Journées

ÉQUATIONS AUX DÉRIVÉES PARTIELLES

Forges-les-Eaux, 6 juin–10 juin 2005

Herbert Koch and Daniel Tataru

Dispersive estimates and absence of embedded eigenvalues

J. É. D. P. (2005), Exposé n° VI, 10 p.

<http://jedp.cedram.org/item?id=JEDP_2005____A6_0>

Journées *Équations aux dérivées partielles*
Forges-les-Eaux, 6 juin–10 juin 2005
GDR 2434 (CNRS)

Dispersive estimates and absence of embedded eigenvalues

Herbert Koch Daniel Tataru

Abstract

In [2] Kenig, Ruiz and Sogge proved

$$\|u\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n)} \lesssim \|Lu\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}$$

provided $n \geq 3$, $u \in C_0^\infty(\mathbb{R}^n)$ and L is a second order operator with constant coefficients such that the second order coefficients are real and nonsingular. As a consequence of [3] we state local versions of this inequality for operators with C^2 coefficients. In this paper we show how to apply these local versions to the absence of embedded eigenvalues for potentials in $L^{\frac{n+1}{2}}$ and variants thereof.

1. Introduction

Let W be a potential which decays at infinity. Then the Schrödinger operator

$$-\Delta_{\mathbb{R}^n} - W$$

has continuous spectrum $[0, \infty)$. In addition its spectrum may contain eigenvalues which could be positive, negative of zero. It is well known that under weak assumptions like

$$\lim_{|x|\to\infty} |x||W(x)| = 0 \tag{1}$$

there are no positive eigenvalues. The argument uses Carleman estimates in three steps as follows. Suppose that

$$-\Delta u - Wu = u$$

with $u \in L^2$, where the eigenvalue is normalized to 1 by scaling. Then one proves that:

1. The eigenfunction u decays faster than polynomially at infinity.

2. If u vanishes faster than polynomially at infinity that u has compact support.

3. If u has compact support then it must vanish.

The assumption (1) on pointwise decay is sharp: There is the famous Wigner-Von Neumann example of a positive eigenvalue and a potential decaying like $1/|x|$ but not better, see [6, 4].

Motivated by the above questions and by other potential applications one seeks to replace the pointwise bound (1) by an L^p bound. In terms of scaling any such bound must necessarily be weaker than (1) due to counterexamples by Jerison and Ionescu ([1]) with potentials concentrated close to $n-1$ dimensional planes.

We obtain absence of positive eigenvalues for a large class of potentials which includes

$$W \in L^{\frac{n+1}{2}}. \tag{2}$$

Our methods are sufficiently robust to allow variable coefficients, gradient potentials and lange range potentials. This extends recent results by Jerison and Ionescu [1] for $W \in L^{n/2}$.

Thus we consider potentials which are the sum of weakly decaying long range potentials V and short range potentials W. We even include the eigenvalue $\lambda > 0$ into the long range potential and study the problem

$$(-\Delta - V)u = Wu. \tag{3}$$

under the following assumptions:

Assumption A1 (The long range potential)**.** *The following inequalities hold.*

$$|V| + |x||DV| + |x|^2|D^2V| \lesssim 1, \tag{4}$$

$$\liminf_{|x|\to\infty} V > 0, \tag{5}$$

and

$$\tau_0 := -\liminf_{|x|\to\infty} \frac{x \cdot \nabla V}{4V} < 1/2. \tag{6}$$

Bounds on derivatives as above will occur at several places in this work. To simplify and to unify the notation we define corresponding function spaces.

Definition 1. $C^2_{\langle x \rangle}$ *is the space of* C^2_{loc} *functions for which the following norm is finite:*

$$\|f\|_{C^2_{\langle x \rangle}} := \max\{\sup_x |f(x)|, \sup \langle x \rangle |Df|, \sup \langle x \rangle^2 |D^2 f|\}$$

The condition (4) can now be written as $V \in C^2_{\sup\langle x \rangle}$.

The bound (5) on V corresponds to the condition $\lambda > 0$ while the last bound (6) says that for large $|x|$ the function $|x|^2$ is strictly convex along the null Hamilton flow for $-\Delta - V$, and thus guarantees nontrapping outside a compact set.

