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Singular Yang-Mills connections


<http://www.numdam.org/item?id=JEDP_1995___A8_0>
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Lecture given at the Partial Differential Equations Meeting
in Saint Jean de Monts, May 29 – June 2, 1995
by Johan Råde at Lund

First I wish to thank the organizers for inviting me to speak at this conference. I will speak about an intriguing partial differential equation that arises in gauge theory. Gauge theory is mainly concerned with the Yang-Mills equation and related equations, such as the Ginzburg-Landau equation (with a magnetic field), the Yang-Mills-Higgs equation and the Seiberg-Witten equation. I will talk about solutions to the Yang-Mills equation with singularities. In a moment I will write down the Yang-Mills equation, in full detail. First I just want to mention the origin of these singular solutions.

The Yang-Mills equation has mainly been studied by topologists and geometers, in particular in connection with the topology of smooth 4-manifolds. In the early 80’s Donaldson showed that Yang-Mills equation could be used as a powerful tool in smooth 4-manifold topology. In particular he defined new invariants for smooth 4-manifolds. These invariants reflect the topology of solution spaces for Yang-Mills equation on the 4-manifold. They are now known as Donaldson polynomials. These developments were a bit of a shock for the 4-manifold topologists. They were suddenly forced to learn about partial differential equations. Many of them did so very successfully. For a brief introduction to the applications of gauge theory to 4-manifold topology see [L] and for a comprehensive text see [DK]. Both books are masterpieces of mathematical exposition.

The Donaldson polynomials were at first extremely hard to calculate. However, a few years ago Kronheimer and Mrowka discovered a method for calculating them in a large number of cases. The key was to introduce a new type of Donaldson polynomials defined using spaces of singular Yang-Mills connections, [K], [KM1], [KM2], see also [R3]. The purpose of my own work has been to understand these singular Yang-Mills connections from the point of view of partial differential equations.

In October last fall a new equation and new invariants were introduced by Seiberg and Witten. Within a few weeks several famous conjectures about 4-manifolds had been settled. Priority often was a matter of days. An interesting account of these developments is given in [T]. It is not clear if 4-manifold topologists are interested in singular Yang-Mills connections any more.
§1. The Yang-Mills equation

Recall that if
\[ \sigma = \sum_{i_1 < \ldots < i_p} \sigma_{i_1 \ldots i_p} dx_{i_1} \wedge \ldots \wedge dx_{i_p} \]
is a differential \(p\)-form, and
\[ \omega = \sum_{j_1 < \ldots < j_q} \omega_{j_1 \ldots j_q} dx_{j_1} \wedge \ldots \wedge dx_{j_q} \]
then the exterior derivative of \(\sigma\) is defined to be the \((p+1)\)-form
\[
(1.1) \quad d\sigma = \sum_j \sum_{i_1 < \ldots < i_p} \frac{\partial}{\partial x_j} \sigma_{i_1 \ldots i_p} dx_j \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_p}.
\]
and the wedge product of \(\sigma\) and \(\omega\) is defined to be the \((p+q)\)-form
\[
(1.2) \quad \sigma \wedge \omega = \sum_{i_1 < \ldots < i_p} \sum_{j_1 < \ldots < j_q} \sigma_{i_1 \ldots i_p} \omega_{j_1 \ldots j_q} dx_{i_1} \wedge \ldots \wedge dx_{i_p} \wedge dx_{j_1} \wedge \ldots \wedge dx_{j_q}.
\]
These operations satisfy the identities
\[
\omega \wedge \sigma = (-1)^{pq} \sigma \wedge \omega \\
d^2 \sigma = 0 \\
d(\sigma \wedge \omega) = d\sigma \wedge \omega + (-1)^p \sigma \wedge dw.
\]
The adjoint of the exterior derivative (with respect to the Euclidean metric \(\sum dx_i^2\)) is given by
\[
d^* \sigma = \sum_{\nu=1}^p \sum_{i_1 < \ldots < i_p} (-1)^{\nu} \frac{\partial}{\partial x_{i_\nu}} \sigma_{i_1 \ldots i_p} dx_{i_1} \wedge \ldots \wedge dx_{i_{\nu-1}} \wedge dx_{i_{\nu+1}} \wedge \ldots \wedge dx_{i_p}.
\]
Now, let \(G\) be a compact Lie group. Let \(\mathfrak{g}\) be the Lie algebra of \(G\). I usually think of \(G\) as a group of matrices; that simplifies the notation a good deal. In particular, then the Lie bracket \([X, Y]\) is simply given by \(XY - YX\). In fact, I will soon restrict my attention to the case \(G = SU(2)\). This will simplify the notation even further.

