}, \) then \( K_2 \) is compact, giving the Proposition. \( \square \)

At this point, because of the lack of the \( L^2 \) conservation law, we cannot go further and a new idea is needed. Following Giga-Kohn [8] in the parabolic case and Merle-Zaag [23] in the hyperbolic case \( \left( (\partial_t^2 - \Delta) u - |u|^{p-1} u = 0, 1 < p < \frac{4}{N-1} + 1 \right) \), we introduce self-similar variables. We set: \( y = \frac{x}{t-\bar{w}}, s = \log \frac{1}{1+t}, 0 < t < 1 \) and define

\[
w(y, s; 0) = (1 - t)^{N-2/2} u(x,t) = e^{-s(N-2)/2} u(e^{-s} y, 1 - e^{-s}),
\]

which is defined for \( 0 \leq s < \infty \), \( \text{supp} w(\cdot,s; 0) \subset \{|y| \leq 1\} \). We also consider, for \( \delta > 0 \), \( u_s(x,t) = u(x,t+\delta) \) and the corresponding \( w \). In other words, we set \( y = \frac{x}{1+t-\bar{w}}, s = \log \frac{1}{1+t} \) and

\[
w(y, s; \delta) = (1 - t)^{N-2/2} u(x,t) = e^{-s(N-2)/2} u(e^{-s} y, 1 + \delta - e^{-s})
\]

which is defined for \( 0 \leq s \leq -\log \delta \), with \( \text{supp} w(\cdot,s, \delta) \subset \{|y| \leq e^{-s-\delta} = \frac{1-t}{1+\delta-t} \leq 1 - \delta \} \). The \( w \) solve, in their domain,

\[
\partial_s^2 w = \frac{1}{\rho} \text{div} (\rho \nabla w - \rho y \cdot \nabla w) y - \frac{N(N-2)}{4} w + |w|^{4/N-2} w - 2y \cdot \nabla \partial_s w - (N-1) \partial_s w,
\]

where \( \rho(y) = (1 - |y|^2)^{-1/2} \).

The elliptic part of this operator degenerates. In fact, \( \frac{1}{\rho} \text{div} (\rho \nabla w - \rho y \cdot \nabla w) y = \frac{1}{\rho} \text{div} (\rho(I - y \otimes y) \nabla w) \), which is elliptic for \( \{|y| < 1 \} \) and degenerates when \( |y| = 1 \).

This new equation gives us a new set of formulas. The reason for introducing \( \delta > 0 \) is that, on \( \text{supp} w(\cdot,s, \delta) \), \( (1 - |y|^2) \geq \delta \), so we stay away from the degeneracy. Bounds on \( w \) (obvious): \( \int_{B_1} |w|^2 + |\nabla w|^2 + |\partial_s w|^2 \leq C, w \in H^1_0(B_1) \) and hence \( \int_{B_1} |\nabla w|^2 \leq C \). All these bounds are uniform in \( \delta, s \).

We introduce an energy, which will provide a Liapunov function for \( v \):

\[
\bar{E}(w(s)) = \int_{B_1} \left\{ \frac{1}{2} (\partial_s w)^2 + |\nabla w|^2 - (y \cdot \nabla w)^2 \right\} \frac{dy}{(1 - |y|^2)^{1/2}}
+ \int_{B_1} \left\{ \frac{N(N-2)}{8} w^2 - \frac{(N-2)}{2N} |w|^{2r} \right\} \frac{dy}{(1 - |y|^2)^{1/2}}
\]

which is finite for \( \delta > 0 \). Our new formulas are \((0 < s_1 < s_2 < \log 1/\delta)\)

i) \( \bar{E}(w(s_2)) - \bar{E}(w(s_1)) = \int_{s_1}^{s_2} \int_{B_1} \frac{(\partial_s w)^2}{(1 - |y|^2)^{3/2}} \frac{dy}{ds} (\bar{E} \uparrow) \).