Assumption A2 (The short range potential)**.** *Multiplication by the potential W has the mapping property*

$$W : W^{\frac{1}{n+1},\frac{2(n+1)}{n-1}} \to W_{loc}^{-\frac{1}{n+1},\frac{2(n+1)}{n+3}}$$

and W can be decomposed as $W = W_1 + W_2$ where

$$\limsup_{j\to\infty} \sup_{u\in C^\infty} \frac{\|W_1 u\|_{W^{-\frac{1}{n+1},\frac{n+1}{n+3}}(\{x|2^j\le|x|\le 2^{j+1}\})}}{\|u\|_{W^{\frac{1}{n+1},\frac{n+1}{n-1}}(\{x|2^j\le|x|\le 2^{j+1}\})}} < \delta \tag{7}$$

$$\limsup_{|x|\to\infty} |x||W_2(x)| < \delta. \tag{8}$$

The W_2 component corresponds to the L^2 Carleman estimates. The class of allowed W_1 potentials includes $L^{\frac{n}{2}}$ and $L^{\frac{n+1}{2}}$ or even better $l^{\frac{n+1}{2}}(L^{\frac{n}{2}})$ where the $l^{\frac{n+1}{2}}$ norm is taken with respect to a partition of $\mathbb{R}^n$ into unit cubes.

Our main result is

Theorem 2. *Assume that V and W satisfy Assumptions A1 and A2, let $\tau_1 > \tau_0$ and assume that δ is sufficiently small. Let $u \in H^1_{loc}(\mathbb{R}^n)$ satisfy* (3) *and $(1+|x|^2)^{\tau_1-\frac{1}{2}}u \in L^2$. Then $u \equiv 0$.*

The proof is trivial if $n = 1$. For $n \ge 2$ it uses Carleman estimates, following the same three steps indicated above. A combined L^2- L^p Carleman inequality replaces the previous L^2 Carleman inequalities. Proving such inequalities is a highly nontrivial task and relies on the bounds established in [3]. Conjugation of the operator $-\Delta - V$ with the weight of the Carleman inequality leads to a non-selfadjoint partial differential equation. A pseudo-convexity type condition is satisfied, but it degenerates for large x. This is related to the fact that the anti-selfadjoint part of the conjugated operator decays for large x in relevant coordinates.

Compared to earlier work and to the steps outlined above, we also consider a different family of weights in the Carleman estimates. Precisely, we begin with weights of the form $h(x) = e^{\tau\sqrt{|x|}}$ for part 2 of the argument, which we then flatten at infinity for part 1. This yields a more robust argument, and also better results in the variable coefficient case.

2. Carleman estimates and embedded eigenvalues

As explained above the proof depends on Carleman inequalities. In this section we explain the Carleman inequalities and their application whereas most of the proofs are postponed to the remaining sections.

Let $1 \le p \le \infty$ and $s \in \mathbb{R}$. We define the Sobolev space $W^{s,p}(\mathbb{R}^n)$ by the norm $\|f\|_{W^{s,p}} = \|(1+|D|^2)^{s/2} f\|_{L^p}$ and $W^{s,p}(U)$ for open subsets U of $\mathbb{R}^n$ through its norm which is the infimum of the norm of extensions.

Given a measurable function f and the Sobolev space $W^{s,q}$ we define the norm

$$\|f\|_{l^p W^{s,q}} = \left(\sum_{j=1}^{\infty} \|f\|^p_{W^{s,q}(\{2^{j-1}\le|x|\le 2^{j+1}\})} \right)^{1/p}$$

with the obvious modification for $p = \infty$.

Our Carleman estimates have the form

$$\|e^{h(\ln(|x|))}v\|_{l^2 W^{\frac{1}{n+1},\frac{2(n+1)}{n-1}}} + \|e^{h(\ln(|x|))}\rho v\|_{L^2} \lesssim \inf_{f_1+f_2=(-\Delta-V)v} \|e^{h(\ln(|x|))}\rho^{-1} f_1\|_{L^2} + \|e^{h(\ln(|x|))} f_2\|_{l^2 W^{-\frac{1}{n+1},\frac{2(n+1)}{n+3}}} \tag{9}$$

where ρ is given by

$$\rho = \left(\frac{h'(\ln(|x|))}{|x|^2} + \frac{h'(\ln(|x|))^2 h''_+(\ln(|x|))}{|x|^4} \right)^{\frac{1}{4}} \tag{10}$$

with h''_+ denoting the positive part of h''. As a general rule, the function h is chosen to be

(a) increasing, $h' \geq \tau_0$, with $h'(0)$ large.