In gauge theory one considers differential forms \(\sigma\) the coefficient \(\sigma_{i_1 \ldots i_p}\) take values in the Lie algebra \(\mathfrak{g}\). These are called \(\mathfrak{g}\)-valued forms. We can still define the exterior
derivative of \( \sigma \) by by (1.1). However, the right hand side of (1.2) is quite meaningless if \( \sigma_{i_1 ... i_p} \) and \( \omega_{j_1 ... j_q} \) are \( g \)-valued forms. Instead we define the bracket of \( \sigma \) and \( \omega \) as

\[
[\sigma, \omega] = \sum_{i_1 < ... < i_p} \sum_{j_1 < ... < j_q} [\sigma_{i_1 ... i_p}, \omega_{j_1 ... j_q}] dx_{i_1} \wedge ... \wedge dx_{i_p} \wedge dx_{j_1} \wedge ... \wedge dx_{j_q}.
\]

Then

\[
[\omega, \sigma] = (-1)^{pq+1}[\sigma, \omega]
\]

\[
d^2 \sigma = 0
\]

\[
d[\sigma, \omega] = [d\sigma, \omega] + (-1)^p[\sigma, d\omega].
\]

**Gauge transformations.** The Lie group \( G \) acts on the Lie algebra \( g \) by conjugation; for \( g \in G \) and \( X \in g \) we can form \( g X g^{-1} \in g \). If \( \sigma \) is a \( g \)-valued \( p \)-form and \( g \) is a \( G \)-valued function, then we can form a new \( g \)-valued \( p \)-form

\[
g.\sigma = \sum_{i_1 < ... < i_p} g \sigma_{i_1 ... i_p} g^{-1} dx_{i_1} \wedge ... \wedge dx_{i_p}.
\]

We say that \( \sigma \) and \( g.\sigma \) are gauge-equivalent. This establishes an equivalence relation on \( g \)-valued \( p \)-forms.

We can now define gauge theory; it is the study of objects that are invariant under gauge transformations. One example is the commutator of \( g \)-valued forms; it is clear that

\[
g.[\sigma, \omega] = [g.\sigma, g.\omega].
\]

**Covariant derivatives.** The exterior derivative is not gauge-invariant; we have

\[
g.d\sigma = \sum_j \sum_{i_1 < ... < i_p} g \left( \frac{\partial}{\partial x_j} \sigma_{i_1 ... i_p} \right) g^{-1} dx_j \wedge dx_{i_1} \wedge ... \wedge dx_{i_p}
\]

but

\[
d(g.\sigma) = \sum_j \sum_{i_1 < ... < i_p} \frac{\partial}{\partial x_j} (g \sigma_{i_1 ... i_p} g^{-1}) dx_j \wedge dx_{i_1} \wedge ... \wedge dx_{i_p}.
\]

In general these differ by terms that involve the derivatives of \( g \). A calculation shows that

\[
g.d\sigma = d(g.\sigma) + [A, g.\sigma]
\]

where

\[
A = -(dg)g^{-1} = -\sum_i \frac{\partial g}{\partial x_i} g^{-1} dx_i.
\]

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This suggests that we define the covariant exterior derivative of $\sigma$ as

\[ d_A = d\sigma + [A, \sigma] = \sum_j \sum_{i_1 < \cdots < i_p} \left( \frac{\partial}{\partial x_j} \sigma_{i_1 \cdots i_p} + [A_j, \sigma_{i_1 \cdots i_p}] \right) dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p}. \]

The covariant derivative depends on the choice of a $\mathfrak{g}$-valued 1-form $A$. We call $A$ a $G$-connection. A short calculation using the Jacobi identity shows that for any connection $A$

\[ d_A[\sigma, \tau] = [d_A\sigma] + (-1)^p[\sigma, d_A\omega]. \]

Another short calculation shows that exterior covariant derivative is gauge-invariant in the sense that

\[ g.d_A\sigma = d_{g.A}(g.\sigma) \]

where

\[ g.A = gA^{-1} - (dg)g^{-1} = \sum_i \left( gA_i g^{-1} - \frac{\partial g}{\partial x_i} g^{-1} \right) dx_i. \]

Note that a connection transforms differently than an ordinary $\mathfrak{g}$-valued 1-form. As before, we say that $A$ and $g.A$ are gauge-equivalent. This establishes an equivalence relation on the set of $G$-connections.