I-30
ii) \[
\frac{1}{2} \int_{B_1} \left[ \partial_s w \cdot w - \frac{(1 + N)}{2} w^2 \right] dy \left| \frac{\partial}{(1 - |y|^2)^{1/2}} \right|_{s_1}^{s_2} = - \int_{s_1}^{s_2} \tilde{E}(w(s)) ds + \frac{1}{N} \int_{s_1}^{s_2} \frac{|w|^{2*}}{(1 - |y|^2)^{1/2}} \]
\[
+ \int_{s_1}^{s_2} \int_{B_1} \left\{ (\partial_s w)^2 + \partial_s w \cdot y \cdot \nabla w + \frac{\partial_s w \cdot w |y|^2}{(1 - |y|^2)} \right\} \frac{dy}{(1 - |y|^2)^{1/2}}.
\]

iii) \[
\lim_{s \to \log \frac{1}{\delta}} \tilde{E}(w(s)) = E = E(u_0, u_1), \text{ so that } \tilde{E}(w(s)) \leq E, \text{ for } 0 \leq s < \log \frac{1}{\delta}.
\]

Our first improvement is (\(\delta > 0\)):

**Lemma.**
\[
\int_0^1 \int_{B_1} \frac{(\partial_s w)^2}{(1 - |y|^2)} dy ds \leq C \log \frac{1}{\delta}.
\]

**Proof.** We notice that
\[
-2 \int_0^1 \frac{(\partial_s w)^2}{(1 - |y|^2)} = \frac{d}{ds} \int \left\{ \frac{1}{2} (\partial_s w)^2 + \frac{1}{2} |\nabla w|^2 - (y \cdot w)^2 + \frac{(N - 2)N}{8} w^2 \right\} dy
\]
\[
- \frac{(N - 2)}{2N} |w|^2 \left( - \log(1 - |y|^2) \right) dy
\]
\[
+ \int \left[ \log(1 - |y|^2) + 2 \right] y \cdot \nabla w \partial_s w
\]
\[
- \int \log(1 - |y|^2) (\partial_s w)^2 - 2 \int (\partial_s w)^2.
\]

We integrate between 0 and 1 and drop the next to last term by sign. One finishes by (C-S), support of \(w(-, s, \delta)\). 

**Corollary.**
\[
\int_0^1 \int_{B_1} \frac{|w|^{2*}}{(1 - |y|^2)^{1/2}} \leq C \left( \log \frac{1}{\delta} \right)^{1/2}
\]
\[
\tilde{E}(w(1)) \geq -C \log \left( \frac{1}{\delta} \right)^{1/2}.
\]

**Proof.** The first estimate follows from ii), iii) above, C-S and the Lemma. Note that (CS) give the \(\frac{1}{2}\) power. The second estimate follows from i) and the fact that
\[
\int_0^1 \tilde{E}(w(s)) ds \geq -C \left( \log \frac{1}{\delta} \right)^{1/2},
\]
which follows from the definition of \(\tilde{E}\) and the first bound. 

Our second improvement is:

**Lemma.**
\[
\int_1^\infty \int_{B_1} \frac{(\partial_s w)^2}{(1 - |y|^2)^{3/2}} \leq C \left( \log \frac{1}{\delta} \right)^{1/2}.
\]

I–31
Proof. Use i), iii) and the second bound in Corollary. Note that the upper limit of integration is not important in the bound. It is chosen for the subsequent applications. □

Corollary. \( \exists \tilde{s}_\delta \in (1, (\log \frac{1}{\delta})^{3/4}) \) s.t.
\[
\int_{\tilde{s}_\delta}^{s_\delta + (\log \frac{1}{\delta})^{1/8}} \int_{B_1} \frac{(\partial_s w)^2}{(1 - |y|^2)^{3/2}} \leq \frac{C}{(\log \frac{1}{\delta})^{1/8}}.
\]

Proof. Split \((1, (\log \frac{1}{\delta})^{3/4})\) into disjoint intervals of length \((\log \frac{1}{\delta})^{1/8}\). Their number is \((\log \frac{1}{\delta})^{5/8}\) and \(\frac{5}{8} - \frac{1}{2} = \frac{1}{2}\). Note the length \(\to +\infty\), the bound \(\to 0\). □