(b) slowly varying on the unit scale, $|h''| \lesssim h'$, $|h'''| \lesssim h'$.

(c) strictly convex for as long as $h'(\ln(|x|)) \gtrsim |x|$.

More precise choices are made later on for convenience, but the estimates are in effect true for all functions h satisfying the above conditions.

The two terms in ρ have different origins. The second one simply measures the effect of the convexity of the function h. The first one, on the other hand, is due to the presence of the long range potential, which provides some extra convexity for large $|x|$.

A simplifying assumption consistent with the choices of weights in this paper is to strengthen (c) to

(c)' $h''(\ln(|x|)) \approx h'(\ln(|x|))$ for as long as $h'(\ln(|x|)) \gtrsim |x|$.

This allows us to simplify the expression of ρ to

$$\rho = \left(\frac{h'(\ln(|x|))}{|x|^2} \left(1 + \frac{h'(\ln(|x|))^2}{|x|^2} \right) \right)^{\frac{1}{4}} \tag{11}$$

Our Carleman estimates use weights which grow exponentially, but also allow for the possibility of leveling off the weight for large enough $|x|$.

Proposition 3. *Suppose that V satisfies Assumption A1. There is a universal constant ε_0 such that with*

$$h'_\epsilon(t) = \tau_1 + (\tau e^{\frac{t}{2}} - \tau_1) \frac{\tau^2}{\tau^2 + \epsilon e^t} \tag{12}$$

(9) *holds with $h = h_\epsilon$ for all $|\varepsilon| \leq \varepsilon_0$, v supported in $|x| > 1$ and satisfying $|x|^{\tau_1 - \frac{1}{2}} v \in L^2$, uniformly with respect to τ large enough.*

Theorem 2 is a standard consequence of this Carleman inequality.

3. A general dispersive estimate for second order operators

In this section we study the second order operator[1]

$$L_\mu = \partial_i a^{ij}(x)\partial_j + \mu^2 c(x) - i\mu(b_j(x)\partial^j + \partial_j b^j(x)),$$

in the unit ball B. Here μ is sufficiently large and plays the role of a semiclassical parameter. Concerning the type and regularity of the coefficients we assume that

$$(REG) \quad \begin{cases} \text{the matrix } (a^{ij}(x)) \text{ is real, symmetric and positive definite} \\ \text{the functions } a^{ij},\ b^i \text{ and } c \text{ are of class } C^2 \end{cases}$$

We define the symbol

$$l(x,\xi) = -\xi_i a^{ij}(x)\xi_j + c(x) + 2b_j\xi_j$$

The real part of l is a second degree polynomial in ξ with characteristic set

$$char_x \Re l(x,\xi) = \{\xi \in \mathbb{R}^n;\ \Re l(x,\xi) = 0\}$$

The geometric assumption on the operator L is

$$(GEOM) \quad \begin{cases} \text{for each } x \text{ the characteristic set } char_x\Re l(x,\xi) \\ \text{is an ellipsoid of size } O(1). \end{cases}$$

Our third hypothesis is concerned with the size of the Poisson bracket of the real and imaginary part of L. We are interested in a principal normality type condition of the form

$$|\{\Re l(x,\xi), \Im l(x,\xi)\}| \lesssim \delta + |\Re l(x,\xi)| + |\Im l(x,\xi)| \tag{13}$$

where the relevant range for δ is $\mu^{-1} < \delta \ll 1$. This would suffice for our purposes if in addition we knew that all the coefficients of l are of class C^3. In general for technical reasons we need to replace the inequality with a decomposition

$$\{\Re l, \Im l\}(x,\xi) = \delta q_0(x,\xi) + q_1^r(x,\xi)\Re l(x,\xi) + q_1^i(x,\xi)\Im l(x,\xi) + q_2(x,\xi) \tag{14}$$