The adjoint of $d_A$ is given by

\[ d_A^*\sigma = \sum_{\nu=1}^p \sum_{i_1 < \cdots < i_p} (-1)^{\nu} \left( \frac{\partial}{\partial x_{i_\nu}} \sigma_{i_1 \cdots i_\nu} + [A_{i_\nu}, \sigma_{i_1 \cdots i_p}] \right) dx_{i_\nu} \wedge \cdots \wedge dx_{i_{\nu-1}} \wedge dx_{i_{\nu+1}} \wedge \cdots \wedge dx_{i_p}. \]

**Curvature.** We do not have $d_A^2\omega = 0$. Instead a short calculation shows that

\[ d_A^2\omega = [F_A, \omega] \]

where $F_A$ is the $\mathfrak{g}$-valued 2-form

\[ F_A = dA + \frac{1}{2}[A, A] = \frac{1}{2} \sum_{ij} \left( \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} + [A_i, A_j] \right) dx_i \wedge dx_j. \]

The 2-form $F_A$ is called the curvature of the connection $A$. Another short calculation shows that curvature is gauge-invariant, i.e.

\[ g.F_A = F_{g.A}. \]
Yet another short computation shows that
\[ d_A F_A = 0. \]
This is known as the Bianchi identity.

**Yang-Mills equation.** Let \( A \) be a connection in a domain \( \Omega \) in \( \mathbb{R}^n \). One defines the energy of the connection \( A \) as
\[
\mathcal{W}(A) = \frac{1}{2} \int_{\Omega} |F_A|^2 dx = \frac{1}{2} \sum_{ij} \int_{\Omega} \left| \frac{\partial A}{\partial x_i} - \frac{\partial A}{\partial x_j} + [A_i, A_j] \right|^2 dx.
\]
A short calculation shows that the Euler-Lagrange equation for this energy functional is
\[ d^* F_A = 0. \]
This equation is known as the Yang-Mills equation. A connection \( A \) that satisfies the Yang-Mills equation fully we get
\[
\sum_{j=1}^{n} \left( \frac{\partial^2 A_i}{\partial x_j^2} - \frac{\partial A_i}{\partial x_j} \frac{\partial A_j}{\partial x_i} + \left[ \frac{\partial A_j}{\partial x_i}, A_i \right] + \left[ \frac{\partial A_i}{\partial x_j}, A_j \right] - 2 \left[ \frac{\partial A_i}{\partial x_j}, A_j \right] + [A_j, [A_j, A_i]] \right) = 0
\]
for \( i = 1, \ldots, n \). The most convenient way to write the equation is
\[ d^* dA + \{ A \otimes \nabla A \} + \{ A \otimes A \otimes A \} = 0. \]
Here we write \( \{ A \otimes \nabla A \} \) for terms that are linear in \( A_i \) and \( \partial A_i / \partial x_j \) et.c.

The Yang-Mills energy is gauge invariant, i.e.
\[
\mathcal{W}(g.A) = \mathcal{W}(A).
\]
Hence the Yang-Mills equation is gauge-invariant. In particular, if \( A \) is a Yang-Mills connection, then \( g.A \) is also a Yang-Mills connection.

To define the Yang-Mills energy and the Yang-Mills equation on a manifold, we need to choose a Riemannian metric. It is easy to verify that in four dimensions the Yang-Mills energy, and hence the Yang-Mills equation, are conformally invariant.
§2. A regularity theorem for Yang-Mills connections

The principal term in Yang-Mills equation is $d^*dA$. The operator $d^*d$ is not elliptic. Thus we cannot expect solutions to be smooth. This is also clear from the gauge invariance. Given a smooth solution we can manufacture a non-smooth solution by applying a suitable non-smooth gauge transformation. Conversely, the best we could hope for is that any solution to Yang-Mills equation is gauge-equivalent to a smooth solution. Such a result was proven by K. Uhlenbeck.

Before discussing her theorem, I want to review a classical geometric result. A connection $A$ is said be trivial if it gauge-equivalent to 0. A connection $A$ is said to be flat if $F_A = 0$. Clearly any trivial connection is flat.

**Lemma 2.1.** If $A$ is a connection defined in a simply connected domain $\Omega$ and $A$ is flat, then $A$ is trivial.

**Proof.** A connection $A$ is trivial if we can solve the equation $g.A = 0$ for $g$. Fully written out, this equation takes the form

\[
\frac{\partial g}{\partial x_i} = g A_i.
\]

This implies

\[
\frac{\partial^2 g}{\partial x_i \partial x_j} = \frac{\partial g}{\partial x_j} A_i + g \frac{\partial A}{\partial x_j} = g A_j A_i + g \frac{\partial A}{\partial x_j}.
\]

The identity

\[
\frac{\partial^2 g}{\partial x_i \partial x_j} = \frac{\partial^2 g}{\partial x_j \partial x_i}
\]

gives rise to the integrability condition

\[
A_j A_i + \frac{\partial A_i}{\partial x_j} = A_i A_j + \frac{\partial A_i}{\partial x_j},
\]

which is equivalent to

\[
F_A = 0.
\]

This condition is clearly necessary for the existence of a solution $g$. By Frobenius theorem it is also sufficient, as long as $\Omega$ is simply connected.