Now it is not hard to see that, since \(\tilde{s}_\delta \in (1, (\log \frac{1}{\delta})^{3/4})\), if \(\tilde{s}_\delta = -\log(1 + \delta - \tilde{t}_\delta)\), \(\frac{(1-\tilde{t}_\delta)}{(1+\delta-\tilde{t}_\delta)} - 1 \leq C \delta^{1/4} \to 0\), which is the point of our choice of \((\log \frac{1}{\delta})^{3/4}\). From this and the compactness of \(\tilde{K}\), one can find \(w^*(y, s)\) which solves our self-similar equation in \(s \in [0, S]\), which is a limit of \(w(y, \tilde{s}_\delta + s, \tilde{\delta}_j)\) as \(\tilde{\delta}_j \to 0\), in \(C([0, S]; H^1_0) \times L^2\). The estimate in the corollary shows that \(w^*\) is independent of \(s\). Moreover, the coercivity of \(u\) shows that \(w^* \neq 0\). Thus, \(w^* \in H^1_0(B_1)\), solves the (degenerate) elliptic equation: \((\rho(y) = (1 - |y|^2)^{-1/2}) , \frac{1}{\rho} \text{div}(\rho \nabla w^* - \rho (y \cdot \nabla w^*) y) - \frac{N(N-2)}{4} w^* + |w^*|^{4/N-2} w^* = 0\). We next show that \(w^*\) satisfies the additional (crucial) estimates:
\[
\int_{B_1} \frac{|w^*|^2}{(1 - |y|^2)^{1/2}} + \int_{B_1} \frac{[|\nabla w^*|^2 - (y \cdot \nabla w^*)^2]}{(1 - |y|^2)^{1/2}} < \infty.
\]
Indeed, for the first estimate, it is enough to show that
\[
\int_{\tilde{s}_\delta}^{\tilde{s}_\delta + S} \int_{B_1} \frac{[w(y, s; \delta_j)]^2}{(1 - |y|^2)^{1/2}} \, dy \, ds \leq C \quad \text{for } j \text{ large}.
\]
But this follows from ii) above once more, together with the choice of \(\tilde{s}_\delta\) (Corollary) and (C-S). The proof of the second estimate is similar, using the first one, iii) and the formula for \(\tilde{E}\).

The conclusion of the proof is obtained by showing that a \(w^*\) in \(H^1_0(B_1)\), solving the degenerate elliptic equation, with the additional bounds, must be 0. To do this, we will use unique continuation. Recall that for \(|y| \leq 1 - \eta_0\), \(\eta_0 > 0\), the linear operator is uniformly elliptic, with smooth coefficient and the non-linearity is critical. An argument going back to Trudinger [31] shows that \(w^*\) is bounded on \(|y| \leq 1 - \eta_0\), for each \(\eta_0 > 0\). Hence, if we show that \(w^* \equiv 0\) near \(|y| = 1\), the standard unique continuation principle [13] will show that \(w^* \equiv 0\). Near \(|y| = 1\), our equation is modelled by (in variables \(z \in \mathbb{R}^{N-1}, r \in \mathbb{R}, r > 0 \) near \(r = 0\))
\[
r^{1/2} \partial_r (r^{1/2} \partial_r w^*) + \Delta_z w^* + |w^*|^{4/N-2} w^* = 0.
\]
In these variables, our information is \(w^* \in H^1_0((0, 1] \times (|z| < 1))\) and the additional estimates are:
\[
\int_0^1 \int_{|z| < 1} |w^*(r, z)|^2 \, \frac{dr}{r^{1/2}} \, dz < \infty,
\]
\[
\int_0^1 \int_{|z| < 1} |\nabla_z w^*(r, z)|^2 \, \frac{dr}{r^{1/2}} \, dz < \infty.
\]
We now take advantage of the degeneracy of the equation. We “desingularize” the problem by writing \(r = a^2\), setting \(v(a, z) = w^*(a^2, z)\), so that \(\partial_a v(a, z) = \frac{d}{da} \int_0^a v(\tau, z) \, d\tau\).
\[2a \partial_r w^*(r, z) = 2r^{1/2} \partial_r w^*(r, z).\] Our equation becomes
\[\partial_a^2 v + \Delta_z v + |v|^{4/N - 2} v = 0,\]
where \(0 < a < 1, |z| < 1\) and our bounds are:
\[\int_0^1 \int_{|z|<1} |\nabla_z v(a, z)|^2 \, da \, dz = \int_0^1 \int_{|z|<1} |\nabla_z w^*(r, z)|^2 \frac{dr}{r^{1/2}} \, dz < \infty\]
and
\[\int_0^1 \int_{|z|<1} \left| \frac{\partial_a v(a, z)}{a} \right|^2 \, da \, dz = \int_0^1 \int_{|z|<1} \left| \partial_r w^*(r, z) \right|^2 \, dr \, dz < \infty,\]
and \(v \in H^1_0((0, 1) \times B_1).\) But, from the additional bound we see that \(\left. \partial_a v(a, z) \right|_{a=0} = 0\). One then extends \(v\) by 0 to \(a < 0\) and checks that the extension is an \(H^1\) solution to the same equation. By Trudinger’s argument, it is bounded. But, since it vanishes for \(a < 0\), by unique continuation \([13]\), \(v \equiv 0\). Hence \(w^* \equiv 0\), reaching our contradiction.

References


I–33


Department of Mathematics, University of Chicago, 5734 University Avenue, Chicago, IL 60637-1514, USA
cenk@math.uchicago.edu