Thus our last assumption has the form

$$(PN) \quad \begin{cases} \text{the Poisson bracket } \{\Re l, \Im l\} \text{ admits a representation (14)} \\ \text{where the } q_i\text{'s are smooth in } \xi, \text{ of class } C^i \text{ in } x \text{ and satisfy} \\ |q_0| \lesssim 1, \qquad |q_1^r| + |q_1^i| \lesssim 1, \qquad |q_2| \lesssim |l| \end{cases}$$

For L in the class of operators described above we are interested in constructing a parametrix T which has good $L^{p'} \to L^p$ and $L^2 \to L^p$ mapping properties, while the errors are always measured in L^2. A dual form of this also allows us to estimate the L^p norm of a function u in terms of the L^2 norms of u and Lu.

In the context of the Carleman estimates such parametrices allow us to superimpose local $L^{p'} \to L^p$ bounds on top of the global $L^2 \to L^2$ estimates in order to obtain a global $L^{p'} \to L^p$ bound.

[1] We use the summation convention here and in the sequel.

Such estimates are dispersive in nature and are strongly related to the spreading of singularities in the parametrix T. This in turn is determined by the nonvanishing curvatures of the characteristic set $char_x \Re l(x,\xi)$.

If L has constant coefficients and real symbol then the theorem below is nothing but a reformulation of the restriction theorem. If L has real symbol but variable coefficients then we are close to the spectral projection estimates of C. Sogge [5]. In the case when L has constant coefficients but complex symbol some bounds of this type were obtained in [2].

In the more general case considered here we rely on bounds and parametrix constructions in author's earlier paper [3]. These apply to principally normal operators. The operator L_μ is principally normal on the unit spatial scale only if $\delta \approx \mu^{-1}$. Otherwise, we use a better spatial localization to the $(\delta\mu)^{-\frac{1}{2}}$ scale. On one hand L_μ is principally normal on this scale, while on the other hand this localization is compatible with the L^2 estimates and this allows us to easily put the pieces back together.

All Sobolev norms in the theorem below are flattened at frequency μ instead of frequency 1 as usual. Hence we introduce the notation

$$W^{s,p}_\mu = \{u \in S';\ (\mu^2 + D^2)^{\frac{s}{2}} u \in L^p\}$$

with the corresponding norm.

We note that the operator L is elliptic at frequencies larger than μ so all the estimates are trivial in that case. All the interesting action takes place at frequency $\lesssim \mu$, where we can identify all Sobolev norms with L^p norms.

Theorem 4. *Suppose that the operator L_μ satisfies the conditions (REG), (GEOM) and (PN) for some $\delta > \mu^{-1}$. Let $\phi \in C(B_2(0))$ have compact support. Then*

A) There exists an operator T such that

$$\begin{aligned} \|Tf\|_{W_\mu^{\frac{1}{n+1},\frac{2(n+1)}{n-1}}} &+ (\delta\mu)^{1/4}\mu^{-1/2}\|Tf\|_{H^1_\mu} \\ &\lesssim \inf_{f=f_1+f_2} (\delta\mu)^{-1/4}\mu^{-1/2}\|f_1\|_{L^2} + \|f_2\|_{W_\mu^{-\frac{1}{n+1},\frac{2(n+1)}{n+3}}} \end{aligned} \tag{15}$$

and

$$\begin{aligned} (\delta\mu)^{-1/4}\mu^{-1/2}\|LT\phi f - \phi f\|_{L^2} &\lesssim \\ \inf_{f=f_1+f_2} (\delta\mu)^{-1/4}\mu^{-1/2}\|f_1\|_{L^2} &+ \|f_2\|_{W_\mu^{-\frac{1}{n+1},\frac{2(n+1)}{n+3}}} \end{aligned} \tag{16}$$

B) For all functions u in $B_2(0)$ we have

$$\begin{aligned} \|\phi u\|_{W_\mu^{\frac{1}{n+1},\frac{2(n+1)}{n-1}}} &\lesssim (\delta\mu)^{1/4}\mu^{1/2}\|u\|_{L^2} \\ &+ \inf_{Lu=f_1+f_2} (\delta\mu)^{-1/4}\mu^{-1/2}\|f_1\|_{L^2} + \|f_2\|_{W_\mu^{-\frac{1}{n+1},\frac{2(n+1)}{n+3}}} \end{aligned} \tag{17}$$