We will not actually use this Lemma. It only serves as a motivation for Uhlenbeck’s good gauge theorem. In fact, Uhlenbeck’s theorem can be viewed as an analyst’s version of Lemma 2.1; it says that if $A$ is a connection, on the unit ball, with small curvature, then there exists a gauge transformation $g$ such that $g.A$ is small.

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For simplicity we now restrict our attention to 4-dimensions. Let $B_1$ denote the unit ball in $\mathbb{R}^4$. Let $\nu$ denote the outward unit normal of $\partial B_1$. Let $L^{p,k}(B_1)$ denote the Sobolev space of functions with $k$ derivatives in $L^p$. We say that a form or a connection is in $L^{p,k}(B_1)$ if all its components are in $L^{p,k}(B_1)$. It is natural to consider connection $A \in L^{2,1}(B_1)$. It follows from the Sobolev embedding $L^{2,1} \to L^4$ that if $A \in L^{2,1}$ then $\mathcal{F}_A \in L^2$ and $\mathcal{W}^1(A) < \infty$.

**Theorem 2.2.** [U1] There exists $\varepsilon > 0$ such that if $A$ is a connection in $L^{2,1}(B_1)$ with

$$\|\mathcal{F}_A\|_{L^2(B_1)} \leq \varepsilon$$

then there exists a gauge transformation $g$ in $L^{2,2}(B_1)$ such that

$$\begin{align*}
\nu \cdot (g.A) &= \sum_i x_i (g.A)_i = 0 \quad \text{on } \partial B_1 \\
^{*}d(g.A) &= \sum_i \frac{\partial}{\partial x_i} (g.A)_i = 0 \quad \text{on } B_1,
\end{align*}$$

(2.6)

and

$$\|g.A\|_{L^{2,1}(B_1)} \leq c\|\mathcal{F}_A\|_{L^2(B_1)}.$$

The conditions (2.6) are called gauge conditions.

The theorem is proven as follows. Assume that $A$ satisfies the gauge conditions. Let $A + b$ be a small perturbation of $A$. We want to show that $A + b$ can be transformed to a connection that satisfies the gauge conditions. This amounts to solving the non-linear boundary value problem

$$\begin{align*}
^{*}d(g.(A + b)) &= 0 \quad \text{on } B_1 \\
\nu \cdot (g.(A + b)) &= 0 \quad \text{on } \partial B_1,
\end{align*}$$

for $g$. If we let $g = \exp \varphi$ and linearize around $\varphi = 0$ and $b = 0$ then we get the linear boundary value problem

$$\begin{align*}
\Delta \varphi + \sum_i \left[ A_i, \frac{\partial \varphi}{\partial x_i} \right] &= -^{*}d^*b \quad \text{on } B_1 \\
\nu \cdot d\varphi &= -\nu \cdot b \quad \text{on } \partial B_1.
\end{align*}$$

(2.4)

This system can clearly be solved if $A$ is small enough; then it is a small perturbation of the Neumann problem for the Laplace operator. It then follows from the implicit function theorem that the non-linear boundary value problem can be solved if $b$ is small enough. The theorem can then be proven by the continuity method. See [U1] for more details.
Theorem 2.3. [U1] There exist constants $c_k$ such that if $A$ in addition to the assumptions in Theorem 2.2 satisfies Yang-Mills equations, then $g.A$ is smooth on the interior of $B_1$ and

$$\|g.A\|_{C^k(B_1/2)} \leq c_k\|F_A\|_{L^2(B_1)}.$$ 

This is seen as follows. Assume that $A$ is Yang-Mills and $d^*A = 0$. We now have that $\Delta A = dd^*A + d^*dA$. Hence it follows that

$$(2.5) \quad \Delta A + \{A \otimes \nabla A\} + \{A \otimes A \otimes A\} = 0.$$ 

This is a semi-linear elliptic equation. If $A \in L^{2,1}$, then we can estimate higher derivatives of $A$ by bootstrapping. In the first iteration step we have to use to usual trick of estimating the difference quotient of $A$.

This situation is common in gauge theory. In order to prove regularity for an equation, one has to supplement it with gauge conditions. Thus, when facing a new equation, the first question is, what is the right gauge condition.

§3. Singular connections

According to a theorem by K. Uhlenbeck, point singularities of finite energy connections are removable. The precise statement is as follows:

**Theorem 3.1.** [U2], [U3] If $A$ is a connection in $L^{2,1}_{\text{loc}}(B_1 \setminus \{0\})$ and $F_A \in L^2(B_1 \setminus \{0\}) = L^2(B_1)$, then there exists a gauge transformation $g \in L^{2,2}(B_1)$ such that $g.A \in L^{2,1}(B_1)$.

This theorem was originally proven under the extra assumption that $A$ be Yang-Mills, [U2]. Later it was discovered that finite energy sufficed, [U3].

According to a theorem of mine, singularities along embedded curves are removable. It suffices to consider the connections on $B_1 \setminus L_1$ where $L_1 = \{(x_1,0,0,0) \mid |x_1| \leq 1\}$.

**Theorem 3.2.** [R2] If $A$ is a connection in $L^{2,1}_{\text{loc}}(B_1 \setminus L_1)$ and $F_A \in L^2(B_1 \setminus L_1) = L^2(B_1)$, then there exists a gauge transformation $g \in L^{2,2}_{\text{loc}}(B_1 \setminus L_1)$ such that $g.A \in L^{2,1}(B_1)$.

The next case is connections on a 4-manifold with singularities along an embedded surface. The local model are then connections on $B_1$ with singularities along $D_1 = \{(x_1,x_2,0,0) \mid x_1^2 + x_2^2 = 1\}$. It is not true that finite energy connections on $B_1 \setminus D_1$ can be extended to connections on $B_1$. Unlike $B_1 \setminus \{0\}$ and $B_1 \setminus L_1$, the domain $B_1 \setminus D_1$ is not simply connected. Hence Lemma 2.1 does not apply to $B_1 \setminus D_1$. Thus, before we attempt to generalize the theorems of §2 and §3 to $B_1 \setminus D_1$ we need to generalize Lemma 2.1 to non-simply-connected domains $\Omega$. This requires the notion of holonomy.
Holonomy and flat connections. Let $A$ be a connection in a region $Q$ in $\mathbb{R}^4$. Let $x_0 \in \Omega$. Let $\gamma : [0,1] \to \Omega$ be a closed smooth curve in $\Omega$ with $\gamma(0) = \gamma(1) = x_0$. The initial value problem

\[
\begin{align*}
\frac{\partial h}{\partial t} + h \sum_i A_i \frac{d\gamma_i}{dt} &= 0, \\
h(0) &= 1
\end{align*}
\]

has a unique solution. The element $h(1) \in G$ is called the holonomy of $A$ around $\gamma$. This initial value problem is gauge-invariant in the sense that

\[
(g.h)(t) = g(x(t))h(t)g(x(t))^{-1}
\]

is a solution for $g.A$. Thus the conjugacy class of the holonomy is gauge-invariant.

If the connection is trivial, then (2.1) has a solution $g$ with $g(x_0) = 1$. Then the solution to (2.3) is given by $h(t) = g(\gamma(t))$. It follows that that the holonomy is $h(1) = g(\gamma(1)) = g(x_0) = g(\gamma(0)) = h(0) = 1$. Thus we get another condition for a connection to be trivial; the holonomy around each loop has to be the identity.

One can show that if $A$ is flat, then the holonomy of $A$ is invariant under smooth deformations of $\gamma$. Thus the holonomy only depends the homotopy class of $\gamma$. Hence it gives a map $\pi_1(\Omega, x_0) \to G$. Here $\pi_1(\Omega, x_0)$ denotes the fundamental group of $\Omega$ with base point $x_0$. It is easily seen that that this map is a homomorphism. If we apply a gauge transformation $g$ to $A$ or if we change the base point, then this homomorphism gets conjugated by an element of $G$.

**Theorem 3.3.** There is a 1-1 correspondence between gauge equivalence classes of flat $G$-connections on $\Omega$ and conjugacy classes of homomorphisms $\pi_1(\Omega) \to G$.

The proof is not hard; see for instance [KN] Prop. 9.3.

In our special case of $B_1 \setminus D_1$, the fundamental group is generated by any loop that goes around $D_1$ once. It follows that flat connections are classified by the holonomy around this loop.

**Corollary 3.4.** There is a 1-1 correspondence between gauge equivalence classes of flat $G$-connections on $B_1 \setminus D_1$ and conjugacy classes in $G$.

Limit Holonomy. As we have seen, flat connections on $B_1 \setminus D_1$ are classified by their holonomy. A non-flat connection does not a uniquely defined holonomy. However, any connection on $B_1 \setminus D_1$ with curvature in $L^2$ has a well-defined limit holonomy.

We introduce cylindrical coordinates $(x_1, x_2, r, \theta)$ on $B_1$, with $x_3 = r \cos \theta$ and $x_4 = r \sin \theta$. In these coordinates $D_1$ is given by $r = 0$. 

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Theorem 3.5. [SS] If $A$ is a $G$-connection in $L^{2,1}_{\text{loc}}(B_1 \setminus D_1)$ with $F_A \in L^2(B_1 \setminus D_1)$, then the holonomy of $A$ around the loop $\gamma(t) = (x_1, x_2, r \cos(2\pi t), r \sin(2\pi t))$ exists for almost all $x_1, x_2$ and $r$. The limit of this holonomy as $r \to 0$ exists for almost all $x_1$ and $x_2$. This limit is independent of $x_1$ and $x_2$ for almost all $x_1$ and $x_2$.