C) Suppose that in addition the problem is pseudoconvex in the sense that

$$q_0(x,\xi) \approx \delta \gg \tau^{-1} \qquad x \in B_2(0) \tag{18}$$

Then for all functions u with compact support in $B_2(0)$ we have

$$\begin{aligned} \|u\|_{W_\mu^{\frac{1}{n+1},\frac{2(n+1)}{n-1}}} &+ (\delta\mu)^{1/4}\mu^{1/2}\|u\|_{L^2} \\ &\lesssim \inf_{Lu=f_1+f_2+f_3} (\delta\mu)^{-1/4}\mu^{-1/2}\|f_1\|_{L^2} + \|f_2\|_{W_\mu^{-\frac{1}{n+1},\frac{2(n+1)}{n+3}}} \end{aligned} \tag{19}$$

4. The L^2 Carleman estimates

In this section we obtain the L^2 Carleman inequalities.

Proposition 5. *Suppose that V satisfies Assumption A1. Let h be as in* (12) *and ρ as in* (10)*. Then for all u satisfying $|x|^{\tau_1-\frac{1}{2}}u \in L^2$ we have*

$$\|e^{h(\ln|x|)}\rho u\|_{L^2} + \left\|\frac{|x|}{h'(\ln|x|)+|x|}e^{h(\ln|x|)}\rho\nabla u\right\|_{L^2} \lesssim \|e^{h(\ln|x|)}\rho^{-1}(\Delta+V)u\|_{L^2}. \tag{20}$$

uniformly with respect to τ sufficiently large and $0 \le \varepsilon \le \varepsilon_0$.

Proof. We use a conformal change of coordinates

$$t = \ln|x|, \qquad y = x/|x| \in \mathbb{S}^{n-1}$$

Denote

$$\Delta u = g$$

and set

$$v(t,y) = e^{(n-2)t/2}u(e^t y), \quad f(t,y) = e^{(n+2)t/2}g(e^t y)$$

A routine computation shows that

$$|x|^{(n+2)/2}(\Delta+V)|x|^{(n+2)/2} = \frac{\partial^2}{\partial t^2} + \Delta_{\mathbb{S}^{n-1}} - ((n-2)/2)^2$$

therefore v solves the equation

$$Lv = f, \qquad L = \partial_t^2 + \Delta_{\mathbb{S}^{n-1}} - ((n-2)/2)^2 + e^{2t}V \tag{21}$$

We also note that part of Assumption A1 in the new coordinates we get

$$-\liminf_{t\to\infty}\frac{V_t}{4V} = \tau_0 < \frac{1}{2}$$

By (22) we slightly readjust τ_0 and choose t_0 so that

$$-\frac{V_t}{4V} \le \tau_0 < \frac{1}{2}, \qquad t > t_0 \tag{22}$$

For any exponential weight h we have

$$\int e^{2h(\ln|x|)}|u|^2dx = \int_{\mathbb{R}}\int_{\mathbb{S}^n} e^{2h(t)+nt}|u(ty)|^2\,dt\,dy = \|e^{h(t)}e^t v\|_{L^2}^2, \tag{23}$$

$$\int e^{2h(\ln|x|)}|g|^2 dx = \int_{\mathbb{R}} \int_{\mathbb{S}^n} e^{2h(t)+nt} |g(ty)|^2 \, dt \, dy = \| e^{h(t)} e^{-t} f \|_{L^2}^2, \tag{24}$$

Hence, in the new coordinates the bound (20) becomes

$$\| e^{h(t)} \rho_1 v \|_{L^2} + \| e^{h(t)} \frac{\rho_1}{e^t + h'(t)} \nabla v \|_{L^2} \lesssim \| e^{h(t)} \rho_1^{-1} f \|_{L^2}, \tag{25}$$

where ∇v is the gradient of v with respect to y and t and, by (11),

$$\rho_1(t) = e^t \rho = \Big(h'(t)(e^{2t} + h'(t)^2) \Big)^{\frac{1}{4}}$$

To prove the above bound one would like to follow a standard strategy. This means conjugating the operator with respect to the exponential weight, and producing a commutator estimate for the self-adjoint and the skew-adjoint part of the conjugated operator. There are problem with this approach in the region where $h'(t)$ is small. A combination of usual commutator argumnent with an energy inequality implies estimate (25). $\square$