This unique limit is called the limit holonomy of the connection.

We can now state the correct analog of Theorem 3.1 and Theorem 3.2 for $B_1 \setminus D_1$. Note that if $G$ is connected, then $\exp : \mathfrak{g} \to G$ is surjective. (Proof: On a complete Riemannian manifold any two points can be connected by a geodesic curve. On a Lie group with an invariant metric, in particular any compact Lie group, the geodesic curves through the identity are precisely the 1-parameter subgroups.)

Theorem 3.6. [R2] If $A$ is a $G$-connection in $L^{2,1}_{\text{loc}}(B_1 \setminus D_1)$ with limit holonomy $\exp(-2\pi X)$, then there exists a gauge transformation $g \in L^{2,2}_{\text{loc}}(B_1 \setminus D_1)$ such that

$$g.A = X\, d\theta + a$$

where $a, \nabla_{X\, d\theta} a \in L_{X\, d\theta}(B_1)$.

Here

$$\nabla_A \sigma = \sum_j \sum_{i_1 < \cdots < i_p} \left( \frac{\partial}{\partial x_j} \sigma_{i_1 \cdots i_p} + [A_j, \sigma_{i_1 \cdots i_p}] \right) dx_j \otimes dx_{i_1} \wedge \cdots \wedge dx_{i_p}.$$

Note that the connection $X\, d\theta + a$ has curvature $d_{X\, d\theta} a + \frac{1}{2}[a, a]$. Hence the condition $a \in L^{2,1}_{X\, d\theta}$ ensures that the curvature lies in $L^2$.

As a consequence of Thm. 3.6, a singularity along a surface of a finite energy connection is removable if and only if the limit holonomy is trivial.

The Yang-Mills connections used by Kronheimer and Mrowka are Yang-Mills connections on a 4-manifold with singularities along an embedded surface. Near any point of the surface they are of the form $X\, d\theta + a$ with $a \in L^{2,1}_{X\, d\theta}(B_1)$.
§4. A regularity theorem for singular Yang-Mills connections

To keep the notation simple, we will now restrict our attention to the Lie group $SU(2)$. This is the group of all unitary $2 \times 2$ matrices with determinant one. These are precisely the matrices

$$
\begin{pmatrix}
  z & w \\
  -\bar{w} & \bar{z}
\end{pmatrix}
$$

where $z$ and $w$ are complex numbers with $|z|^2 + |w|^2 = 1$.

The corresponding Lie algebra $su(2)$ consists of the skew-hermitian $2 \times 2$ matrices with trace zero. These are precisely the matrices

$$
\begin{pmatrix}
  it & z \\
  -\bar{z} & -it
\end{pmatrix}
$$

with $t$ real and $z$ complex.

Each conjugacy class in $SU(2)$ contains exactly one element of the form

$$
\begin{pmatrix}
  \exp(-2\pi i\alpha) & 0 \\
  0 & \exp(2\pi i\alpha)
\end{pmatrix}
$$

with $0 \leq \alpha \leq 1/2$. It then follows from Theorem 3.6 that the natural class of connections on $B_1 \setminus D_1$ are connections of the form

$$
\begin{pmatrix}
  i\alpha & 0 \\
  0 & -i\alpha
\end{pmatrix}
d\theta + a
$$

with $0 \leq \alpha \leq 1/2$. Here I will only discuss the case $0 < \alpha < 1/2$. In the case of $\alpha = 0$, the singularity is removable, and we are back to the case discussed in §2. In the case $\alpha = 1/2$, the singularity is removable as far as the local analysis is concerned; however there can be topological obstructions to removing the singularity globally on a 4-manifold, see [KM1].

If $\sigma$ is an $su(2)$-valued $p$-form, then we can decompose $\sigma$ as

$$
\sigma = \begin{pmatrix}
  i\sigma_D & \sigma_T \\
  -\sigma_T & -i\sigma_D
\end{pmatrix}
$$

where $\sigma_D$ is a real valued $p$-form and $\sigma_T$ is a complex valued $p$-form. We have

$$
\nabla (\begin{pmatrix}
  i\alpha & 0 \\
  0 & -i\alpha
\end{pmatrix}) d\theta \sigma = \nabla \sigma + \left(\begin{pmatrix}
  i\alpha & 0 \\
  0 & -i\alpha
\end{pmatrix} d\theta, \sigma\right) = \begin{pmatrix}
  i\nabla \sigma_D & \nabla_{2i\alpha d\theta} \sigma_T \\
  -\nabla_{2i\alpha d\theta} \sigma_T & -i\nabla \sigma_D
\end{pmatrix}
$$

Thus $\nabla (\begin{pmatrix}
  i\alpha & 0 \\
  0 & -i\alpha
\end{pmatrix}) d\theta$ acts on $\sigma_D$ as $\nabla$ and on $\sigma_T$ as $\nabla_{2i\alpha d\theta}$. Let $d_{2i\alpha d\theta}$ denote the covariant exterior derivative given by the connection $2\alpha d\theta$. Let $d_{2i\alpha d\theta}^*$ denote the adjoint of $d_{2i\alpha d\theta}$.

We then have the following analog of Theorem 2.2.
Theorem 4.1. [R1] For any $\alpha$ with $2\alpha \notin \mathbb{Z}$ there exists $\varepsilon > 0$ such that if

$$A = \begin{pmatrix} i\alpha & 0 \\ 0 & -i\alpha \end{pmatrix} \, d\theta + a$$

is a connection with $a \in L^{2,1}_{\partial \mathcal{B}}(\mathcal{B}_1)$ and

$$\|F_A\|_{L^2(\mathcal{B}_1 \setminus D_1)} \leq \varepsilon,$$

then there exists a gauge transformation $g \in L^{2,2}_{\partial \mathcal{B}}(\mathcal{B}_1)$ such that

$$g \cdot A = \begin{pmatrix} ia & 0 \\ 0 & -i\alpha \end{pmatrix} \, d\theta + a'$$

where

$$\begin{cases} d^* a'_\rho = 0 & \text{on } B_1 \\ d_{2i\alpha d\theta}^*(r^{-2} a'_r) = 0 & \text{on } B_1 \setminus D_1 \\ \nu - a' = 0 & \text{on } \partial B_1 \end{cases}$$

and

$$\|a'\|_{L^{2,1}_{\partial \mathcal{B}}(\mathcal{B}_1)} \leq c\|F_A\|_{L^2(\mathcal{B}_1)}.$$ 

We also have the following analog of Theorem 2.3.

Theorem 4.2. [R1] If in addition to the assumptions of Theorem 4.1 the connection is Yang-Mills, then

$$\begin{cases} |a'_\rho| + |\nabla a'_\rho| \leq c\|F_A\|_{L^2(\mathcal{B}_1)} & \text{on } B_{1/2} \\ |\nabla^k a'_r| \leq cr^{2\min\{2\alpha, 1-2\alpha\} - k}\|F_A\|_{L^2(\mathcal{B}_1)} & \text{on } B_{1/2} \end{cases}.$$ 

These seemingly strange theorems demand an explanation. The key is to understand the function space $L^{2,1}_{\partial \mathcal{B}}(\mathcal{B}_1)$ in more detail. It follows from (1.1) that

$$a \in L^{2,1}_{\partial \mathcal{B}}(\mathcal{B}_1) \iff \begin{cases} a_\rho \in L^{2,1}(\mathcal{B}_1) \\ a_r \in L^{2,1}_{2i\alpha d\theta}(\mathcal{B}_1) \end{cases}$$

Here $a_r \in L^{2,1}_{2i\alpha d\theta}$ means that $a_r, \nabla_{2i\alpha d\theta} a_r \in L^2(\mathcal{B}_1)$. Fully written out

$$\|\nabla_{2i\alpha d\theta} \sigma_r\|_{L^2(\mathcal{B}_1)}^2 = \int_{\mathcal{B}_1} \left( \left| \frac{\partial \sigma_r}{\partial x_1} \right|^2 + \left| \frac{\partial \sigma_r}{\partial x_2} \right|^2 + \left| \frac{\partial \sigma_r}{\partial r} \right|^2 + r^{-2} \left| \frac{\partial \sigma_r}{\partial \theta} + 2i\alpha \sigma_r \right|^2 \right) r \, dx_1 \, dx_2 \, dr \, d\theta$$

Now,

$$\int_{S^1} f^2 \, d\theta \leq \left( \min\{2\alpha, 1-2\alpha\} \right)^{-1} \int_{S^1} (df/d\theta + 2i\alpha f)^2 \, d\theta.$$
It follows that
\[ \| r^{-1} \sigma_T \|_{L^2(B_1)} \leq c \| \nabla_{2\omega d\theta} \sigma_T \|_{L^2(B_1)}. \]
On the other hand, it is clear that
\[ \| \nabla_{2\omega d\theta} \sigma_T \|_{L^2(B_1)} \leq c(\| \nabla \sigma_T \|_{L^2(B_1)} + \| r^{-1} \sigma_T \|_{L^2(B_1)}). \]
Hence
\[ a \in L^{2,1}_{\left( \frac{i}{\omega} - \frac{i\omega}{\theta} \right)}(B_1) \Leftrightarrow \begin{cases} a_D \in L^{2,1}(B_1) \\ \nabla a_T, r^{-1} a_T \in L^2(B_1) \end{cases} \]
So now you think I'm going to talk about analysis on weighted Sobolev spaces with singular weights. I'm not.