5. The L^p Carleman inequality

In this section we prove Proposition 3. We first conjugate with respect to the exponential weight. If we set $w = e^{h(\ln(|x|))} v$ then we can rewrite (9) in the form

$$\|w\|_{l^2 W^{\frac{1}{n+1}, \frac{2(n+1)}{n-1}}} + \|\rho w\|_{L^2} \lesssim \inf_{L_h w = f_1 + f_2} \|\rho^{-1} f_1\|_{L^2} + \|f_2\|_{l^2 W^{-\frac{1}{n+1}, \frac{2(n+1)}{n+3}}}$$

where

$$L_h = \Delta + Vw + h'(\ln|x|)^2 |x|^{-2} - h'(\ln|x|) \Big[\nabla \frac{x}{|x|^2} + \frac{x}{|x|^2} \nabla \Big]$$

We want to apply Theorem 4 on dyadic annuli

$$A_j = \{ x | 2^{j-1} < |x| < 2^{j+1} \}$$

The rescaling $y = 2^{-j} x$ transforms this set to A_0 and the operator L_h to

$$L_h^j = \Delta + 2^{2j} \tilde{V} + h'(\ln(2^j |y|))^2 |y|^{-2} - h'(\ln(2^j|y|)) \Big[\nabla \frac{y}{|y|^2} + \frac{y}{|y|^2} \nabla \Big]$$

We verify that we can apply Theorem 4 to L_j. Since h' varies slowly on the unit scale we can take the corresponding value for μ to be

$$\mu_j = \sqrt{2^{2j} + h'(j \ln 2)^2}$$

The coefficients b and c are given by

$$c = \mu_j^{-2} (2^{2j} V + h'(\ln(2^j|y|))^2 / |y|^2), \qquad b_j = -\frac{h'(\ln(2^j|y|))}{\mu_j} \frac{y_j}{|y|^2}$$

and are clearly of class C^2 and size $O(1)$. We have

$$\Re l_h^j(x, \xi) = -\xi^2 + c, \qquad \Im l_h^j(x, \xi) = 2b \cdot \xi$$

Their Poisson bracket has the form

$$
\begin{aligned}
\{-|\xi|^2+c, b\cdot\xi\} =& \frac{h'(t)}{\mu|y|^2}(-|\xi|^2+c) + 2y\cdot\xi\left(\frac{1}{|y|^4} - \frac{h''(t)}{h'(t)|y|^3}\right) b\cdot\xi \\
& - \frac{2^{2j}h'(t)}{|y|^2\mu_j^3} y\cdot\nabla V - \frac{2h'(t)^2h''(t)}{|y|^4\mu_j^3}, \qquad t=\ln(2^j|y|)
\end{aligned}
$$

Then we can apply Theorem 4 with δ comparable to the size of the third term. For our choice of h we have $|h''| \lesssim h'$ and also

$$
h''(t) < 0 \implies h'(t) \ll e^t
$$

Hence we can choose

$$
\delta_j = \mu_j^{-3}\left(2^{2j}h'(j\ln 2) + h'(j\ln 2)^2 h''_+(j\ln 2)\right)
$$

Let $\phi \in C_0^\infty(\mathbb{R})$ be a nonnegative function supported in $[-1,1]$ with

$$
\sum_{j=-\infty}^{\infty} \phi^2(t-j) = 1
$$

and let $\phi_j(x) = \phi(\ln|x| - j)$. After rescaling, part A of Theorem 4 yields a parametrix T_j for L_h in A_j with the property that

$$
\begin{aligned}
&\|T_j g\|_{W^{\frac{1}{n+1},\frac{2(n+1)}{n-1}}} + \|\rho T_j g\|_{L^2} + \|\rho\frac{|x|}{h'(\ln|x|)+|x|}\nabla(T_j g)\|_{L^2} \\
&+\|\rho^{-1}(L_h T_j - 1)\phi_j g\|_{L^2} \lesssim \inf_{g=g_1+g_2} \|\rho^{-1}g_1\|_{L^2(A_j)} + \|g_2\|_{W^{-\frac{1}{n+1},\frac{2(n+1)}{n+3}}(A_j)}.
\end{aligned}
$$

We define a parametrix for L_h by

$$
T = \sum_{j=0}^{\infty} \phi_j T_j \phi_j
$$

Summing up the bounds on T_j we obtain a bound for T,

$$
\|Tg\|_{l^2W^{\frac{1}{n+1},\frac{2(n+1)}{n-1}}} + \|\rho Tg\|_{L^2} \lesssim \inf_{g=g_1+g_2} \|\rho^{-1}g_1\|_{L^2} + \|g_2\|_{l^2W^{-\frac{1}{n+1},\frac{2(n+1)}{n+3}}}.
$$

We also compute the error

$$
1 - L_h T = \sum_{j=0}^{\infty} \phi_j(1-L_hT_j)\phi_j - \sum_{j=0}^{\infty}[L_h,\phi_j]T_j\phi_j
$$

Since

$$
[L_h,\phi_j] = O(|x|^{-1})\nabla + O(h'(\ln|x|)|x|^{-2})
$$

and

$$
|x|^{-1} \lesssim \rho^2\frac{|x|}{h'(\ln|x|)+|x|}, \qquad h'(\ln|x|)|x|^{-2} \lesssim \rho^2
$$

we can bound the error by

$$\|\rho^{-1}(1-LT)g\|_{L^2} \lesssim \inf_{g=g_1+g_2} \|\rho^{-1}g_1\|_{L^2} + \|g_2\|_{l^2W^{-\frac{1}{n+1},\frac{2(n+1)}{n+3}}}.$$

Now, after the construction of the parametrix the assertion of Proposition 3. We sketch the argument. Split w into

$$w = v + TLw$$

Then the second term satisfies the desired bounds while for the first we know that

$$\|\rho^{-1}Lv\|_{L^2} = \|\rho^{-1}(LT-1)Lw\|_{L^2} \lesssim \inf_{Lw=g_1+g_2} \|\rho^{-1}g_1\|_{L^2} + \|g_2\|_{l^2W^{-\frac{1}{n+1},\frac{2(n+1)}{n+3}}}.$$

Proposition 5 allows us to also estimate $\|\rho v\|_{L^2}$.

On the other hand by Theorem 4, B rescaled and applied to v in A_j we get

$$\|\phi_j v\|_{W^{\frac{1}{n+1},\frac{2(n+1)}{n-1}}} \lesssim \|\rho v\|_{L^2(A_j)} + \|\rho^{-1}Lv\|_{L^2(A_j)}$$

and after summation in j,

$$\|v\|_{l^2W^{\frac{1}{n+1},\frac{2(n+1)}{n-1}}} \lesssim \|\rho v\|_{L^2} + \|\rho^{-1}Lv\|_{L^2}$$

thereby concluding the proof.

References

[1] A. D. Ionescu and D. Jerison. On the absence of positive eigenvalues of Schrödinger operators with rough potentials. *Geom. Funct. Anal.*, 13(5):1029–1081, 2003.

[2] C. E. Kenig, A. Ruiz, and C. D. Sogge. Uniform Sobolev inequalities and unique continuation for second order constant coefficient differential operators. *Duke Math. J.*, 55(2):329–347, 1987.

[3] Herbert Koch and Daniel Tataru. Dispersive estimates for principally normal pseudodifferential operators. *Comm. Pure Appl. Math.*, 58(2):217–284, 2005.

[4] Michael Reed and Barry Simon. *Methods of modern mathematical physics. IV. Analysis of operators*. Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1978.

[5] Christopher D. Sogge. *Fourier integrals in classical analysis*, volume 105 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1993.

[6] J. von Neumann and E.P. Wigner. Über merkwürdige diskrete Eigenwerte. *Z. Phys.*, 30:465–467, 1929.

Fachbereich Mathematik, Universität Dortmund

Department of Mathematics, University of California, Berkeley