As I mentioned before, the finite energy condition and the Yang-Mills equation are conformally invariant in 4 dimensions. Thus we can replace the standard metric
\[ \sum_i dx_i^2 = dx_1^2 + dx_2^2 + dr^2 + r^2 d\theta^2 \]
with any conformal metric. A natural choice is the metric
\[ r^{-2} \sum_i dx_i^2 = r^{-2}(dx_1^2 + dx_2^2 + dr^2) + d\theta^2. \]
With this metric $D_1$ is moved out to infinity. We recognize $r^{-2}(dx_1^2 + dx_2^2 + dr^2)$ as the upper half space model of hyperbolic 3-space. Thus $\mathbb{R}^4$ with the metric $r^{-2} \sum dx_i^2$ is isometric with $H^3 \times S^1$, the cartesian product of hyperbolic 3-space and the unit circle. The unit ball with this metric is isometric with $H^3_+ \times S^1$, the cartesian product of one half of hyperbolic 3-space and the unit circle. Thus we can view $\sigma_T$ as a differential form on $H^3_+ \times S^1$. A short calculation shows that $r^{-1} \sigma_T, \nabla \sigma_T \in L^{2,1}$ if and only if $\sigma_T \in L^{2,1}(H^3_+ \times S^1)$. Thus
\[ a \in L^{2,1}_{\left( \frac{i}{\omega} - \frac{i\omega}{\theta} \right)}(B_1) \Leftrightarrow \begin{cases} a_D \in L^{2,1}(B_1) \\ a_T \in L^{2,1}(H^3_+ \times S^1) \end{cases} \]
Thus we should view $a_D$ as a differential form on $B_1$ and $a_T$ as a differential form on $H^3_+ \times S^1$. Let $d^*_{2\omega d\theta}$ denote the adjoint of $d_{2\omega d\theta}$ with respect to the metric $r^{-1} \sum dx_i^2$. Moreover, a short calculation shows that
\[ d^*_{2\omega d\theta} a_T = r^4 d^*_{2\omega d\theta}(r^{-2} a_T). \]
In other words, the gauge condition says that $a_D$ is coclosed on $B_1$ and $a_T$ is coclosed on $H^3_+ \times S^1$. 

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Theorem 4.1 is now proven along the same lines as Thm. 2.1. Instead of the equation (2.4) we get the equations

\[
\begin{align*}
\Delta \varphi_D + \sum_i (a_T)_i \frac{\partial \varphi_T}{\partial x_i} &= -d^* b_D & \text{on } B_1 \\
\Delta_{h,2io d\theta} \varphi_T + \Re \sum_i ((a_T)_i (d \varphi_D)_i + (a_D)_i (d_{2io d\theta} \varphi_T)_i) &= -d^*_h,2iad6\theta_T & \text{on } H^3_+ \times S^1 \\
\nu \cdot d\varphi_D &= \nu \cdot b_D & \text{on } \partial B_1 \\
\nu_h \cdot d_{2io d\theta} \varphi_T &= \nu_h \cdot b_T & \text{on } \partial H^3_+ \times S^1
\end{align*}
\]

where $\Delta_{h,2io d\theta} = d_{2io d\theta}^* d_{h,2io d\theta}^* + d_{h,2io d\theta}^* d_{2io d\theta}$ is the covariant Hodge Laplacian for 1-forms on $H^3_+ \times S^1$ given by the connection $2io d\theta$, and $\nu_h = r\nu$ is the outward unit normal of $H^3_+ \times S^1$. Thus we get a small perturbation of the Neumann problem for $\Delta$ on $B_1$ and the Neumann problem for $\Delta_{h,2io d\theta}$ on $H^3_+ \times S^1$. The theory for the former is well known. The latter is analyzed in [R1] by elementary methods.

Theorem 4.2 is now proven along the same lines as Thm. 2.3. Instead of the equation (2.5) we get the system

\[
\begin{align*}
\Delta a_D + \{a_T \otimes \nabla_{2io d\theta} a_T\} + \{a_D \otimes a_T \otimes a_T\} &= 0 & \text{on } B_1 \\
\Delta_{h,2io d\theta} a_T + \{a_D \otimes \nabla_{h,2io d\theta} a_T\} + \{a_T \otimes \nabla_h a_D\} + \{a_T \otimes a_T \otimes a_T\} &= 0 & \text{on } H^3_+ \times S^1.
\end{align*}
\]

The 1-form $a$ can now be estimated by a bootstrapping procedure. On the first equation we apply standard elliptic estimates for the usual Laplacian $\Delta$ on $B_1$. On the second equation we apply decay estimates at infinity for the covariant Hodge Laplacian $\Delta_{h,2io d\theta}$ on $H^3_+ \times S^1$. These decay estimates are derived in [R1] by elementary methods.
References


More references on singular Yang-Mills fields can be found in [R1].

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May 25, 1995

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